An algorithm to solve any tropical linear system $A \odot x = B \odot x$

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Abstract

An algorithm to solve any tropical linear system $A \odot x = B \odot x$ is presented. The given system is converted into two classical linear systems: a system of equations and a system of inequalities, each item (equation or inequality) involving exactly two variables, one with coefficient 1, and another with coefficient -1. The two classical linear systems are solved, essentially, by triangulation and backward substitution.

1 Introduction

Consider the set $\mathbb{R} \cup \{-\infty\}$, denoted $\mathbb{T}$ for short, endowed with *tropical addition* $\oplus$ and *tropical multiplication* $\odot$, where these operations are defined as follows:

$$a \oplus b = \max\{a, b\}, \qquad a \odot b = a + b,$$

for $a, b \in \mathbb{T}$. Here, $-\infty$ is the neutral element for tropical addition and 0 is the neutral element for tropical multiplication. Notice that $a \oplus a = a$, for all a, i.e., tropical addition is *idempotent*. Notice also that a has no inverse with respect to $\oplus$. We will write $\oplus$ or $\max$, (resp. $\odot$ or $+$) at our convenience. In this paper we will use the adjective *classical* as opposed to *tropical*. Most definitions in tropical mathematics just mimic the classical ones. Very often, working with $(\mathbb{T}, \oplus, \odot)$ leads to working with $\min$, which will be denoted $\oplus'$.

Assume $n > 1$. Given matrices $A, B \in \mathcal{M}_{m\times n}(\mathbb{T})$, we want to compute all $x \in \mathbb{T}^n$ such that $A \odot x = B \odot x$. This means

$$\max\{a_{ij} + x_j : 1 \le j \le n\} = \max\{b_{ij} + x_j : 1 \le j \le n\}, \quad i = 1, 2, \dots, m.$$

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Using the notion of *win sequence* (introduced in this paper; see definition 4), the given problem is reduced to solving two classical linear systems: a system of equations and a system of inequalities, each item (equation or inequality) being *bivariate*, i.e., it involves exactly two variables, one with coefficient 1, and another with coefficient -1. Then we give a procedure (a Gaussian–like elimination method) to solve these two systems. More precisely, by elementary transformations of rows, we can triangulate and then apply backward substitution to the system of equations. This is exactly the Gaussian method. For the system of inequalities, strictly speaking, we cannot achieve a triangular form, but nearly so: we can reduce the given matrix to a *sub–special matrix* (see definition 6) by row transformations, and then we can solve the resulting system of inequalities by backward substitution. We need not use the simplex algorithm or other well–known ones to solve our system of inequalities.

The problem $A \odot x = B \odot x$ has been addressed before. Indeed, in [3], a strongly polynomial algorithm is found which either finds a solution or it tells us that no solution exists. In [1] sec. 3.5, it is solved by a technique called symmetrization and resolution of balances. In [2], the problem is solved by finding generators for the solution set. The idea of finding (a minimal family of) generators is pursued in [9, 10] for the closely related problem $A \odot x \leq B \odot x$. An iterative method in presented in [5] for another closely related problem, namely $A \odot x = B \odot y$, where x and y are unknown, here. Also, there is a technique in [11] to solve the problem $A \odot x \oplus a = B \odot x \oplus b$, relying on a recursive formulation of the closure operator (also called Kleene star operator) on matrices. In [6] ch. 4, the closely related problem $A \odot x \oplus b = x$ (similar to the classical Jacobi iterative method) is solved using Kleene stars.

Let $m, n, p \in \mathbb{N}$ be given. The following are kindred problems in tropical linear algebra:

- $P1$: $A \odot x = 0$,
- $P2$: $A \odot x = b$,
- $P3$: $A \odot x \leq b$,
- $P4$: $A \odot x = B \odot x$,
- $P5$: $A \odot x \leq B \odot x$,
- $P6$: $C \odot x = D \odot y$,
- $P7$: $A \odot x \oplus a = B \odot x \oplus b$.

Here the data are matrices $A, B \in \mathcal{M}_{m\times n}(\mathbb{T})$, $C \in \mathcal{M}_{p\times n}(\mathbb{T})$, $D \in \mathcal{M}_{p\times m}(\mathbb{T})$ and vectors a, b, c, d over $\mathbb{T}$, and the jth problem is finding **all** vectors $x \in \mathbb{T}^n$,

$y \in \mathbb{T}^m$, such that Pj holds. Some references since 1984 are [1, 2, 4, 5, 9, 10, 11]. Earlier books and papers can be found there.

Of course, $x = -\infty$, $y = -\infty$ are solutions to $P3, P4, P5$ and $P6$. These are the *trivial solutions*.

Only if the vector b is real, problem $P2$ reduces to $P1$. More generally, one must realize that, contrary to classical linear algebra, problems $P7$ and $P4$ do not reduce to problem $P2$ or $P1$, because there are no inverses for tropical addition, so there is no tropical analogue for the matrix $-B$. Nevertheless, there are well–known connections among these problems, i.e., being able to solve some of them is equivalent to being able to solve some other. By *deciding a problem* we mean finding **all** the solutions, if any, or declaring that the problem has no (non–trivial) solution.

