Contemporary Mathematics
Volume **495**, 2009

Tropical conics for the layman

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ABSTRACT. We present a simple and elementary procedure to sketch the tropical conic given by a degree–two homogeneous tropical polynomial. These conics are trees of a very particular kind. Given such a tree, we explain how to compute a defining polynomial. Finally, we characterize those degree–two tropical polynomials which are reducible and factorize them. We show that there exist irreducible degree–two tropical polynomials giving rise to pairs of tropical lines.

1. Introduction

In recent years, there has been a growing interest in *projective tropical geometry*, [**3, 8, 9, 12, 20, 22, 23, 25, 27, 28, 29, 30, 32, 33, 34, 36, 41, 42**]. This new geometry is related to toric geometry, [**15, 19, 31**]. Several authors have searched for tropical versions of some classical theorems of projective geometry, [**13, 35, 37, 38, 39, 40**]. Some of these old theorems involve conics.

The aim of this paper is to present tropical conics to non–experts, using only tropical algebra (also called max–algebra, max–plus algebra, semirings, moduloïds, dioïds, pseudorings, pseudomodules, band spaces over belts, idempotent mathematics). But first, one word of advise is in order. Tropical conics are, of course, fairly well understood by experts (in terms of combinatorics: secondary polytopes of matrices, Gale dual spaces, etc.), see [**10**]. Also, there exist algorithms and computer programs to deal with them. Our point is, nonetheless, that all of this can be done in elementary terms, easily and fast, just by hand.

This paper originated as an attempt to explain in full detail and give proofs for all statements made in example 3.4 in [**32**].

Our polynomials will be either homogeneous in three variables or non–necessary homogeneous in two variables. To a degree–two tropical polynomial p, we associate a point in the tropical plane and a triple of non–negative real numbers, $s_{21}^+, s_{32}^+, s_{31}^+$, which completely determine the tropical conic $\mathcal{C}(p)$. These data are simply computed from p and they are all that is needed to know in order to sketch $\mathcal{C}(p)$. It is known that the regular subdivision of the Newton polygon of p determines the combinatorial type of $\mathcal{C}(p)$ but, to our knowledge, nothing precise has been said about the exact coordinates of the vertices of $\mathcal{C}(p)$.

There are two types (with several sub–types) of tropical conics: degenerate and non–degenerate ones. We explain how they are classified according to the values of the invariants $s_{21}^+, s_{32}^+, s_{31}^+$ and certain alternating sums d_1, d_2, d_3 of the s_{ij}^+'s. Degenerate (also called improper) tropical conics are classified in theorem 2.8. It turns out that pairs of tropical lines are degenerate tropical conics, but the converse is not true. And non–degenerate

[1]We would like to thank the anonymous referee for pointing out a better way to present this material.
2000 *Mathematics Subject Classification.* Primary 05C05; Secondary 12K99.
Key words and phrases. Tropical conics, factorization of tropical polynomials, tropically singular matrix.
Partially supported by MTM 2005–02865 and by UCM 910444.

(also called proper) tropical conics are classified in theorem 2.6, into *one–point central* and *two–point central* ones.

Given a degree–two tropical polynomial p, the values $s_{21}^+, s_{32}^+, s_{31}^+$ can be arranged into a 3×3 symmetric non–negative real matrix denoted $\operatorname{shape}(p)^+$. We characterize those tropical conics $\mathcal{C}(p)$ having tropically singular associated matrix $\operatorname{shape}(p)^+$ (corollary 2.11). These are pairs of tropical lines and, surprisingly enough, one–point central conics.

In the last section of the paper, we address the question of irreducibility of degree–two tropical polynomials, also in elementary terms. We show that there exist irreducible degree–two tropical polynomials giving rise to pairs of tropical lines.

Some of the results in this paper have already appeared in [**2**], while other are new. The idea of considering shape matrices comes, somehow, from [**21**]. The values s_{ij} come from [**32**].

Many results in tropical algebra have been discovered since the late fifties so that the literature on this topic is vast. Some references are the books [**4, 6, 16, 17, 45**] and the papers [**1, 5, 7, 14, 43, 44**]. The factorization problem for tropical polynomials in one variable has been investigated in [**24**]. The tropical version of the existence and uniqueness of a tropical conic passing through five given points in the plane in general position can be found in [**32**].

2. Tropical conics

2.1. Tropical planes. *Tropical geometry* arises when one works over $\mathbb{T}$, the *tropical semi–field*. By definition, $\mathbb{T}$ is the set $\mathbb{R} \cup \{-\infty\}$ endowed with two operations: $\max$ and $+$. *Tropical addition* is $\max$ and $+$ is taken as *tropical multiplication*. They are denoted $\oplus$ and $\odot$, respectively. The neutral element for tropical addition is $-\infty$ and zero is the neutral element for tropical multiplication. It is noticeable that $a \oplus a = a$, for $a \in \mathbb{T}$, that is, tropical addition is *idempotent*. However, there does not exist an inverse element, with respect to $\oplus$, for $a \in \mathbb{T}$. This is all that $\mathbb{T}$ lacks in order to be a field.

$\mathbb{R}_{\geq 0}$ will denote the set of non–negative real numbers. For $a \in \mathbb{T}$, we will set $a^+ = \max\{a, 0\} = a \oplus 0$, the *non–negative part of* a. For a matrix A, the matrix obtained by replacing every entry a of A by a^+ will be denoted A^+. For a polynomial P, the polynomial obtained by replacing every coefficient a of P by a^+ will be denoted P^+.

The *tropical affine* 2*–space* is $\mathbb{T}^2$, where addition and multiplication are defined coordinatewise. Here we can define *translations* in the standard way; every point $(t_1, t_2) \in \mathbb{R}^2$ defines the map: $(X, Y) \mapsto (X + t_1, Y + t_2)$.

In the space $\mathbb{T}^3 \setminus \{(-\infty, -\infty, -\infty)\}$ we define an equivalence relation $\sim$ by letting $(b_1, b_2, b_3) \sim (c_1, c_2, c_3)$ if there exists $\lambda \in \mathbb{R}$ such that

$$(b_1 + \lambda, b_2 + \lambda, b_3 + \lambda) = (c_1, c_3, c_3).$$

The equivalence class of (b_1, b_2, b_3) is denoted $[b_1, b_2, b_3]$. Its elements are obtained by adding multiples of the vector $(1, 1, 1)$ to the point (b_1, b_2, b_3). The *tropical projective* 2*–space*, $\mathbb{TP}^2$, is the set of such equivalence classes. Notice that, at least, one of the coordinates of any point in $\mathbb{TP}^2$ must be finite.

