CHARACTERIZATION OF A BANACH-FINSLER MANIFOLD IN TERMS OF THE ALGEBRAS OF SMOOTH FUNCTIONS

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Abstract. In this note we give sufficient conditions to ensure that the weak Finsler structure of a complete $C^k$ Finsler manifold $M$ is determined by the normed algebra $C^b_k(M)$ of all real-valued, bounded and $C^k$ smooth functions with bounded derivative defined on $M$. As a consequence, we obtain: (i) the Finsler structure of a finite-dimensional and complete $C^k$ Finsler manifold $M$ is determined by the algebra $C^b_k(M)$; (ii) the weak Finsler structure of a separable and complete $C^k$ Finsler manifold $M$ modeled on a Banach space with a Lipschitz and $C^k$ smooth bump function is determined by the algebra $C^b_k(M)$; (iii) the weak Finsler structure of a $C^1$ uniformly bumpable and complete $C^1$ Finsler manifold $M$ modeled on a Weakly Compactly Generated (WCG) Banach space is determined by the algebra $C^b_1(M)$; and (iv) the isometric structure of a WCG Banach space $X$ with an $C^1$ smooth bump function is determined by the algebra $C^b_1(X)$.

1. Introduction and Preliminaries

In this note, we are interested in characterizing the Finsler structure of a Finsler manifold $M$ in terms of the space of real-valued, bounded and $C^k$ smooth functions with bounded derivative defined on $M$. The problem of the interrelation of the topological, metric and smooth structure of a space $X$ and the algebraic and topological structure of the space $C(X)$ (the set of real-valued continuous functions defined on $X$) has been largely studied. These results are usually referred to as Banach-Stone type theorems. Recall the celebrated Banach-Stone theorem, asserting that the compact spaces $K$ and $L$ are homeomorphic if and only if the Banach spaces $C(K)$ and $C(L)$ endowed with the sup-norm are isometric. For more information on Banach-Stone type theorems see the survey [10] and references therein.

The Myers-Nakai theorem states that the structure of a complete Riemannian manifold $M$ is characterized in terms of the Banach algebra $C^1_b(M)$ of all real-valued, bounded and $C^1$ smooth functions with bounded derivative defined on $M$ endowed with the sup-norm of the function and its derivative. More specifically, two complete Riemannian manifolds $M$ and $N$ are equivalent as Riemannian manifolds, i.e. there is a $C^1$ diffeomorphism $h : M \to N$ such that

$$\langle dh(x)(v), dh(x)(w) \rangle_{h(x)} = \langle v, w \rangle_x$$

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for every $x \in M$ and $v, w \in T_x M$ if and only if the Banach algebras $C^1_b(M)$ and $C^k_b(N)$ are isometric. This result was first proved by S. B. Myers [22] for a compact and Riemannian manifold and by M. Nakai [23] for a finite-dimensional Riemannian manifold. Very recently, I. Garrido, J.A. Jaramillo and Y.C. Rangel [12] gave an extension of the Myers-Nakai theorem for every infinite-dimensional, complete Riemannian manifold. A similar result for the so-called finite-dimensional Riemannian-Finsler manifolds is given in [14] (see also [26]).

Our aim in this work is to extend the Myers-Nakai theorem to the context of Finsler manifolds. On the one hand, we obtain the Myers-Nakai theorem for (i) finite-dimensional and complete Finsler manifolds, and (ii) WCG Banach spaces with a $C^1$ smooth bump function. On the other hand, we study for $k \geq 1$ the algebra $C^k_b(M)$ of all real-valued, bounded and $C^k$ smooth functions with bounded first derivative defined on a complete Finsler manifold $M$. We prove that these algebras determine the weak Finsler structure of a complete Finsler manifold when $k = 1$ and the Finsler structure when $k \geq 2$. In particular, we obtain a weaker version of the Myers-Nakai theorem for (i) separable and complete Finsler manifolds modeled on a Banach space with a Lipschitz and $C^k$ smooth bump function, and (ii) $C^1$ uniformly bumpable and complete Finsler manifolds modeled on WCG Banach spaces. In the proof of these results we will use the ideas of the Riemannian case [12].

The notation we use is standard. The norm in a Banach space $X$ is denoted by $\| \cdot \|$. The dual space of $X$ is denoted by $X^*$ and its dual norm by $\| \cdot \|_*$. The open ball with center $x \in X$ and radius $r > 0$ is denoted by $B(x, r)$. A $C^k$ smooth bump function $b : X \to \mathbb{R}$ is a $C^k$ smooth function on $X$ with bounded, non-empty support, where $\text{supp}(b) = \{x \in X : b(x) \neq 0\}$. If $M$ is a Banach manifold, we denote by $T_x M$ the tangent space of $M$ at $x$. Recall that the tangent bundle of $M$ is $TM = \{(x, v) : x \in M$ and $v \in T_x M\}$. We refer to [6], [8], [19] and [7] for additional definitions. We will say that the norms $\| \cdot \|_1$ and $\| \cdot \|_2$ defined on a Banach space $X$ are $K$-equivalent ($K \geq 1$) whether $\frac{1}{K} \| v \|_1 \leq \| v \|_2 \leq K \| v \|_1$, for every $v \in X$.

Let us begin by recalling the definition of a $C^k$ Finsler manifold in the sense of Palais as well as some basic properties (for more information about these manifolds see [25], [7], [27], [24], [13] and [18]).

**Definition 1.1.** Let $M$ be a (paracompact) $C^k$ Banach manifold modeled on a Banach space $(X, \| \cdot \|)$, where $k \in \mathbb{N} \cup \{\infty\}$. Let us consider the tangent bundle $TM$ of $M$ and a continuous map $\| \cdot \|_M : TM \to [0, \infty)$. We say that $(M, \| \cdot \|_M)$ is a $C^k$ Finsler manifold in the sense of Palais if $\| \cdot \|_M$ satisfies the following conditions:

(P1) For every $x \in M$, the map $\| \cdot \|_x : \| \cdot \|_{T_x M} : T_x M \to [0, \infty)$ is a norm on the tangent space $T_x M$ such that for every chart $\varphi : U \to X$ with $x \in U$, the norm $v \in X \mapsto \|d\varphi^{-1}(\varphi(x))(v)\|_x$ is equivalent to $\| \cdot \|$ on $X$.

