FFF#153. The Schwarz-Cauchy Inequality.

M. J. de la Puente of Universidad Complutense in Madrid, Spain found a linear algebra student who can use the associativity of the inner product to reverse a standard inequality. Let u and v be two vectors of a real inner product space; since the inequality we derive is trivial when $v = 0$, we suppose that $v \neq 0$ and that $\lambda = u \cdot u / v \cdot v$. Then

$$0 \leq (u - \lambda v) \cdot (u - \lambda v) = u \cdot u - 2u \cdot \lambda v + \lambda^2 v \cdot v$$

$$= u \cdot u - \frac{2u \cdot (u \cdot v) \cdot v}{v \cdot v} + \frac{(u \cdot v)^2 (v \cdot v)}{(v \cdot v)^2}$$

$$= u \cdot u - \frac{2(u \cdot u) \cdot (v \cdot v)}{v \cdot v} + \frac{(u \cdot v)^2}{v \cdot v} = -u \cdot u + \frac{(u \cdot v)^2}{v \cdot v}$$

whence $(u \cdot u)(v \cdot v) \leq (u \cdot v)^2$. ■

De la Puente points out that assuming associativity of the inner product actually leads to equality:

$$(u \cdot u) \cdot (v \cdot v) = u \cdot (u \cdot v) \cdot v = u \cdot (v \cdot u) \cdot v = u \cdot (v \cdot u) \cdot v = (u \cdot v) \cdot (u \cdot v).$$

FFF#154. How the factorial works.

Norton Starr of Amherst College in Massachusetts has forwarded copies of some pages in the book, *Go Figure*, by Clint Brookhart (Contemporary Books, 1998). Some of the mishaps he indicates are just unedifying sloppiness, but a couple are rather mysterious. For example, the author shows how one can compute $248.3e^{0.0076(60)}$ with a scientific calculator that lacks a e^x key but does have inverse and natural log keys. Is there such a calculator?

More interestingly, on pages 34 and 35, the author explains "how the n factorial works." This quantity occurs "throughout mathematical formulas and expressions, particularly in many types of series (the sum of a usually infinite sequence of numbers). ⋯ Because sums in these series increase rapidly, it is useful to be able to approximate when dealing with large values of n." The tool for this, of course, is Stirling's approximation formula, quoted as

$$n! = \left(\frac{n}{e}\right)^n (2\pi n)^{\frac{1}{2}}.$$

"Let's see," writes the author, "how well Stirling's formula works when $n!$ grows exponentially." He then goes on to calculate $12!(= 4.7569 \times 10^8)$ and $20!(= 2.42278 \times 10^{18})$, and concludes with the comment:

Finally, let's compare the two factorials we computed:

$$\frac{20!}{12!} = \frac{(2.422787 \times 10^{18})}{(4.7569 \times 10^8)} = 5.1 \times 10^9.$$

The summation does grow exponentially!