Points in $\mathbb{T}^2$ (resp. $\mathbb{TP}^2$) having finite coordinates will be called *interior points*. The rest of the points will be called *boundary points*. The *boundary* of $\mathbb{T}^2$ (resp. $\mathbb{TP}^2$) is the union of its boundary points. We will use X, Y, Z as variables in $\mathbb{TP}^2$. Any permutation of the variables X, Y, Z provides a *change of projective tropical coordinates*. *Translations* are also natural changes of projective tropical coordinates: given $[t_1, t_2, t_3] \in \mathbb{R}^3$, the point $[X, Y, Z]$ maps to $[X', Y', Z'] = [X + t_1, Y + t_2, Z + t_3]$. We may write

$$[X', Y', Z'] = [X, Y, Z] \odot D, \qquad D = \begin{pmatrix} t_1 & -\infty & -\infty \\ -\infty & t_2 & -\infty \\ -\infty & -\infty & t_3 \end{pmatrix}. \tag{2.1}$$

A particular case is the *tropical identity matrix*

$$I = \begin{pmatrix} 0 & & \\ -\infty & 0 & \\ -\infty & -\infty & 0 \end{pmatrix}.$$

Here, tropical matrix multiplication is defined in the usual way, but using $\oplus$ and $\odot$.

The plane $\mathbb{TP}^2$ is covered by three copies of $\mathbb{T}^2$ as follows. There exist injective maps

$$j_3 : \mathbb{T}^2 \to \mathbb{TP}^2, \quad (x, y) \mapsto [x, y, 0], \qquad j_2 : \mathbb{T}^2 \to \mathbb{TP}^2, \quad (x, z) \mapsto [x, 0, z],$$
$$j_1 : \mathbb{T}^2 \to \mathbb{TP}^2, \qquad (y, z) \mapsto [0, y, z]$$

and $\mathbb{TP}^2 = \operatorname{im} j_3 \cup \operatorname{im} j_2 \cup \operatorname{im} j_1$. The complementary set of, say, $\operatorname{im} j_3$ is

$$\{[x, y, -\infty] : x, y \in \mathbb{T}\}.$$

Moreover, we have $j_3(x, x) = [x, x, 0] = [0, 0, -x]$, for $x \in \mathbb{T}$. This means that the coordinate axis Z in $\mathbb{TP}^2$ is transformed by j_3^{-1} into the usual line $X = Y$ in $\mathbb{T}^2$. The negative Z half–axis in $\mathbb{TP}^2$ corresponds to the north–east direction in $\mathbb{T}^2$. Similarly for j_2, j_1.

It is easy to check that the set of interior points of $\mathbb{TP}^2$ equals the intersection $\operatorname{im} j_3 \cap \operatorname{im} j_2 \cap \operatorname{im} j_1$.

For simplicity, we will consider the Euclidean metric in $\mathbb{T}^2$. Notice that the composite maps $j_l^{-1} \circ j_k$ are NOT isometries, for $k, l \in \{1, 2, 3\}$, $k \neq l$.

The projective tropical coordinates of a point in $\mathbb{TP}^2$ are not unique. In order to avoid this inconvenience, we choose a *normalization*, that is we fix a rule that allows us to have unique coordinates for all (but perhaps a small set of) points in $\mathbb{TP}^2$, according to this rule. For instance, setting the last coordinate equal to zero is a normalization. We call it the $Z = 0$ *normalization* and say that *we work in* $Z = 0$. To consider the $Z = 0$ normalization is the same thing as passing to the affine tropical plane, via j_3. Other possible normalizations are setting $Y = 0$, or setting $X = 0$, or setting X, Y, Z all non–negative and, at least, one equal to zero, or setting $X + Y + Z = 0$, etc.

2.2. Tropical conics are trees. A tropical polynomial is a tropical sum of tropical monomials. For instance, a tropical homogeneous degree–two polynomial in the variables X, Y, Z is

(2.2)

$$P(X, Y, Z) =$$
$$a_{11} \odot X^{\odot 2} \oplus a_{22} \odot Y^{\odot 2} \oplus a_{33} \odot Z^{\odot 2} \oplus a_{21} \odot X \odot Y \oplus a_{32} \odot Y \odot Z \oplus a_{31} \odot X \odot Z$$
$$= \max\{a_{11} + 2X, a_{22} + 2Y, a_{33} + 2Z, a_{21} + X + Y, a_{32} + Y + Z, a_{31} + X + Z\}.$$

For us, degree–two means that the Newton polygon of P is the triangle determined by the points $(2, 0), (0, 2), (0, 0)$; in other words, that $a_{21}, a_{32}, a_{31} \in \mathbb{T}$ but $a_{11}, a_{22}, a_{33} \in \mathbb{R}$. The *tropical conic* $\mathcal{C}(P)$ given by P is, by definition, the set of points in $\mathbb{TP}^2$ where the *maximum is attained, at least, twice.* A simple computer program (done in MAPLE, for instance) may be used in order to sketch this conic, say in $Z = 0$. But we want to show that one can easily sketch $\mathcal{C}(P)$ without a computer! Indeed, it is well–known that $\mathcal{C}(P)$ is a tree, see [**12, 27, 29, 32**] and so, all we need to compute is the coordinates of its vertices.

So let us recall here some facts about *trees*; see [**11, 18**] for details. A *graph* G is an ordered pair (V, E), where V is a finite set of points, called *vertices* of G, and E is a set of cardinality–two subsets of V. The elements of E are called *edges* of G. The edge joining vertices u, w will be denoted $\overline{uw}$. The *degree* of a vertex w of G is the number of edges of G incident with w. Degree–one vertices are called *pendant vertices* and edges incident to pendant vertices are called *pendant edges*.