(P2) For every $x_0 \in M$, every $\varepsilon > 0$ and every chart $\varphi : U \to X$ with $x_0 \in U$, there is an open neighborhood $W$ of $x_0$ such that if $x \in W$ and $v \in X$, then

\[
(1.1) \quad \frac{1}{1 + \varepsilon} \|d\varphi^{-1}(\varphi(x_0))(v)\|_{x_0} \leq \|d\varphi^{-1}(\varphi(x))(v)\|_x \leq (1 + \varepsilon)\|d\varphi^{-1}(\varphi(x_0))(v)\|_{x_0}.
\]
In terms of equivalence of norms, the above inequalities yield the fact that the norms $||d\varphi^{-1}(\varphi(x))(\cdot)||_x$ and $||d\varphi^{-1}(\varphi(x_0))(\cdot)||_{x_0}$ are $(1+\varepsilon)$-equivalent.

Let us recall that Banach spaces and Riemannian manifolds are $C^\infty$ Finsler manifolds in the sense of Palais [25].

Let $M$ be a Finsler manifold, we denote by $T_x M^*$ the dual space of the tangent space $T_x M$. Let $f : M \to \mathbb{R}$ be a differentiable function at $p \in M$. The norm of $df(p) \in T_p M^*$ is given by

$$||df(p)||_p = \sup\{|df(p)(v)| : v \in T_p M, ||v||_p \leq 1\}.$$ 

Let us consider a differentiable function $f : M \to N$ between Finsler manifolds $M$ and $N$. The norm of the derivative at the point $p \in M$ is defined as

$$||df(p)||_p = \sup\{|df(p)(v)| : v \in T_p M, ||v||_p \leq 1\} = \sup\{|\xi(df(p)(v)) : \xi \in T_p M^*, v \in T_p M \text{ and } ||v||_p = 1 = ||\xi||_{T_p M^*}\},$$

where $||\cdot||_{T_p M^*}$ is the dual norm of $||\cdot||_{T_p M}$. Recall that if $(M, ||\cdot||_M)$ is a Finsler manifold, the length of a piecewise $C^1$ smooth path $c : [a, b] \to M$ is defined as $\ell(c) := \int_a^b ||c'(t)||_{c(t)} dt$. Besides, if $M$ is connected, then it is connected by piecewise $C^1$ smooth paths, and the associated Finsler metric $d_M$ on $M$ is defined as

$$d_M(p, q) = \inf\{\ell(c) : c \text{ is a piecewise } C^1 \text{ smooth path connecting } p \text{ and } q\}.$$ 

It was shown in [25] that the Finsler metric is consistent with the topology given in $M$. The open ball of center $p \in M$ and radius $r > 0$ is denoted by $B_M(p, r) := \{q \in M : d_M(p, q) < r\}$. The Lipschitz constant $\text{Lip}(f)$ of a Lipschitz function $f : M \to N$, where $M$ and $N$ are Finsler manifolds, is defined as $\text{Lip}(f) = \sup\{\frac{d_N(f(x), f(y))}{d_M(x, y)} : x, y \in M, x \neq y\}$. We shall only consider connected manifolds. Let us recall the following “mean value inequality” for Finsler manifolds [1, 18].

**Lemma 1.2.** Let $M$ and $N$ be $C^1$ Finsler manifolds (in the sense of Palais) and $f : M \to N$ a $C^1$ smooth function. Then, $f$ is Lipschitz if and only if $||df||_\infty := \sup\{|df(x)|_x : x \in M\} < \infty$. Furthermore, $\text{Lip}(f) = ||d_M||_\infty$.

We will also need the following result related to the $(1 + \varepsilon)$-bi-Lipschitz local behavior of the charts of a $C^1$ Finsler manifold in the sense of Palais [18, Lemma 2.4].

**Lemma 1.3.** Let us consider a $C^1$ Finsler manifold $M$ (in the sense of Palais). Then, for every $x_0 \in M$ and every chart $(U, \varphi)$ with $x_0 \in U$ satisfying inequality (1.1), there exists an open neighborhood $V \subset U$ of $x_0$ satisfying

$$\frac{1}{1 + \varepsilon} d_M(p, q) \leq ||\varphi(p) - \varphi(q)|| \leq (1 + \varepsilon) d_M(p, q), \quad \text{for every } p, q \in V,$$

where $||\cdot||_V$ is the (equivalent) norm $||d\varphi^{-1}(\varphi(x))(\cdot)||_{x_0}$ defined on $X$.

Now, let us recall the concept of uniformly bumpable manifold, introduced by D. Azagra, J. Ferrera and F. López-Mesas [1] for Riemannian manifolds. A natural extension to Finsler manifolds is defined in the same way [18].

**Definition 1.4.** A $C^k$ Finsler manifold in the sense of Palais $M$ is $C^k$ uniformly bumpable whenever there are $R > 1$ and $r > 0$ such that for every $p \in M$ and $\delta \in (0, r)$ there exists a $C^k$ smooth function $b : M \to [0, 1]$ such that:
Definition 1.5. Let $X$ and $Y$ be Banach spaces and consider a function $f : U \to Y$, where $U$ is an open subset of $X$. The function $f$ is said to be weakly $C^k$ smooth at the point $x_0$ whenever there is an open neighborhood $U_{x_0}$ of $x_0$ such that $y^* \circ f$ is $C^k$ smooth at $U_{x_0}$, for every $y^* \in \mathcal{Y}^*$. The function $f$ is said to be weakly $C^k$ smooth on $U$ whenever $f$ is weakly $C^k$ smooth at every point $x \in U$.

