ALCOVED CONVEX POLYHEDRA: NEW SHAPES FOR DESIGN

María Jesús de la Puente

Fac. Matemáticas, Depto. Álgebra, Geometría y Topología, Universidad Complutense de Madrid UCM, Spain

Two descriptions of alcoved polyhedra

- by facets: $a_{i4} \leq x_i \leq -a_{4i}$ and $a_{ij} \leq x_i - x_j \leq -a_{ji}$ or
- by 4 generators in $\mathbb{R}^3$: $\underline{1}, \underline{2}, \underline{3}, \underline{4}$

Alcoved polyhedra to matrices

- facet equation constants a_{ij} give a matrix A
- 4 generators give columns of a matrix A_0 (with zero last row)
- to pass from A to A_0 and back is trivial
- can always assume A is NI (normal idempotent: $a_{ii} = 0$, $a_{ij} \leq 0$ and $a_{ij} + a_{jk} \leq a_{ik}$
- sometimes can assume $A = A_0$

Generators? In tropical sense

- $\oplus := \max$ is tropical addition and $\odot := +$ is tropical multiplication, coordinatewise
- to span is to make tropical linear combinations

Tropical Planes

- three generic points span a unique plane
- a generic plane in 3–space has one vertex
- label this vertex taxonomically (parents/children)

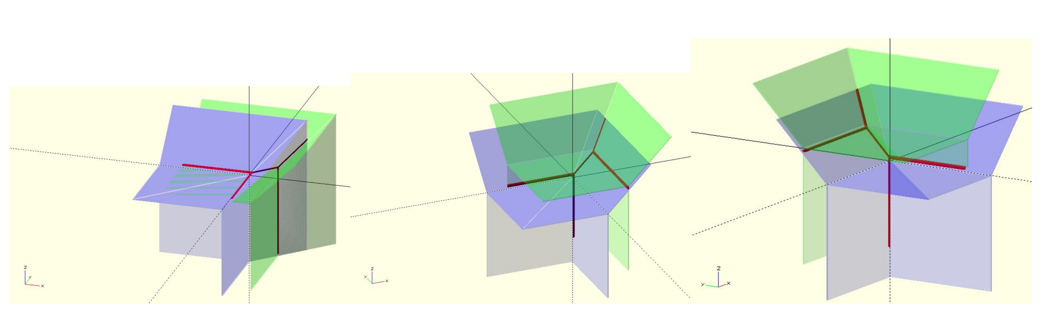

Figure 1: Tropical lines as intersection of pairs of tropical planes.

Tropical Lines

- two different points span a unique line
- a generic line in 3–space has two vertices
- label these vertices taxonomically (parents/children) + proximity criterion
- the intersection of two generic planes is a line

Objectives

- Alcoved polyhedra are convex bodies having facet equations of only two types: $x_i = cnst$ or $x_i - x_j = cnst$
- Represent an alcoved polyhedron by a 4×4 matrix
- Compute with matrices/manipulate alcoved polyhedra
- Translate properties from alcoved polyhedra to matrices and back: make a dictionary

Orientation of an alcoved polyhedron in 3–space

- Place South Pole (i.e., vertex $\underline{4}$) and North Pole (i.e., vertex $\underline{123}$), Notice Polar Axis
- Locate other generators: $\underline{1}$ minimum in right facet, $\underline{2}$ minimum in back facet and $\underline{3}$ minimum in top facet
- Locate 3–generated vertices and 2–generated vertices

Maximal alcoved polyhedra are Dodecahedra

An alcoved polyhedron has, at most, 12 facets, 30 edges and 20 vertices. Facets can be 4–gons, 5–gons or 6–gons.