A *tree* is a connected graph without cycles. A tree $G = (V, E)$ naturally carries a *discrete metric*; it is the function $\mathrm{d} : V \times V \to \mathbb{N}$, where $\mathrm{d}(u, w)$ is the least number of edges to be passed through when going from u to w. If $\mathrm{d}(u, w) = 1$, we say that u, w are *consecutive vertices*. The *eccentricity* of a vertex w is $\mathrm{e}(w) = \max_{u\in V} \mathrm{d}(u, w)$ and the *radius* of the graph G is $\mathrm{r}(G) = \min_{w\in V} \mathrm{e}(w)$. A vertex w in G is *central* in G if $\mathrm{e}(w) = \mathrm{r}(G)$ and the *center* of G is the set of all central points in G. It is known that *every tree has a center and it consists either of just one vertex or two consecutive vertices.* This explains the names *one–point central* and *two–point central conics*, given below in theorem 2.6.

A tropical projective plane curve $\mathcal{C}$ of degree $d > 0$ is a weighted tree of a very particular sort. Each vertex of $\mathcal{C}$ is determined by its tropical projective coordinates. The pendant vertices of $\mathcal{C}$ are precisely the points in $\mathcal{C}$ which lie on the boundary of $\mathbb{TP}^2$. There are $3d$ such vertices, counted with multiplicity. They are grouped in 3 families of d vertices each: d vertices have the X (resp. Y) (resp. Z) coordinate equal to $-\infty$. Every pendant edge in $\mathcal{C}$ has infinite length. There are $3d$ such edges, counted with multiplicity, and they are grouped in 3 families of d edges each. The rest of the edges in $\mathcal{C}$ have finite lengths. Edges in $\mathcal{C}$ may carry a multiplicity, which is a natural number, no greater than d. The multiplicity of a vertex is deduced from the multiplicities of the edges incident to it.

A tropical projective plane curve $\mathcal{C}$ can be represented in $Z = 0$ (or in $Y = 0$ or $X = 0$). More precisely, this means that we represent $j_3^{-1}(\mathcal{C})$ (and still denote it $\mathcal{C}$) (or $j_2^{-1}(\mathcal{C})$ or $j_1^{-1}(\mathcal{C})$) in $\mathbb{T}^2$. Say, we choose to work in $Z = 0$. Then the *slope* of every edge of finite length in $\mathcal{C}$ is a rational number and at each non–pendant vertex w the *balance condition* holds. This means that $\sum_{j=1}^{s} \lambda_j e_j = 0$, where $u_1, \ldots, u_s$ are all the vertices in $\mathcal{C}$ consecutive to w, $\lambda_1, \ldots, \lambda_s \in \mathbb{N}$ are the weights of the edges $\overline{wu_1}, \ldots \overline{wu_s}$ and $e_1, \ldots, e_s \in \mathbb{Z}^2$ are the primitive integral vectors at the point w in the directions of such edges.

2.3. Matrices and points associated to a tropical degree–two polynomial. Let

$$P = \max\{a_{11} + 2X, a_{22} + 2Y, a_{33} + 2Z, a_{21} + X + Y, a_{32} + Y + Z, a_{31} + X + Z\}$$

be a homogeneous tropical polynomial of degree two. As explained in subsection 2.2, the tropical conic $\mathcal{C}(P)$ has six pendant edges, counted with multiplicities. These multiplicities are either one or two. Without loss of generality, we may work in $Z = 0$. Then $\mathcal{C}(P)$ has two pendant edges in the west direction, two in the south direction and two in the north-east direction, all counted with multiplicity. In order to sketch the conic $\mathcal{C}(P)$ we must determine the non–pendant vertices of $\mathcal{C}(P)$. We will see that there are four such points, at most.

Just like in usual geometry, to P we associate the symmetric matrix $A(P) = (a_{ij})$, bearing in mind that we need not divide the coefficients of mixed terms by two, since tropical addition is idempotent. Conversely, to such a matrix A, we can associate a polynomial $P(A)$ and, eventually, a tropical conic $\mathcal{C}(A)$.

Most matrices considered in this paper are 3×3 and have entries in $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$ but their diagonal elements belong to $\mathbb{R}$ (the only exception appears in the definition of tropical determinant) and are symmetric. Therefore, we only write their lower triangular parts. To the symmetric matrix

$$A = \begin{pmatrix} a_{11} & & \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

we associate the diagonal matrix

$$D = D(A) = \begin{pmatrix} a_{11}/2 & & \\ -\infty & a_{22}/2 & \\ -\infty & -\infty & a_{33}/2 \end{pmatrix},$$

which corresponds to a translation of coordinates, as we have seen in p. 3. The tropical inverse matrix of D is obtained by negating the signs of its diagonal entries. Obviously, it corresponds to the inverse translation. We define the *shape matrix* associated to A as $S = \operatorname{shape}(A) = D^{\odot -1} \odot A \odot D^{\odot -1}$. Clearly, **the shape matrix corresponds to the given conic $\mathcal{C}(P)$, after translation**. It is crucial and easy to check that *the shape matrix $S = (s_{ij})$ is symmetric and has zero diagonal.* The remaining entries of S are related to A by the following formulas:

(2.3) $2s_{21} = 2a_{21} - a_{11} - a_{22}, \quad 2s_{32} = 2a_{32} - a_{22} - a_{33}, \quad 2s_{31} = 2a_{31} - a_{33} - a_{11}.$

Therefore the shape matrix is *invariant*, in the sense that it does not change if A is replaced by $A = \alpha + U$, for any $\alpha \in \mathbb{R}$, where U denotes the 3×3 matrix all whose entries are one. Notice also that the matrices A and S are/are not simultaneously real. Back to the polynomial P, let $\operatorname{shape}(P)$, $D(P)$ denote the polynomials associated to the matrices S and D.

The *tropical determinant* of an arbitrary 3×3 matrix $A = (a_{ij})$ is defined as

$$|A|_{trop} = \max_{\sigma \in S_3}\{a_{1\sigma(1)} + a_{2\sigma(2)} + a_{3\sigma(3)}\},$$

where S_3 denotes the permutation group in 3 symbols. A matrix is *tropically singular* if the *maximum in the tropical determinant is attained, at least, twice*. For the matrices above we have

$$2|D|_{trop} = a_{11} + a_{22} + a_{33}$$

and D is tropically non–singular. Moreover, A and S are/are not simultaneously tropically singular, because

$$\sum_{i=1}^{3} a_{i\sigma(i)} = \sum_{i=1}^{3} s_{i\sigma(i)} + 2|D|_{trop},$$

for all $\sigma \in S_3$. The tropical determinant of S is $\max\{0, s, s, 2s_{21}, 2s_{32}, 2s_{31}\}$, where

$$s = s_{21} + s_{32} + s_{31}. \tag{2.4}$$

In addition, $|A|_{trop} = 2|D|_{trop} + |S|_{trop}$.