On the one hand, J. M. Gutiérrez and J. L. G. Llavona [15] proved that if $f : U \to Y$ is weakly $C^k$ smooth on $U$, then $g \circ f \in C^k(U)$ for all $g \in C^k(Y)$. They also proved that if $f : U \to Y$ is weakly $C^k$ smooth on $U$, then $f \in C^{k-1}(U)$. For $k = 1$, the above yields that every weakly $C^1$ smooth function on $U$ is continuous on $U$. Also, for $k = \infty$, every weakly $C^\infty$ smooth function on $U$ is $C^\infty$ smooth on $U$. M. Bachir and G. Lancien [4] proved that, if the Banach space $Y$ has the Schur property, then the concept of weakly $C^k$ smoothness coincides with the concept of $C^k$ smoothness. On the other hand, there are examples of weakly $C^1$ smooth functions that are not $C^1$ smooth (see [15] and [4]).

Definition 1.6. Let $M$ and $N$ be $C^k$ Finsler manifolds and $U \subset M$, $O \subset N$ open subsets of $M$ and $N$, respectively. A function $f : U \to N$ is said to be weakly $C^k$ smooth at the point $x_0$ of $U$ if there exist charts $(W, \varphi)$ of $M$ at $x_0$ and $(V, \psi)$ of $N$ at $f(x_0)$ such that $\psi \circ f \circ \varphi^{-1}$ is weakly $C^k$ smooth at $\varphi(W)$. We say that $f : U \to N$ is weakly $C^k$ smooth in $U$ if $f$ is weakly $C^k$ smooth at every point $x \in U$. We say that a bijection $f : U \to O$ is a weakly $C^k$ diffeomorphism if $f$ and $f^{-1}$ are weakly $C^k$ smooth on $U$ and $O$, respectively. Notice that these definitions do not depend on the chosen charts.
Let us note that there are homeomorphisms which are weakly $C^1$ smooth but not differentiable. Indeed, we follow [15, Example 3.9] and define $g : \mathbb{R} \to c_0(\mathbb{N})$ and $h : c_0(\mathbb{N}) \to c_0(\mathbb{N})$ by $g(t) = (0, \frac{1}{2} \sin(2t), \ldots, \frac{1}{n} \sin(nt), \ldots)$ and $h(x) = x + g(x_1)$ for every $t \in \mathbb{R}$ and $x = (x_1, \ldots, x_n, \ldots) \in c_0$. The function $h$ is an homeomorphism, $h^{-1}(y) = y - g(y_1)$ for every $y \in c_0$, and $h$ is weakly $C^1$ smooth on $c_0(\mathbb{N})$. Notice that if $h$ were differentiable at a point $x \in c_0$ with $x_1 = 0$, then

$$h'(x)(1,0,0,\ldots) = (1,1,1,\ldots) \in \ell_\infty \setminus c_0,$$

which is a contradiction.

Now, let us consider different definitions of isometries between $C^k$ Finsler manifolds.

**Definition 1.7.** Let $(M, || \cdot ||_M)$ and $(N, || \cdot ||_N)$ be $C^k$ Finsler manifolds and a bijection $h : M \to N$.

- **(MI)** We say that $h$ is a **metric isometry** for the Finsler metrics, if
  $$d_N(h(x), h(y)) = d_M(x, y), \quad \text{for every } x, y \in M.$$

- **(FI)** We say that $h$ is a $C^k$ **Finsler isometry** if it is a $C^k$ diffeomorphism satisfying
  $$||dh(x)(v)||_{h(x)} = ||(h(x), dh(x)(v))||_N = ||(x, v)||_M = ||v||_x,$$
  for every $x \in M$ and $v \in T_x M$. We say that the Finsler manifolds $M$ and $N$ are $C^k$ **equivalent as Finsler manifolds** if there is a $C^k$ Finsler isometry between $M$ and $N$.

- **($\omega$-FI)** We say that $h$ is a **weak $C^k$ Finsler isometry** if it is a weakly $C^k$ diffeomorphism and a metric isometry for the Finsler metrics. We say that the Finsler manifolds $M$ and $N$ are **weakly $C^k$ equivalent as Finsler manifolds** if there is a weak $C^k$ Finsler isometry between $M$ and $N$.

**Proposition 1.8.** Let $M$ and $N$ be $C^k$ Finsler manifolds. Let us assume that there is a $C^k$ diffeomorphism and metric isometry (for the Finsler metrics) $h : M \to N$. Then $h$ is a $C^k$ Finsler isometry.

**Proof.** Let us fix $x \in M$ and $y = h(x) \in N$. For every $\varepsilon > 0$, there are $r > 0$ and charts $\varphi : B_M(x, r) \subset M \to X$ and $\psi : B_N(y, r) \subset N \to Y$ satisfying inequalities (1.1) and (1.2). Since $h : M \to N$ is a metric isometry, $h$ is a bijection from $B_M(x, r)$ onto $B_N(y, r)$.

Let us consider the equivalent norms on $X$ and $Y$ defined as $||| \cdot |||_x := ||d\varphi^{-1}(\varphi(x))(\cdot)||_x$ and $||| \cdot |||_y := ||d\psi^{-1}(\psi(y))(\cdot)||_y$, respectively.