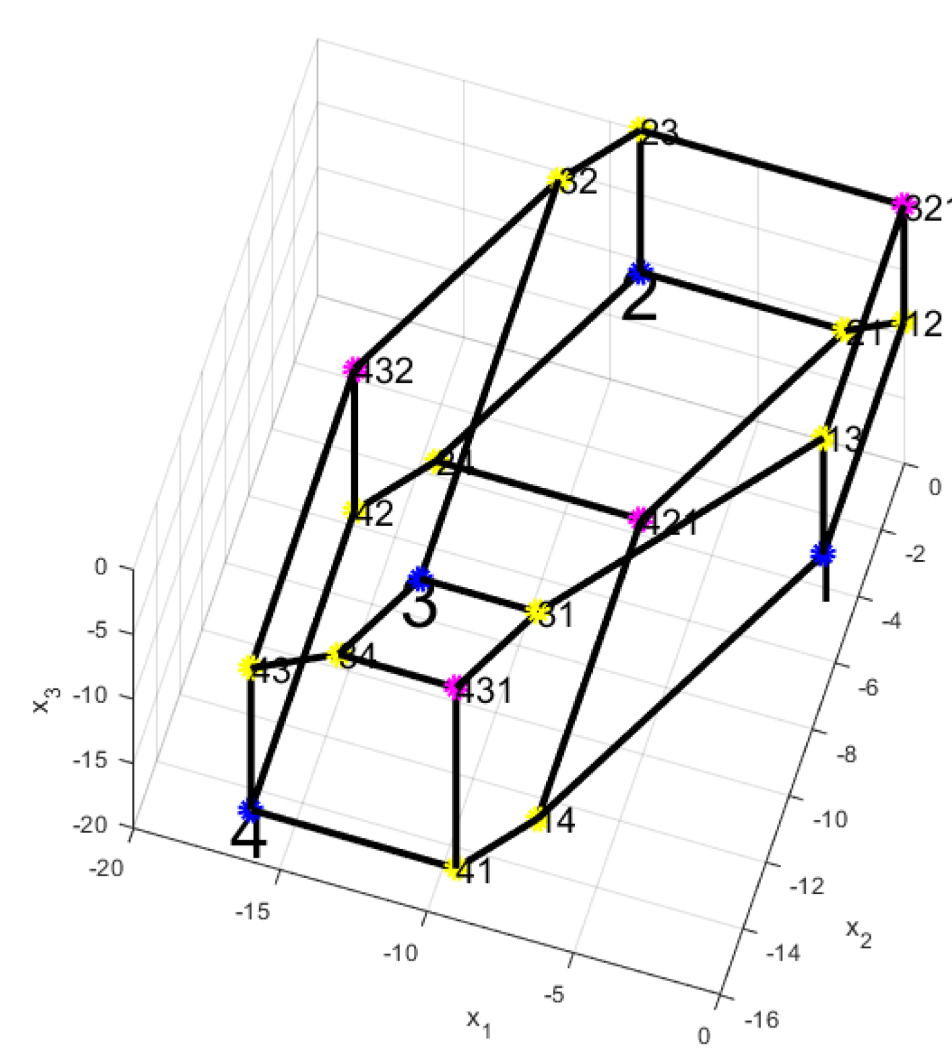

Figure 2: Green model: $\mathcal{N}$ is (4.5.6), $\mathcal{S}$ is (4.5.6), EB is (4.5.6.4.5.6)