LEMMA 2.1. $\operatorname{shape}(\operatorname{shape}(A)) = \operatorname{shape}(A)$.

PROOF. It follows from the formulas (2.3). □

In the following, we assume $A = \operatorname{shape}(A)$ (or equivalently, $P = \operatorname{shape}(P)$), meaning that $a_{11} = a_{22} = a_{33} = 0$ and $a_{ij} = s_{ij}$, if $i \neq j$. Now, the next crucial lemma tells us that *the matrices A and A^{+} give rise to the same tropical conic.*

LEMMA 2.2. *If $P = \operatorname{shape}(P)$, then $\mathcal{C}(P) = \mathcal{C}(P^+)$.*

PROOF. By hypothesis,

$$P = \max\{2X, 2Y, 2Z, s_{21} + X + Y, s_{32} + Y + Z, s_{31} + X + Z\}.$$

If $-\infty \le s_{21} \le 0$ then

$$P^+ = \max\left\{2X, 2Y, 2Z, X + Y, s_{32}^+ + Y + Z, s_{31}^+ + X + Z\right\}.$$

It is obvious that

$$\max\{2X, 2Y, s_{21} + X + Y\} = \max\{2X, 2Y\} = \max\{2X, 2Y, X + Y\}.$$

Moreover, these three maxima are attained at least twice at exactly the same points in $\mathbb{R}^2$. Therefore, the term $s_{21} + X + Y$ is irrelevant in P, as far as $\mathcal{C}(P)$ is concerned.

We can reason similarly with s_{32}, s_{31}, and thus conclude that the tropical conics $\mathcal{C}(P), \mathcal{C}(P^+)$ are equal. □

In the former paragraphs, we have reduced the study of tropical conics to the case $A = \operatorname{shape}(A)^+$, a non–negative real matrix. Now, what does such a tropical conic $\mathcal{C}(A)$ look like, say in $Z = 0$? To answer this question, we define the points $v^1(A), v^2(A), v^3(A)$ which arise from the rows of A:

$$v^1(A) = [-s_{11}, -s_{21}, -s_{31}],\ v^2(A) = [-s_{21}, -s_{22}, -s_{32}],\ v^3(A) = [-s_{31}, -s_{32}, -s_{33}]$$

and one more point $v^0(A)$ by

$$v^0(A) = [s_{32}, s_{31}, s_{21}].$$

The points will be denoted v^0, v^1, v^2, v^3, for short.

LEMMA 2.3. *Suppose $A = \operatorname{shape}(A)^+$. Then, in $Z = 0$,*

1. *the segment $\overline{v^0v^1}$ is parallel to the X axis,*
2. *the segment $\overline{v^0v^2}$ is parallel to the Y axis,*
3. *the segment $\overline{v^0v^3}$ is parallel to the line $X = Y$.*

PROOF. Taking differences, we have $v^1 - v^0 = [-s_{32}, -s_{21} - s_{31}, -s_{21} - s_{31}]$ and the coordinates of this point in $Z = 0$ are $(-s_{32} + s_{21} + s_{31}, 0)$. The rest is similar: $v^2 - v^0 = (0, -s_{31} + s_{32} + s_{21})$ and $v^3 - v^0 = (-s_{31} - s_{32} + s_{21}, -s_{31} - s_{32} + s_{21})$. □

Notice how the lengths of the segments $\overline{v^0v^j}$ depend on alternating sums of the entries of the matrix $A = \operatorname{shape}(A)^+$. More precisely, set

$$\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} s_{21} \\ s_{32} \\ s_{31} \end{pmatrix} \tag{2.5}$$

in terms of the ordinary matrix multiplication. Hence

$$s_{ij} = \frac{d_i + d_j}{2}, \quad i \ne j. \tag{2.6}$$

The length of $\overline{v^0v^j}$ is $|d_j|$, for $j = 1, 2$, and the length of $\overline{v^0v^3}$ is $\sqrt{2}|d_3|$ (the factor $\sqrt{2}$ is due to our choice of normalization $Z = 0$). Moreover, the angle $\angle v^1v^0v^2$ is $\dfrac{\pi}{2}$. In addition, $\angle v^1v^0v^3$ is $\frac{3\pi}{4}$ (resp. $\frac{\pi}{4}$) if $d_1d_3 > 0$ (resp. $d_1d_3 < 0$). Notice that the vertices v^1, v^0, v^3 determine a right triangle in $Y = 0$. Similarly, the vertices v^2, v^0, v^3 determine a right triangle in $X = 0$.

LEMMA 2.4. *If $A = \operatorname{shape}(A)^+$, then d_j is negative for, at most, one $j \in \{1, 2, 3\}$.*

PROOF. Suppose $d_1 < 0$. By the hypothesis and relations (2.6), $0 \le d_1 + d_2$ and $0 \le d_1 + d_3$, whence $0 < -d_1 \le d_2$ and $0 < -d_1 \le d_3$. The other cases are similar. □

LEMMA 2.5. *If $A = \operatorname{shape}(A)^+$, then the following are equivalent:*

(1) *A is tropically singular,*
(2) *the maximum of s_{21}, s_{32}, s_{31} is no greater than the sum of the other two,*
(3) *d_1, d_2, d_3 are all non–negative.*

PROOF. Equivalence between (2) and (3) follows from (2.5). We show that (1) and (2) are equivalent. Note that only one of the numbers $2s_{21}, 2s_{32}, 2s_{31}$ can be greater than $s = s_{21}+s_{32}+s_{31}$, so that the maximum is attained twice in $\max\{0, s, s, 2s_{21}, 2s_{32}, 2s_{31}\}$ if and only if it equals s. Now note that this happens if and only if (2) is satisfied. □

Any tropical conic $\mathcal{C}$ has some non–pendant vertices. These are the points in $\mathcal{C}$ where the maximum is attained, at least, three times.