Since $h$ is a metric isometry, we obtain from Lemma 1.3, for $p, q$ in an open neighborhood of $\varphi(x)$,

$$|||\psi \circ h \circ \varphi^{-1}(p) - \psi \circ h \circ \varphi^{-1}(q)|||_y \leq (1 + \varepsilon)d_N(h \circ \varphi^{-1}(p), h \circ \varphi^{-1}(q)) =
(1 + \varepsilon)d_M(\varphi^{-1}(p), \varphi^{-1}(q)) \leq (1 + \varepsilon)^2|||p - q|||_x.$$
Thus, \( \sup \{ ||d(\psi \circ h \circ \varphi^{-1})(\varphi(x))(w)||_y : ||w||_x \leq 1 \} \leq (1 + \varepsilon)^2 \). Now, for every \( v \in T_x M \) with \( v \neq 0 \), let us write \( w = d\varphi(x)(v) \in X \). We have

\[
||dh(x)(v)||_y = ||d\varphi^{-1}(\psi(y))d\psi(y)dh(x)(v)||_y = ||d(\psi \circ h)(x)||_y = \leq (1 + \varepsilon)^2||w||_x = (1 + \varepsilon)^2||v||_x.
\]

Since this inequality holds for every \( \varepsilon > 0 \) and the same argument works for \( h^{-1} \), we conclude that \( ||dh(x)(v)||_y = ||v||_x \) for all \( v \in T_x M \). Thus, \( h \) is a \( C^k \) Finsler isometry. \( \square \)

Let us now turn our attention to the Banach algebra \( C^1_b(M) \), the algebra of all real-valued, \( C^1 \) smooth and bounded functions with bounded derivative defined on a \( C^1 \) Finsler manifold \( M \), i.e.

\[
C^1_b(M) = \{ f : M \to \mathbb{R} : f \in C^1(M), \ ||f||_\infty < \infty \text{ and } ||df||_\infty < \infty \},
\]

where \( ||f||_\infty := \sup\{||f(x)||_x : x \in M\} \) and \( ||df||_\infty := \sup\{||df(x)||_\infty : x \in M\} \). The usual norm considered on \( C^1_b(M) \) is \( ||f||_{C^1_b} = \max\{||f||_\infty, ||df||_\infty \} \) for every \( f \in C^1_b(M) \) and \( (C^1_b(M), ||\cdot||_{C^1_b(M)}) \) is a Banach space. Let us notice that, by Lemma 1.2, we have \( ||df||_\infty = \text{Lip}(f) \). Recall that \((C^1_b(M), 2||\cdot||_{C^1_b(M)})\) is a Banach algebra.

For \( 2 \leq k \leq \infty \) and a \( C^k \) Finsler manifold \( M \), let us consider the algebra \( C^k_b(M) \) of all real-valued, \( C^k \) smooth and bounded functions that have bounded first derivative, i.e.

\[
C^k_b(M) = \{ f : M \to \mathbb{R} : f \in C^k(M), \ ||f||_\infty < \infty \text{ and } ||df||_\infty < \infty \} = C^k(M) \cap C^1_b(M).
\]

with the norm \( ||\cdot||_{C^1_b} \). Thus, \( C^k_b(M) \) is a subalgebra of \( C^1_b(M) \). Nevertheless, it is not a Banach algebra.

A function \( \varphi : C^k_b(M) \to \mathbb{R} (1 \leq k \leq \infty) \) is said to be an algebra homomorphism whether for all \( f, g \in C^k_b(M) \) and \( \lambda, \eta \in \mathbb{R} \),

(i) \( \varphi(\lambda f + \eta g) = \lambda \varphi(f) + \eta \varphi(g) \), and

(ii) \( \varphi(f \cdot g) = \varphi(f) \varphi(g) \).

Let us denote by \( H(C^k_b(M)) \) the set of all nonzero algebra homomorphisms, i.e.

\[
H(C^k_b(M)) = \{ \varphi : C^k_b(M) \to \mathbb{R} : \varphi \text{ is an algebra homomorphism and } \varphi(1) = 1 \}.
\]

Let us list some of the basic properties of the algebra \( C^k_b(M) \) and the algebra homomorphisms \( H(C^k_b(M)) \). They can be checked as in the Riemannian case (see [11], [12] and [17]).

(a) If \( \varphi \in H(C^k_b(M)) \), then \( \varphi \neq 0 \) if and only if \( \varphi(1) = 1 \).

(b) If \( \varphi \in H(C^k_b(M)) \), then \( \varphi \) is positive, i.e. \( \varphi(f) \geq 0 \) for every \( f \geq 0 \).

(c) If the \( C^k \) Finsler manifold \( M \) is modeled on a Banach space that admits a Lipschitz and \( C^k \) smooth bump function, then \( C^k_b(M) \) is a unital algebra that separates points and closed sets of \( M \). Let us briefly give the proof for completeness. Let us take \( x \in M \), and \( C \subset M \) a closed subset of \( M \) with \( x \not\in C \). Let us take \( r > 0 \) small enough so that \( C \cap B_M(x, r) = \emptyset \) and a chart \( \varphi : B_M(x, r) \to X \) satisfying inequality (1.1). Let us take \( s > 0 \) small enough so that \( \varphi(x) \in B(\varphi(x), s) \subset \varphi(B(x, r/2)) \subset X \) and a Lipschitz and
Proposition 1.9. Let $C$ of Finsler manifolds. It states that the algebra structure of Definition 2.1.

The next proposition can be proved in a similar way to the Riemannian case [12].

Proposition 1.9. Let $M$ be a complete $C^k$ Finsler manifold that is $C^k$ uniformly bumbable. Then, $\varphi \in H(C^k_b(M))$ has a countable neighborhood basis in $H(C^k_b(M))$ if and only if $\varphi \in M$.

2. A Myers-Nakai Theorem

Our main result is the following Banach-Stone type theorem for a certain class of Finsler manifolds. It states that the algebra structure of $C^k_b(M)$ determines the $C^k$ Finsler manifold. Recall that two normed algebras $(A, || \cdot ||_A)$ and $(B, || \cdot ||_B)$ are equivalent as normed algebras whenever there exists an algebra isomorphism $T : A \to B$ satisfying $||T(a)||_B = ||a||_A$ for every $a \in A$. Let us begin by defining the class of Banach spaces where the Finsler manifolds shall be modeled.