We need some notations:

- For $c, d \in \mathbb{T}$, $c \oplus' d$ means $\min\{c, d\}$ and $c \odot' d$ means $c + d$.
- For $c, d \in \mathbb{T}^n$, $c \odot' d^T$ means $\min\{c_1 + d_1, c_2 + d_2, \ldots, c_n + d_n\}$.
- If $A = (a_{ij}) \in \mathcal{M}_{m\times n}(\mathbb{R})$ then $A^* = (-a_{ji})$ is the *conjugate matrix*.

The relationship among these problems is as follows:

- *Deciding $P3$ is possible, if A is real.*

  Indeed, $x^{\#} = A^* \odot' b$ is a solution (called *principal solution*) and $x \leq x^{\#}$ if and only if x is a solution; see [4], p. 31; in [1] this process is called *residuation*.

- *Deciding $P3$ helps with deciding $P3$, if A is real.*

  Indeed, $P3$ might be incompatible but, if it has a solution, then $x^{\#}$ is the greatest one; see [4], p. 31.

- *Deciding $P6$ implies deciding $P2$.*

  Given A and b, we decide $A \odot x = I \odot y$. For each pair of solutions x, y, if any, we set $y = b$, if possible.

- *Deciding $P4$ is equivalent to deciding $P6$.*

  Suppose x is a solution to $P4$ and write $A \odot x = y$. Concatenating matrices, write $C = \begin{bmatrix} A \\ B \end{bmatrix} \in \mathcal{M}_{2m\times n}(\mathbb{T})$, $D = \begin{bmatrix} I \\ I \end{bmatrix} \in \mathcal{M}_{2m\times n}(\mathbb{T})$, where I is the tropical identity matrix, so that $C \odot x = D \odot y$. Therefore, if we can decide $P6$, then we can decide $P4$.

Suppose now x, y are solutions to $P6$ and write $z = \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$, $A = \begin{bmatrix} C & -\infty \\ -\infty & D \end{bmatrix}$, $B = \begin{bmatrix} -\infty & D \\ C & -\infty \end{bmatrix}$ so that $A \odot z = B \odot z$. Therefore, if we can decide $P4$, then we can decide $P6$.

- *Deciding $P5$ implies deciding $P4$.*

  Obvious.

- *Deciding $P3$ and $P4$ implies deciding $P5$.*

  If x is a solution to $P4$ and we write $A \odot x = y$, then we find all z such that $A \odot z \leq y$.

- *Deciding $P4$ implies deciding $P7$.*

  We introduce a new scalar variable z and write $A \odot x \oplus a \odot z = B \odot x \oplus b \odot z$. Concatenating matrices, write $t = \left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right]$, $C = [A, a]$, $D = [B, b]$ so that $C \odot t = D \odot t$. After solving $P4$, set $t_{n+1} = z = 0$.

2 The problem

Assume $n > 1$. Given matrices $A, B \in \mathcal{M}_{m \times n}(\mathbb{T})$, we want to compute all $x \in \mathbb{T}^n$ such that

$$A \odot x = B \odot x. \tag{1}$$

Notations:

- $[n] = \{1, \ldots, n\}$, for $n \in \mathbb{N}$.
- For any $c \in \mathbb{T}$, $x = c \in \mathbb{T}^n$ means $x_j = c$, for all $j \in [n]$.
- $\mathbf{A} = (\mathbf{a}_{ij})$, $\mathbf{B} = (\mathbf{b}_{ij})$, with

$$\mathbf{a}_{ij} = \begin{cases} a_{ij} & \text{if } a_{ij} \geq b_{ij}, \\ -\infty & \text{otherwise} \end{cases} \qquad \mathbf{b}_{ij} = \begin{cases} b_{ij} & \text{if } a_{ij} \leq b_{ij}, \\ -\infty & \text{otherwise} \end{cases}$$

- $M = A \oplus B = \mathbf{A} \oplus \mathbf{B} = (m_{ij})$ is the *maximum matrix*.
- If $k \in [m]$ and $j, l \in [n]$, then

$$\operatorname{dif}(M; j, l)_k = \begin{cases} m_{kj} - m_{kl}, & \text{if } m_{kl} \neq -\infty, \\ \text{undetermined}, & \text{otherwise.} \end{cases}$$

  The undetermined case will never appear in the following. Notice that for fixed k, $\operatorname{dif}(M; j, l)_k$ has the *cocycle properties*:

1. $\mathrm{dif}(M; j, l)_k = -\mathrm{dif}(M; l, j)_k$,
2. $\mathrm{dif}(M; i, j)_k + \mathrm{dif}(M; j, l)_k = \mathrm{dif}(M; i, l)_k$.