If $\mathcal{C}$ has more than two non–pendant vertices, let us consider two consecutive ones u^1, u^2. If these points come together, a new tropical conic $\mathcal{C}'$ arises. Obviously, if $\mathcal{C}$ has parallel pendant edges e^1, e^2 such that e^j is incident to u^j, then e^1 is a pendant edge with multiplicity two in $\mathcal{C}'$. Let $\mathcal{C}'$ be a tropical conic which can be obtained from $\mathcal{C}$ by successively collapsing one or more pairs of consecutive non–pendant vertices. Then we will say that *$\mathcal{C}'$ is a degeneration of $\mathcal{C}$*. Such a conic $\mathcal{C}'$ is called *degenerate*.

Now we get our two main theorems. In page 9 we explain why theorem 2.6 deals with *non–degenerate tropical conics* while theorem 2.8 classifies *degenerate tropical conics*.

In the second part of the following theorem, superscripts work modulo 3, and $t^{i,j}$ stands for the point in $\mathbb{TP}^2$ whose i–th coordinate is $-2s_{ij}$ and the rest are null.

THEOREM 2.6. *Let $A = \operatorname{shape}(A)^+ = (s_{ij})$. Suppose that $s_{ij} > 0$ for all $i \ne j$ and $d_j \ne 0$, for $j = 1, 2, 3$. Then the following mutually exclusive cases arise, for the tropical conic $\mathcal{C} = \mathcal{C}(A)$.*

(1) One–point central conic. *If d_1, d_2, d_3 are all positive, then $\mathcal{C}$ has four non–pendant vertices; these are v^1, v^2, v^3 and v^0.*
(2) Two–point central conic. *If $d_j < 0$ for some $j \in \{1, 2, 3\}$, then $\mathcal{C}$ has four non–pendant vertices; these are $v^{j-1}, v^{j+1}, w^{j-1} = v^{j-1} + t^{j-1,j}$ and $w^{j+1} = v^{j+1} + t^{j+1,j}$.*

PROOF. We may assume that $d_1 > 0$ and $d_2 > 0$ by a permutation of variables and lemma 2.4. For simplicity, let us work in $Z = 0$ and let us evaluate

$$P = \max\{2X, 2Y, 0, s_{21} + X + Y, s_{31} + X, s_{32} + Y\}$$

in $v^1 = (s_{31}, s_{31} - s_{21})$ and $v^2 = (s_{32} - s_{21}, s_{32})$. Using that $d_1 > 0$ and $d_2 > 0$, we obtain that

$$\max\{2s_{31}, 2(s_{31} - s_{21}), 0, 2s_{31}, 2s_{31}, d_3\} = 2s_{31}$$
$$\max\{2(s_{32} - s_{21}), 2s_{32}, 0, 2s_{32}, d_3, 2s_{32}\} = 2s_{32}$$

both attained three times. This means that v^1 and v^2 are non–pendant vertices of $\mathcal{C}$. Now we evaluate P in $v^3 = (-s_{31}, -s_{32})$ and $v^0 = (s_{32} - s_{21}, s_{31} - s_{21})$ and obtain

$$\max\{-2s_{31}, -2s_{32}, 0, -d_3, 0, 0\}$$
$$\max\{2(s_{32} - s_{21}), 2(s_{31} - s_{21}), 0, d_3, d_3, d_3\}.$$

It follows that v^3 and v^0 are also non–pendant vertices of $\mathcal{C}$, if $d_3 > 0$ and, no further non–pendant vertices of $\mathcal{C}$ arise, by symmetry in the variables; see figure 1, right. The center of

$\mathcal{C}$ is v^0 and we say that $\mathcal{C}$ is a *one–point central conic*. Six pendant edges hang from the v^1, v^2, v^3 as explained in subsection 2.2, completing the picture of $\mathcal{C}$; see figure 2 line 1, column 3. If we work in $X = 0$, (resp. $Y = 0$) we obtain other representations of $\mathcal{C}$; see figure 2 line 1, column 1 (resp. column 2).

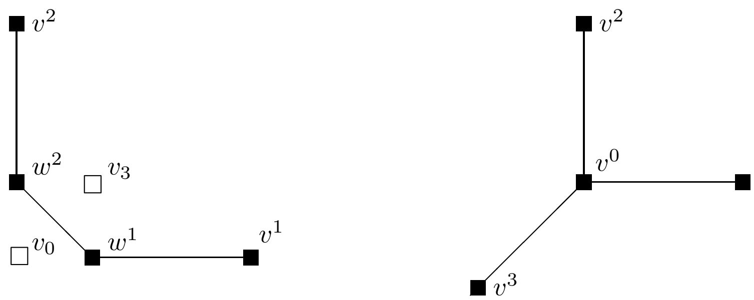

FIGURE 1. Non–pendant vertices: cases $d_3 < 0$ and $d_3 > 0$.

Now, if $d_3 < 0$, we consider $w^1 = v^1 + [-2s_{31}, 0, 0]$ and $w^2 = v^2 + [0, -2s_{32}, 0]$. Working in $Z = 0$ and evaluating P in $w^1 = (-s_{31}, s_{31} - s_{21})$ and $w^2 = (s_{32} - s_{21}, -s_{32})$ we get

$$\max\{-2s_{31}, 2(s_{31} - s_{21}), 0, 0, 0, d_3\} = 0$$

$$\max\{2(s_{32} - s_{21}), -2s_{32}, , 0, 0, d_3, 0\} = 0$$

both attained three times. It follows that w^1 and w^2 are non–pendant vertices of $\mathcal{C}$ (in addition to v^1 and v^2), if $d_3 < 0$. No more non–pendant vertices of $\mathcal{C}$ arise also in this case. In particular, v^3, v^0 are NOT vertices in $\mathcal{C}$, if $d_3 < 0$; see figure 1, left. The center of $\mathcal{C}$ consists of w^1 and w^2 and we say that $\mathcal{C}$ is a *two–point central conic*. Six pendant edges of $\mathcal{C}$ hang from v^1, v^2, w^1, w^2. Such a tropical conic is represented in figure 2 line 2, column 3.