Definition 2.1. A Banach space $(X, || \cdot ||)$ is said to be $k$-admissible if for every equivalent norm $| \cdot |$ and $\varepsilon > 0$, there are an open subset $B \supset \{x \in X : |x| \leq 1\}$ of $X$ and a $C^k$ smooth function $g : B \to \mathbb{R}$ such that

(i) $|g(x) - |x|| < \varepsilon$ for $x \in B$, and

(ii) $\text{Lip}(g) \leq (1 + \varepsilon)$ for the norm $| \cdot |$.

It is easy to prove the following lemma.

Lemma 2.2. Let $X$ be a Banach space with one of the following properties:

(A.1) Density of the set of equivalent $C^k$ smooth norms: every equivalent norm on $X$ can be approximated in the Hausdorff metric by equivalent $C^k$ smooth norms [6].

(A.2) $C^k$ -fine approximation property ($k \geq 2$) and density of the set of equivalent $C^1$ smooth norms: For every $C^1$ smooth function $f : X \to \mathbb{R}$ and every
Let \( \varepsilon > 0 \), there is a \( C^k \) smooth function \( g : X \to \mathbb{R} \) satisfying \(|f(x) - g(x)| < \varepsilon\) and \(|f'(x) - g'(x)| < \varepsilon\) for all \( x \in X \) (see [16], [2] and [20]); also, every equivalent norm defined on \( X \) can be approximated in the Hausdorff metric by equivalent \( C^1 \) smooth norms (see [6, Theorem II 4.1]).

Then \( X \) is \( k \)-admissible.

Banach spaces satisfying condition (A.2) are, for instance, separable Banach spaces with a Lipschitz \( C^k \) smooth bump function. Banach spaces satisfying condition (A.1) for \( k = 1 \) are, for instance, Weakly Compactly Generated (WCG) Banach spaces with a \( C^1 \) smooth bump function.

**Theorem 2.3.** Let \( M \) and \( N \) be complete \( C^k \) Finsler manifolds that are \( C^k \) uniformly bumpable and are modeled on \( k \)-admissible Banach spaces. Then \( M \) and \( N \) are \( C^k \) equivalent as Finsler manifolds if and only if \( C^k_b(M) \) and \( C^k_b(N) \) are equivalent as normed algebras. Moreover, every normed algebra isomorphism \( T : C^k_b(N) \to C^k_b(M) \) is of the form \( T(f) = f \circ h \) where \( h : M \to N \) is a weak \( C^k \) Finsler isometry. In particular, \( h \) is a \( C^{k-1} \) Finsler isometry whenever \( k \geq 2 \).

In order to prove Theorem 2.3, we shall follow the ideas of the Riemmanian case [12]. Let us divide the proof into several propositions.

**Proposition 2.4.** Let \( M \) and \( N \) be \( C^k \) Finsler manifolds such that \( N \) is modeled on a \( k \)-admissible Banach space \( Y \). Let \( h : M \to N \) be a map such that \( T : C^k_b(N) \to C^k_b(M) \) given by \( T(f) = f \circ h \) is continuous. Then \( h \) is \( ||T|| \)-Lipschitz for the Finsler metrics.

**Proof.** For every \( y \in N \), let us take a chart \( \psi_y : V_y \to Y \) with \( \psi_y(y) = 0 \). Let us consider the equivalent norm on \( Y \), \( ||| \cdot |||_y := ||d\psi_y^{-1}(\cdot)||_y \) and fix \( \varepsilon > 0 \). Let us define the ball \( B_{||| \cdot |||_y}(z, t) := \{ w \in Y : |||w - z|||_y < t \} \).

**Fact.** For every \( r > 0 \) such that \( B_{||| \cdot |||_y}(0, r) \subset \psi_y(V_y) \) and every \( \bar{\varepsilon} > 0 \), there exists a \( C^k \) smooth and Lipschitz function \( f_y : Y \to \mathbb{R} \) such that

1. \( f_y(0) = r \),
2. \( ||f_y||_\infty := \sup\{|f_y(z)| : z \in Y\} = r \),
3. \( \text{Lip}(f_y) \leq (1 + \varepsilon)^2 \) for the norm \( ||| \cdot |||_y \),
4. \( f_y(z) = 0 \) for every \( z \in Y \) with \( |||z|||_y \geq r \), and
5. \( ||z||_y \leq r - f_y(z) + \bar{\varepsilon} \) for every \( ||z||_y \leq r \).

Let us prove the Fact. First of all, let us take \( r > 0, \bar{\varepsilon} > 0 \) and \( 0 < \alpha < \min\{1, \frac{\varepsilon}{2}, \frac{2\varepsilon}{3r}\} \). Since \( N \) is a \( C^k \) Finsler manifold modeled on a \( k \)-admissible Banach space \( Y \), there are an open subset \( B \supset \{ x \in Y : |||x|||_y \leq 1 \} \) of \( Y \) and a \( C^k \) smooth function \( g : B \to \mathbb{R} \) such that

1. \( |g(x) - ||x||_y| < \alpha/2 \) on \( B \), and
2. \( \text{Lip}(g) \leq (1 + \alpha/2) \) for the norm \( ||| \cdot |||_y \).