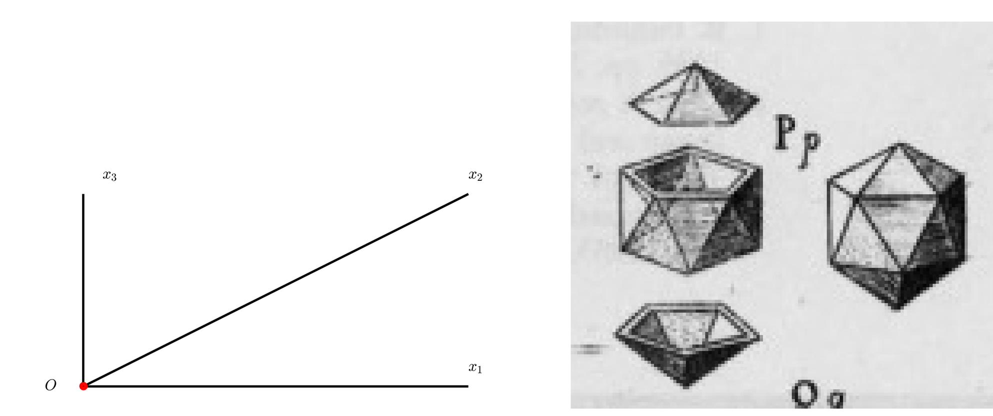

Figure 3: Coordinate axes in $\mathbb{R}^3$; the red dot marks the origin (left). North and South Casks and Equatorial Belt in an icosahedron in figure from Kepler's *Harmonices Mundi*, 1691 (right).

Polar Casks and Equatorial Belt

- North Cask is the union of facets meeting the North Pole
- South Cask is the union of facets meeting the South Pole
- Remaining facets make Equatorial Belt: a cycle of facets $(q_1, q_2, \ldots, q_6)$, i.e., a q_1–gon followed by a q_2–gon, ... q_6–gon, traveling westwards, beginning front.

Polar Cask types

- type is (4.5.6) if $x_1 = cnst$ is 4–gon, $x_2 = cnst$ is 5–gon and $x_3 = cnst$ is 6–gon,
- permutations of above
- type is (5.5.5) if $x_1 = cnst$, $x_2 = cnst$ and $x_3 = cnst$ are 5–gons (occurs in left or right states)

Theorems

- Theorem 1: North and South Cask types determine and alcoved polyhedron
- Theorem 2: Volume formula exists

```
%{
DATA MATRIX

     0    -9   -11   -16
    -7     0   -14   -16
    -9   -11     0   -16
     0     0     0     0

MATLAB RESULTS FOR MATRIX ABOVE OF GREEN MODEL ARE:

3-GENERATED (MAGENTA DOTS)

   -16    -9    -7     0
    -7   -16    -5     0
    -5    -2   -16     0
     0     0     0     0
     2     1     1     1
     3     3     2     2
     4     4     4     3

2-GENERATED (YELLOW DOTS)

     0    -2    -9   -11    -7     0    -9    -7   -16   -14   -16   -13
     0     0     0    -2   -14    -7   -16   -14    -7    -5   -16   -16
    -9   -11     0     0     0     0   -16   -16   -16   -16    -5    -2
     0     0     0     0     0     0     0     0     0     0     0     0
     2     1     3     2     1     3     1     4     2     4     3     4
     1     2     2     3     3     1     4     1     4     2     4     3
     0     0     0     0     0     0     0     0     0     0     0     0

PERTURBATION MATRIX  E =

     0    -7    -5
    -9     0    -2
    -7    -5     0

CANT TUPLE  c = -2    -5    -7    -5    -7    -9    -2

p_vector = 0     4     4     4

h_vector = 0     0     0     1

N_cask_type = 4     5     6     1

S_cask_type = 4     5     6     1

EB = 4     5     6     4     5     6

volume is (rounded)  W =  2621

baricenter =  -8.6  -8.1  -7.5

%}
```

Figure 4: Matlab computations for model in figure 2

Box, perturbation and cant tuple

- To cant means to bevel, to form an oblique surface upon something
- A box is a convex polyhedron with facet equations $x_i = cnst$
- To obtain an alcoved polyhedron, one must cant six edges in a box: those not meeting the poles
- An alcoved polyhedron is a perturbed box, a canted box
- If $A_0 = A$ express $A = B - E$ with B (box matrix) and E (perturbation matrix)
- Cant tuple (i.e., list of cant parameters) obtained from E

Acknowledgements

I thank P.L. Clavería for providing figure 1 and 3D models. Work partially supported by Proyecto I+D MTM 2016–76808–P and 910444 UCM group.

- Web: http://www.mat.ucm.es/ mpuente/
- Email: mpuente@ucm.es