If $d_1 < 0$ or $d_2 < 0$, other two–point central conics are obtained, and they are represented in figure 2 line 2, columns 1 and 2. Notice that a factor $\sqrt{5}$ appears in the length of edges of slope $\frac{1}{2}$ or 2, due to our choice of Euclidean metric. □

COROLLARY 2.7. *Let $A = \text{shape}(A)^+ = (s_{ij})$. Suppose that $s_{ij} > 0$ for all $i \neq j$, $d_1 > 0$, $d_2 > 0$ and $d_3 \neq 0$. Then in $Z = 0$, the tropical conic $\mathcal{C} = \mathcal{C}(A)$ has two different pendant edges in the north–east direction (resp. west direction) (resp. south direction) and they are $\sqrt{d_1^2 + d_2^2}$ (resp. $2s_{32}$) (resp. $2s_{31}$) apart.*

PROOF. The previous theorem applies and the statement follows from the equalities (2.6). □

Notice that $\sqrt{d_1^2 + d_2^2}$ tends to zero if and only if $2s_{21} = d_1 + d_2$ tends to zero.

Suppose that $A = \text{shape}(A)^+$ and $s_{ij} > 0$, for $i \neq j$ and $d_j \neq 0$, for $j = 1, 2, 3$. Then the degenerations of the tropical conic $\mathcal{C}(A)$ arise by letting $s_{ij} = 0$ or $d_j = 0$ for some indices. We have the following classification theorem.

THEOREM 2.8. *If $A = \text{shape}(A)^+$ and $s_{ij} = 0$ or $d_j = 0$ for some indices, then the following cases arise (up to a permutation of variables) for the tropical conic $\mathcal{C}(A)$:*

(1) $s_{21} > 0, s_{32} > 0, s_{31} = 0, d_1 > 0, d_2 > 0$ *and* $d_3 < 0$.
(2) $s_{21} > 0, s_{32} = s_{31} = 0, d_1 > 0, d_2 > 0$ *and* $d_3 < 0$.
(3) Double tropical line. $s_{21} = s_{32} = s_{31} = 0$ *(equivalently,* $d_1 = d_2 = d_3 = 0$ *or, yet equivalently,* $d_3 = s_{32} = s_{31} = 0$*).*
(4) Pair of tropical lines. $s_{21} > 0, s_{32} > 0, s_{31} > 0, d_1 > 0, d_2 > 0$ *and* $d_3 = 0$.
(5) Pair of tropical lines. $s_{21} > 0, s_{32} = 0, s_{31} > 0, d_1 > 0$ *and* $d_2 = d_3 = 0$.

PROOF. (1) This situation arises when v^1 and w^1 collapse, in a two–point central conic.
(2) This situation arises when, in addition to the former, v^2 and w^2 collapse, in a two–point central conic.
(3) This situation arises when v^1, w^1, v^2 and w^2 all collapse to one point, in a two–point central conic. It also arises when v^j all collapse to one point, for $j = 0, 1, 2, 3$, in a one–point central conic.
(4) This situation arises when w^1 and w^2 collapse, in a two–point central conic. It also arises when v^0 and v^3 collapse, in a one–point central conic.
(5) This situation arises when v^2, w^2 and w^1 all collapse, in a two–point central conic. It also arises when v^0, v^2 and v^3 collapse, in a one–point central conic.

□

These conics are represented in figure 2, lines 3 to 8, where a thick segment represents a multiplicity–two edge.

Let us summarize. Up to translation, tropical conics are determined by a non–negative real matrix $S^+ = (s_{ij})$ with zero diagonal. We have gone through all the possibilities for the s_{ij}, in the two theorems above. This means that no more tropical conics do exist! Therefore, theorem 2.6 classifies *non–degenerate tropical conics*, while theorem 2.8 classifies *degenerate tropical conics*.

A **procedure to sketch, say in $Z = 0$, the tropical conic $\mathcal{C}(P)$ defined by an arbitrary homogeneous degree–two polynomial** P is the following

- From P, compute the matrices A and $S^+ = \text{shape}(A)^+$.
- Sketch the conic $\mathcal{C}(S^+)$, according to the classification given by the theorems above and translate this conic to the point $\frac{1}{2}(a_{33} - a_{11}, a_{33} - a_{22})$ in $\mathbb{R}^2$ to obtain $\mathcal{C}(P)$.

The following are all direct consequences of our discussion.

COROLLARY 2.9. *A tropical conic is non–degenerate if and only if it is not the union of two tropical lines and all of its pendant edges have multiplicity one.* □

COROLLARY 2.10 (Pairs of tropical lines). *For a tropical conic $\mathcal{C} = \mathcal{C}(A)$, the following statements are equivalent:*

- *$\mathcal{C}$ is a pair of lines,*
- *d_1, d_2, d_3 are all non–negative and, at least, one d_j equals zero,*
- *the maximum of $s^+_{21}, s^+_{32}, s^+_{31}$ equals the sum of the other two,*
- $v^0 \in \{v^1, v^2, v^3\}$ *for the matrix* $\text{shape}(A)^+$. □

Notice that the number of different pendant edges in a pair of tropical lines is six, five or three. Pairs of tropical lines are represented in figure 2, lines 6 to 8.