Now, let us take a \( C^\infty \) smooth and Lipschitz function \( \theta : \mathbb{R} \to [0, 1] \) such that

1. \( \theta(t) = 0 \) whenever \( t \leq \alpha \),
2. \( \theta(t) = 1 \) whenever \( t \geq 1 - \alpha \),
3. \( \text{Lip}(\theta) \leq (1 + \varepsilon) \), and
4. \( |\theta(t) - t| \leq 2\alpha \) for every \( t \in [0, 1 + \alpha] \).
Let us define 

\[ f(x) = \begin{cases} \theta(g(x)) & \text{if } x \in B, \\ 1 & \text{if } x \in Y \setminus B. \end{cases} \]

It is straightforward to verify that \( f \) is well-defined, \( C^k \) smooth, \( f(x) = 1 \) whenever \( |||x|||_y \geq 1 \) and \( f(x) = 0 \) whenever \( |||x|||_y \leq \alpha/2 \). Let us now consider \( f_y : Y \to [0, r] \) as \( f_y(z) = r(1 - f(\frac{z}{r})) \), which is \( C^k \) smooth, Lipschitz and satisfies:

(i) \( f_y(0) = r \),

(ii) \( |||f_y|||_\infty = r \),

(iii) \( |f_y(z) - f_y(x)| \leq (1 + \varepsilon)(1 + \alpha/2)|||z - x|||_y \leq (1 + \varepsilon)^2|||z - x|||_y \),

(iv) \( f_y(z) = 0 \) for every \( z \in Y \) with \( |||z|||_y \geq r \),

(v) \( \frac{r(\bar{z})}{2} + g(\bar{z}) \leq \frac{r}{2} + 2\alpha + f(\bar{z}) \) for every \( |||z|||_y \leq r \). Thus, \( |||z|||_y \leq r(\frac{\bar{z}}{2} + 2\alpha) + r - f_y(z) \leq \bar{z} + r - f_y(z) \) for every \( |||z|||_y \leq r \).

Let us now prove Proposition 2.4. Let us fix \( p_1, p_2 \in M \) and \( \varepsilon > 0 \). Let us consider \( \sigma : [0, 1] \to M \) a piecewise \( C^1 \) smooth path in \( M \) joining \( p_1 \) and \( p_2 \), with \( \ell(\sigma) \leq d_M(p_1, p_2) + \varepsilon \). Since \( h : M \to N \) is continuous, the path \( \hat{\sigma} := h \circ \sigma : [0, 1] \to N \), joining \( h(p_1) \) and \( h(p_2) \), is continuous as well. For every \( q \in \hat{\sigma}([0, 1]) \), there is \( 0 < r_q < 1 \) and a chart \( \psi_q : \nu_q \to Y \) such that \( \psi_q(0) = 0 \), \( B_N(q, r_q) \subset \nu_q \) and the bijection \( \psi_q : \nu_q \to \psi_q(\nu_q) = (1 + \varepsilon)\)-bi-Lipschitz for the norm \( |||d\psi_q^{-1}(0)\cdot|||_q \) in \( Y \) (see Lemma 1.3). Since \( \hat{\sigma}([0, 1]) \) is a compact set of \( N \), there is a finite family of points \( 0 = t_1 < t_2 < \cdots < t_m = 1 \) and a family of open intervals \( \{I_k\}_{k=1}^m \) covering the interval \( [0, 1] \) so that, if we define \( q_k := \hat{\sigma}(t_k) \) and \( r_k := r_{q_k} \), for every \( k = 1, \ldots, m \), we have

(a) \( \hat{\sigma}(I_k) \subset B_N(q_k, r_k/(1 + \varepsilon)) \),

(b) \( I_j \cap I_k \neq \emptyset \) if and only if \( |j - k| \leq 1 \).

It is clear that \( \hat{\sigma}([0, 1]) \subset \bigcup_{k=1}^m B_N(q_k, \frac{r_k}{1 + \varepsilon}) \). Now, let us select a point \( s_k \in I_k \cap I_{k+1} \) such that \( t_k < s_k < t_{k+1} \), for every \( k = 1, \ldots, m - 1 \). Let us write \( a_k := \hat{\sigma}(s_k) \), for every \( k = 1, \ldots, m - 1 \), \( \psi_k := \psi_{q_k} \), \( V_k := V_{q_k} \) and \( \|\| : \|\|_{k} := |||d\psi_k^{-1}(0)\cdot|||_{q_k} \), for every \( k = 1, \ldots, m \). Notice that \( a_k \in B_N(q_k, \frac{r_{q_k}}{1 + \varepsilon}) \cap B_N(q_{k+1}, \frac{r_{q_{k+1}}}{1 + \varepsilon}) \), for every \( k = 1, \ldots, m - 1 \). Since \( \psi_k : V_k \to \psi_k(V_k) = (1 + \varepsilon)\)-bi-Lipschitz for the norm \( |||\cdot|||_k \) in \( Y \), we deduce that \( \psi_k(a_k) \in B_{|||\cdot|||_k}(0, r_k) \), for every \( k = 1, \ldots, m - 1 \).

Now, let us apply the above Fact to \( r_k, \varepsilon \) and \( \bar{z} = \varepsilon/2m \) to obtain functions \( f_k : Y \to [0, r_k] \) satisfying properties (1)–(5), \( k = 1, \ldots, m \). Let us define the \( C^k \) smooth and Lipschitz functions \( g_k : N \to [0, r_k] \) as \( g_k(z) = f_k(\psi_k(z)) \) for every \( z \in V_k \) and \( g_k(z) = 0 \) for \( z \not\in V_k \), \( k = 1, \ldots, m \). Then,

(i) \( g_k \in C^k_b(N) \),

(ii) \( g_k(q_k) = r_k \),

(iii) \( |g_k(z) - g_k(x)| \leq (1 + \varepsilon)^3d_N(z, x) \) for all \( z, x \in N \);

(iv) If \( z \in \psi_k^{-1}(B_{|||\cdot|||_k}(0, r_k)) \), then \( |||\psi_k(z)|||_k \leq r_k \) and from condition (5) on the Fact, we obtain

\[
d_N(z, q_k) \leq (1 + \varepsilon)|||\psi_k(z) - \psi_k(q_k)|||_k = (1 + \varepsilon)|||\psi_k(z)|||_k \leq (1 + \varepsilon)(r_k - g_k(z) + \varepsilon/2m).
\]

The Lipschitz constant of \( g_k \circ h \), for \( k = 1, \ldots, m \), is the following

\[
\text{Lip}(g_k \circ h) \leq ||g_k \circ h||_{C^k_b(M)} = ||T(g_k)||_{C^k_b(M)} \leq ||T|| ||g_k||_{C^k_b(N)} = ||T|| \max\{||g_k||_\infty, ||dg_k||_\infty\} \leq ||T||(1 + \varepsilon)^3.
\]
Now, since $r_k = g_k(q_k) = g_k(h(\sigma(t_k)))$ and $\psi_k(h(\sigma(s_k))) \in B_{||\cdot||_k}(0, r_k)$, we have
\[
\begin{align*}
    d_N(h(p_1), h(p_2)) & \leq \sum_{k=1}^{m-1} [d_N(h(\sigma(t_k)), h(\sigma(s_k))) + d_N(h(\sigma(s_k)), h(\sigma(t_{k+1})))] \\
    & \leq \sum_{k=1}^{m-1} (1 + \varepsilon)[g_k(q_k) - g_k(h(\sigma(s_k))) + g_k(\sigma(s_k)) + g_k(\sigma(s_k)) + \varepsilon/m] \\
    & \leq \sum_{k=1}^{m-1} (1 + \varepsilon)[\text{Lip}(g_k \circ h)d_M(\sigma(t_k), \sigma(s_k))] + \\
    & \quad \text{Lip}(g_k \circ h)d_M(\sigma(t_{k+1}), \sigma(s_k)) + \varepsilon/m] \\
    & \leq \sum_{k=1}^{m-1} ||T||(1 + \varepsilon)^4[d_M(\sigma(t_k), \sigma(s_k)) + d_M(\sigma(\alpha(t_k)), \sigma(s_k))] + \varepsilon(1 + \varepsilon) \\
    & \leq \sum_{k=1}^{m-1} ||T||(1 + \varepsilon)^4[\varepsilon(1 + \varepsilon)] = ||T||(1 + \varepsilon)^4[\varepsilon(1 + \varepsilon)] \\
    & \leq ||T||(1 + \varepsilon)^4(d_M(p_1, p_2) + \varepsilon) + \varepsilon(1 + \varepsilon)
\end{align*}
\]
for every $\varepsilon > 0$. Thus, $h$ is $||T||$-Lipschitz.

**Lemma 2.5.** Let $M$ and $N$ be $C^k$ Finsler manifolds such that $N$ is modeled on a Banach space with a Lipschitz $C^k$ smooth bump function. Let $h : M \to N$ be a homeomorphism such that $f \circ h \in C^k_b(M)$ for every $f \in C^k_b(N)$. Then, $h$ is a weakly $C^k$ smooth function on $M$.

**Proof.** Let us fix $x \in M$ and $\varepsilon = 1$. There are charts $\varphi : U \to X$ of $M$ at $x$ and $\psi : V \to Y$ of $N$ at $h(x)$ satisfying inequalities (1.1) and (1.2) on $U$ and $V$, respectively. We can assume that $h(U) \subset V$. Since $Y$ admits a Lipschitz and $C^k$ smooth bump function and $\psi(h(U))$ is an open neighborhood of $\psi(h(x))$ in $Y$, there are real numbers $0 < s < r$ such that $B(\psi(h(x)), s) \subset B(\psi(h(x)), r) \subset \psi(h(U))$ and a Lipschitz and $C^k$ smooth function $\alpha : Y \to \mathbb{R}$ such that $\alpha(y) = 1$ for $y \in B(\psi(h(x)), s)$ and $\alpha(y) = 0$ for $y \notin B(\psi(h(x)), r)$. Let us define $U_0 := h^{-1}(\psi^{-1}(B(\psi(h(x)), s))) \subset U$, which is an open neighborhood of $x$ in $M$.

Let us check that $y^* \circ (\psi \circ h \circ \varphi^{-1})$ is $C^k$ smooth on $\varphi(U_0) \subset X$ for all $y^* \in Y^*$. Following the proof of [9, Theorem 4], we define $g : N \to \mathbb{R}$ as $g(y) = 0$ whenever $y \notin V$ and $g(y) = \alpha(\psi(y)) \cdot y^*(\psi(y))$ whenever $y \in V$. It is clear that $g \in C^k_b(N)$ and, by assumption, $g \circ h \in C^k_b(M)$. Now, it follows that $\psi(h(\varphi^{-1}(z))) \in B(\psi(h(x)), s)$ for every $z \in \varphi(U_0)$. Thus
\[
y^* \circ (\psi \circ h \circ \varphi^{-1})(z) = y^*(\psi(h(\varphi^{-1}(z)))) = \alpha(\psi(h(\varphi^{-1}(z))))y^*(\psi(h(\varphi^{-1}(z)))) = g(h(\varphi^{-1}(z))) = g \circ h \circ \varphi^{-1}(z),
\]
for every $z \in \varphi(U_0)$. Since $(g \circ h) \circ \varphi^{-1}$ is $C^k$ smooth on $\varphi(U_0)$, we have that $y^* \circ (\psi \circ h \circ \varphi^{-1})$ is $C^k$ smooth on $\varphi(U_0)$. Thus $\psi \circ h \circ \varphi^{-1}$ is weakly $C^k$ smooth on $\varphi(U_0)$ and $h$ is weakly $C^k$ smooth on $M$. \qed
Proof of Theorem 2.3. If $h : M \to N$ is a weak $C^k$ Finsler isometry, we can define the operator $T : C^k_b(N) \to C^k_b(M)$ by $T(f) = f \circ h$. Let us check that $T$ is well defined. For every $x \in M$, there are charts $\varphi : U \to X$ of $M$ at $x$ and $\psi : V \to Y$ of $N$ at $h(x)$, such that $h(U) \subset V$ and $\psi \circ h \circ \varphi^{-1}$ is weakly $C^k$ smooth on $\varphi(U) \subset X$. Also, $f \circ \psi^{-1}$ is $C^k$ smooth on $\psi(V) \subset Y$. Thus, by [15, Proposition 4.2], $(f \circ \psi^{-1}) \circ (\psi \circ h \circ \varphi^{-1}) = f \circ h \circ \varphi^{-1}$ is $C^k$ smooth on $\varphi(U)$. Therefore, $f \circ h$ is $C^k$ smooth on $U$. Since this holds for every $x \in M$, we deduce that $f \circ h$ is $C^k$ smooth on $M$. Moreover, $T$ is an algebra isomorphism with $\|T(f)\|_{C^k_b(M)} = \|f \circ h\|_{C^k_b(M)} = \|f\|_{C^k_b(\mathbb{N})}$ for every $f \in C^k_b(N)$.