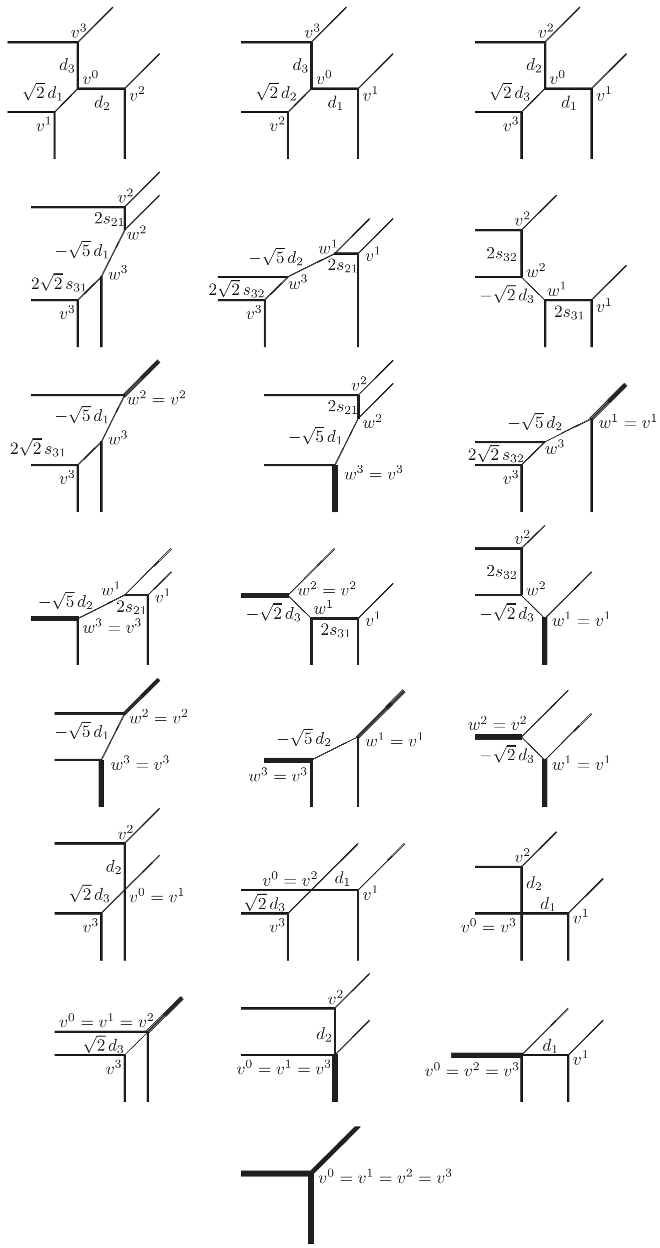

FIGURE 2. Tropical conics. Line 1 is occupied by one–point central conics, line 2 is occupied by two–point central conics, lines 3 to 8 are occupied by degenerate conics, where lines 6 to 8 are occupied by pairs of lines.

COROLLARY 2.11. *A tropical conic $\mathcal{C} = \mathcal{C}(P)$ has tropically singular associated matrix* $\operatorname{shape}(P)^+$ *if and only if $\mathcal{C}$ is either a pair of tropical lines or a one–point central conic.*

PROOF. This follows from lemma 2.5, part 1 of theorem 2.6 and corollary 2.10. □

A tropical conic $\mathcal{C}(P)$ is determined by a triple $(s_{21}^+, s_{32}^+, s_{31}^+)$ of real non–negative numbers and any row of the matrix $A = A(P)$. The null triple corresponds to a double tropical line. Let $(s_{21}^+, s_{32}^+, s_{31}^+) \neq (0,0,0)$ be the coordinates of a point in the non–negative octant $\mathcal{O} = \mathbb{R}^3_{\geq 0}$. In figure 3 we see the plane section of $\mathcal{O}$ given by $s_{21}^+ + s_{32}^+ + s_{31}^+ = s$, for some positive s. According to corollary 2.11, tropical conics having tropically singular matrix $\operatorname{shape}(A)^+$ correspond to the shaded closed triangle, the boundary of which corresponds to pairs of lines. Other degenerate tropical conics correspond to the boundary of the section.

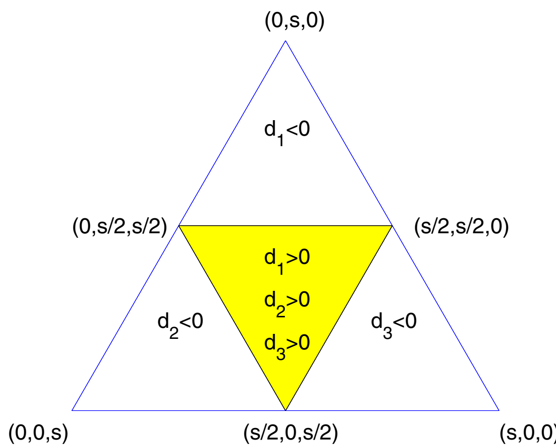

FIGURE 3. Section of octant $\mathcal{O}$ given by $s_{21}^+ + s_{32}^+ + s_{31}^+ = s$.

It is known that every balanced weighted tree is a tropical curve, see [**12, 26, 27**]. When $d = 2$, here is a **procedure to find a defining polynomial** P **for a balanced graph** $\mathcal{C}$.

- From the edges of $\mathcal{C}$, compute the values $s_{21}^+, s_{32}^+, s_{31}^+$ and classify $\mathcal{C}$ (degenerate or non–degenerate and type).
- From the vertices of $\mathcal{C}$, compute symmetric matrices A and $\operatorname{shape}(A)$, using as many unknowns as necessary.
- Solve for the unknowns, according to the classification.

EXAMPLE 2.12. In $Z = 0$, let the weighted tree $\mathcal{C}$ in figure 4 be given. Here thick segments represent edges of multiplicity two. The non–pendant vertices of $\mathcal{C}$ are $v^1 = (4,2) = [4,2,0]$, $v^3 = (0,0) = [0,0,0]$ and the balance condition is satisfied at both. Indeed, at v^1 the primitive vectors are $(1,1), (0,1), (-2,-1)$ and $2(1,1) + (0,1) + (-2,-1) = (0,0)$. Similarly, for v^3. Therefore, this tree corresponds to a tropical conic. It is a degenerate tropical conic (not a pair of lines) and, by corollary 2.7 and theorem 2.8, $s_{21}^+ = s_{32}^+ = 0$ and $s_{31}^+ = 2$. Then s_{21}, s_{32} are non–positive and $s_{31} = 2$. We fill the negated coordinates of v^1, v^3 into the rows of a symmetric matrix A and compute the matrices $D = D(A)$ and $\operatorname{shape}(A) = D^{\odot -1} \odot A \odot D^{\odot -1}$ obtaining:

$$A = \begin{pmatrix} -4 & & \\ -2 & a_{22} & \\ 0 & 0 & 0 \end{pmatrix}, \qquad \operatorname{shape}(A) = \begin{pmatrix} 0 & & \\ -a_{22}/2 & 0 & \\ 2 & -a_{22}/2 & 0 \end{pmatrix},$$

for some $a_{22} \in \mathbb{R}$. Therefore $s_{21} = s_{32} = -a_{22}/2 \leq 0$. The points associated to $\operatorname{shape}(A)$ are $v^{1'} = [0, a_{22}/2, -2] = [2, 2 + a_{22}/2, 0]$ and $v^{3'} = [-2, a_{22}/2, 0]$ and the slope of the segment $\overline{v^{3'}v^{1'}}$ is $\frac{1}{2}$ (in $Z = 0$), independently of the precise value of a_{22}. Then, any $a_{22} \geq 0$ will do. We may take $a_{22} = 0$ and we conclude that $\mathcal{C}$ is given by the tropical polynomial $P = (-4) \odot X^{\odot 2} \oplus Y^{\odot 2} \oplus Z^{\odot 2} \oplus (-2) \odot X \odot Y \oplus Y \odot Z \oplus X \odot Z$.