Conversely, let $T : C^k_b(N) \to C^k_b(M)$ be a normed algebra isometry. Then, we can define the function $h : H(C^k_b(M)) \to H(C^k_b(N))$ by $h(\varphi) = \varphi \circ T$ for every $\varphi \in H(C^k_b(M))$. The function $h$ is a bijection. Moreover, $h$ is an homeomorphism. Recall that we identify $x \in M$ with $\delta_x \in H(C^k_b(M))$. Thus, $h(x) = h(\delta_x) = \delta_x \circ T$. Since $h$ is an homeomorphism, by Proposition 1.9, we obtain for every $p \in N$ a unique point $x \in M$ such that $h(\delta_x) = \delta_p$. Let us check that $T(f) = f \circ h$ for all $f \in C^k_b(N)$. Indeed, for every $x \in M$ and every $f \in C^k_b(N)$,

$$T(f)(x) = \delta_x(T(f)) = (\delta_x \circ T)(f) = h(\delta_x)(f) = \delta_{h(x)}(f) = f(h(x)) = f \circ h(x).$$

Now, from Proposition 2.4 and Lemma 2.5 we deduce that $h$ is a weak $C^k$ Finsler isometry.

\[\square\]

Remark 2.6. It is worth mentioning that, for Riemannian manifolds, every metric isometry is a $C^1$ Finsler isometry. This result was proved by S. Myers and N. Steenrod [21] in the finite-dimensional case and by I. Garrido, J.A. Jaramillo and Y.C. Rangel [12] in the general case. Also, S. Deng and Z. Hou [5] obtained a version for finite-dimensional Riemannian-Finsler manifolds. Nevertheless, there is no a generalization, up to our knowledge, of the Myers-Steenrod theorem for all Finsler manifolds. Thus, for $k = 1$ we can only assure that the metric isometry obtained in Theorem 2.3 is weakly $C^1$ smooth.

Let us finish this note with some interesting corollaries of Theorem 2.3. First, recall that every separable Banach space with a Lipschitz $C^k$ smooth bump function satisfies condition (A.2) and every WCG Banach space with a $C^1$ smooth bump function satisfies condition (A.1) for $k = 1$.

Corollary 2.7. Let $M$ and $N$ be complete, $C^1$ Finsler manifolds that are $C^1$ uniformly bumpy and are modeled on WCG Banach spaces. Then $M$ and $N$ are weakly $C^1$ equivalent as Finsler manifolds if, and only if, $C^1_b(M)$ and $C^1_b(N)$ are equivalent as normed algebras. Moreover, every normed algebra isomorphism $T : C^1_b(N) \to C^1_b(M)$ is of the form $T(f) = f \circ h$ where $h : M \to N$ is a weak $C^1$ Finsler isometry.

Notice that the assumptions of Corollary 2.7 hold if $M$ and $N$ are modeled on Banach spaces with separable dual.

Corollary 2.8. Let $M$ and $N$ be complete, separable $C^k$ Finsler manifolds that are modeled on Banach spaces with a Lipschitz and $C^k$ smooth bump function. Then $M$ and $N$ are weakly $C^k$ equivalent as Finsler manifolds if and only if $C^k_b(M)$ and
$C^k_b(N)$ are equivalent as normed algebras. Moreover, every normed algebra isomorphism $T : C^k_b(N) \to C^k_b(M)$ is of the form $T(f) = f \circ h$ where $h : M \to N$ is a weak $C^k$ Finsler isometry. In particular, $h$ is a $C^{k-1}$ Finsler isometry whenever $k \geq 2$.

Since every weakly $C^k$ smooth function with values in a finite-dimensional normed space is $C^k$ smooth and every finite-dimensional $C^k$ Finsler manifold is $C^k$ uniformly bumpable \cite{18}, we obtain the following Myers-Nakai result for finite-dimensional $C^k$ Finsler manifolds.

**Corollary 2.9.** Let $M$ and $N$ be complete and finite dimensional $C^k$ Finsler manifolds. Then $M$ and $N$ are $C^k$ equivalent as Finsler manifolds if, and only if, $C^k_b(M)$ and $C^k_b(N)$ are equivalent as normed algebras. Moreover, every normed algebra isomorphism $T : C^k_b(N) \to C^k_b(M)$ is of the form $T(f) = f \circ h$ where $h : M \to N$ is a $C^k$ Finsler isometry.

We obtain an interesting application of Finsler manifolds to Banach spaces. Recall the well-known Mazur-Ulam Theorem establishing that every surjective isometry between two Banach spaces is affine.

**Corollary 2.10.** Let $X$ and $Y$ be WCG Banach spaces with $C^1$ smooth bump functions. Then $X$ and $Y$ are isometric if, and only if, $C^1_b(X)$ and $C^1_b(Y)$ are equivalent as normed algebras. Moreover, every normed algebra isomorphism $T : C^1_b(Y) \to C^1_b(X)$ is of the form $T(f) = f \circ h$ where $h : X \to Y$ is a surjective isometry. In particular, $h$ and $h^{-1}$ are affine isometries.

**References**


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