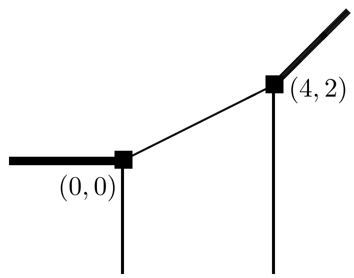

FIGURE 4. A weighted tree $\mathcal{C}$ in $Z = 0$.

3. Factorization of degree–two tropical polynomials

A tropical polynomial p (homogeneous or not) in any number of variables is called *reducible* if it is the tropical product of two non–constant tropical polynomials. A tropical hypersurface $\mathcal{C}$ (affine or projective) is called *reducible* if it is the union of two hypersurfaces (affine or projective, accordingly) $\mathcal{C}_1, \mathcal{C}_2$ with $\mathcal{C}_1 \neq \mathcal{C} \neq \mathcal{C}_2$. It is clear that the reducibility of a polynomial causes the reducibility of the corresponding hypersurface but, in the tropical setting, the converse is NOT true; see corollary 3.3 below.

Let P be a homogeneous degree–two tropical polynomial in three variables. The simplest example of a reducible polynomial arises when $a_{ij} = 0$, for all $i \neq j$. Then P is the tropical square of the linear form $a_{11}/2 \odot X \oplus a_{22}/2 \odot Y \oplus a_{33}/2 \odot Z$, because in tropical algebra the *freshman's dream* $(a \oplus b)^{\odot n} = a^{\odot n} \oplus b^{\odot n}$ holds for all n! The corresponding matrices and points are easy to compute:

$$A = \begin{pmatrix} a_{11} & & \\ -\infty & a_{22} & \\ -\infty & -\infty & a_{33} \end{pmatrix} = D^{\odot 2},$$

$\operatorname{shape}(A) = I$ is the tropical identity matrix, $\operatorname{shape}(A)^+$ is the zero matrix and the tropical conic $\mathcal{C}(P)$ is a *double line* with vertex at $v = \frac{1}{2}[-a_{11}, -a_{22}, -a_{33}]$.

LEMMA 3.1. *P is reducible if and only if* $\operatorname{shape}(P)$ *is.*

PROOF. Consider the associated matrix $A = A(P)$. The factorization $A = D \odot S \odot D$ corresponds to a change of variables $[X, Y, Z] \mapsto [X', Y', Z'] = [X, Y, Z] \odot D$. □

The former lemma allows us to reduce our discussion to the case $P = \operatorname{shape}(P)$.

THEOREM 3.2. *If* $P = \text{shape}(P) = X^{\odot 2} \oplus Y^{\odot 2} \oplus Z^{\odot 2} \oplus s_{21} \odot X \odot Y \oplus s_{32} \odot Y \odot Z \oplus s_{31} \odot X \odot Z$*, then the following statements hold.*

(1) *If* $-\infty \leq s_{ij} < 0$*, for some* $i \neq j$*, then* P *is irreducible.*
(2) *If* $s_{ij} \geq 0$*, for all* $i \neq j$*, then* P *is reducible if and only if the maximum of* s_{21}, s_{32}, s_{31} *equals the sum of the other two.*

PROOF. Up to tropical multiplication by a real constant, a tropical factorization of P must have the form

$$(a \odot X \oplus b \odot Y \oplus Z) \odot ((-a) \odot X \oplus (-b) \odot Y \oplus Z), \tag{3.1}$$

for $a, b \in \mathbb{R}$, where $s_{21} = |a-b|$, $s_{32} = |b|$, $s_{31} = |a| \in \mathbb{R}_{\geq 0}$. The irreducibility statement now follows. For the second statement, let us assume that $s_{ij} \geq 0$, for all $i \neq j$ and, without loss of generality, that $s_{31} = \max\{s_{21}, s_{32}, s_{31}\}$. Suppose that $s_{31} = s_{21} + s_{32}$. Then we take $a = s_{31}$ and $b = s_{32}$, so that P equals the product (3.1). The converse is easy. □

COROLLARY 3.3. *If* $P = \text{shape}(P)$ *and* $-\infty \leq s_{ij} < 0$*, for all* $i \neq j$*, then the polynomial* P *is irreducible, but the conic* $\mathcal{C}(P)$ *is a double line.* □

Summing up, here is a **procedure to determine whether a given tropical degree–two homogeneous polynomial P in three variables is reducible and, in such a case, to obtain a factorization**.

- Compute the polynomial $\text{shape}(P)$ and decide whether it is reducible or not, using theorem 3.2.
- If $\text{shape}(P)$ is reducible, we can factor it, as explained in the proof of theorem 3.2. Then, a change of coordinates provides a factorization of P, by lemma 3.1.

EXAMPLE 3.4. Let $P = X^{\odot 2} \oplus 12 \odot Y^{\odot 2} \oplus Z^{\odot 2} \oplus 7 \odot X \odot Y \oplus 6 \odot Y \odot Z \oplus 1 \odot X \odot Z$. The associated matrices are

$$A = \begin{pmatrix} 0 & & \\ 7 & 12 & \\ 1 & 6 & 0 \end{pmatrix}, \ D = \begin{pmatrix} 0 & & \\ -\infty & 6 & \\ -\infty & -\infty & 0 \end{pmatrix}, \ S = S^+ = \begin{pmatrix} 0 & & \\ 1 & 0 & \\ 1 & 0 & 0 \end{pmatrix}.$$

Then $s_{21} = s_{31} = 1$, $s_{32} = 0$ and $\max\{s_{21}, s_{32}, s_{31}\} = s_{31} = s_{21} + s_{32}$. By theorem 3.2, the polynomial $\text{shape}(P) = X^{\odot 2} \oplus Y^{\odot 2} \oplus Z^{\odot 2} \oplus 1 \odot X \odot Y \oplus Y \odot Z \oplus 1 \odot X \odot Z$ is reducible and a factorization is given by (3.1) with $a = 1$, $b = 0$. Then the translation given by the point $[0, -6, 0]$ provides $(1 \odot X \oplus (-6) \odot Y \oplus Z) \odot ((-1) \odot X \oplus 6 \odot Y \oplus Z)$ as a factorization of P.

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