- $|I|$ denotes the underlying set of an ordered pair I.

Let

$$\mathbf{A} \odot x = \mathbf{B} \odot x. \tag{2}$$

Notice that (1) is equivalent to (2) and (2) is simpler than (1) because it involves fewer real coefficients. Thus, we will assume that $A = \mathbf{A}$ and $B = \mathbf{B}$, in the following.

Remark 1. *1. $x = -\infty$ satisfies (1). This is the* trivial solution.

2. If $\mathrm{row}(A, i) > \mathrm{row}(B, i)$ or $\mathrm{row}(A, i) < \mathrm{row}(B, i)$ for some $i \in [m]$, then $x = -\infty$ is the only solution to (1).

3. If $\mathrm{row}(A, i) = \mathrm{row}(B, i)$ for some $i \in [m]$, then these two rows can be erased, so that m can be decreased to $m - 1$.

4. If $\mathrm{col}(A, j) = \mathrm{col}(B, j) = -\infty$ for some $j \in [n]$, then no restriction is imposed on x_j. Then these two columns and x_j can be erased, so that n decreases to $n - 1$.

We will assume that $\mathrm{row}(A, i) \neq \mathrm{row}(B, i)$, $\mathrm{row}(A, i) \not< \mathrm{row}(B, i)$ and $\mathrm{row}(A, i) \not> \mathrm{row}(B, i)$, for all $i \in [m]$, and $\mathrm{col}(A, j) = \mathrm{col}(B, j) = -\infty$, for no $j \in [n]$, in the following.

The sets in the next definition are denoted I, J, K, L in [2].

Definition 1. *For each $i \in [m]$, let*

1. $WA(i) = \{j : a_{ij} > b_{ij}\}$, $WB(i) = \{j : a_{ij} < b_{ij}\}$.

2. $E(i) = \{j : a_{ij} = b_{ij} \neq -\infty\}$, $F(i) = \{j : a_{ij} = b_{ij} = -\infty\}$.

3. $\mathrm{win}(i) = (WA(i) \times WB(i)) \cup (E(i) \times E(i)) \subset [n] \times [n]$. Each element of $\mathrm{win}(i)$ is called a winning pair.

For each $i \in [m]$, $WA(i) \cup WB(i) \cup E(i) \cup F(i) = [n]$ is a *disjoint* union. By our assumptions, $F(i) \neq [n]$ and $\mathrm{win}(i) \neq \emptyset$. Therefore, if system (1) has a non–trivial solution then $\mathrm{win}(i) \neq \emptyset$, for all $i \in [m]$. *The converse is not true;* some *compatibility* is required among pairs in $\mathrm{win}(i)$, for different i's, in order for (1) to have a non–trivial solution.

Notice that $\cap_{h=1}^{m} F(h) = \emptyset$, by our assumptions.

How are non–trivial solutions to (1) related to winning pairs? Let us see. Recall that $M = A \oplus B = (m_{ij})$.

Definition 2. *Consider $i \in [m]$ and $I \in \operatorname{win}(i)$. Let $x \in \mathbb{T}^n$, $y \in \mathbb{T}^m$ be any vectors satisfying $A \odot x = B \odot x = y$. We say that* the solution x to (1) arises from I *if*

$$m_{ij} + x_j \leq y_i, \tag{3}$$

*for all $j \in [n] \setminus F(i)$, with **equality for all** $j \in |I|$.*

Write $|I| = \{\underline{i}_1, \underline{i}_2\}$. Then $m_{i\underline{i}_1} + x_{\underline{i}_1} = y_i = m_{i\underline{i}_2} + x_{\underline{i}_2}$, whence we obtain the *bivariate equation*

$$x_{\underline{i}_2} = \operatorname{dif}(M; \underline{i}_1, \underline{i}_2)_i + x_{\underline{i}_1}. \tag{4}$$

Notice that $j \in F(i)$ if and only if $m_{ij} = -\infty$. Thus, if $j \in F(i)$, then the inequality (3) is obvious and, if $j \notin F(i)$, then $m_{ij} \neq -\infty$ and, working in $\mathbb{T}$, we will be able to subtract m_{ij} from the right–hand–side. Also, if $\underline{i}_1 = \underline{i}_2$, *we need not consider this equation*, since it is trivially true.

Remark 2. *Suppose $i, k \in [m]$, $i < k$, $I \in \operatorname{win}(i), K \in \operatorname{win}(k)$. Assume that the solution x arises from I and from K. Then for all $\underline{i} \in |I|$ and $\underline{k} \in |K|$*

$$\begin{aligned} m_{i\underline{i}} + x_{\underline{i}} &= y_i, \\ m_{i\underline{k}} + x_{\underline{k}} &\leq y_i, \\ m_{k\underline{k}} + x_{\underline{k}} &= y_k, \\ m_{k\underline{i}} + x_{\underline{i}} &\leq y_k. \end{aligned}$$

Adding up, $m_{i\underline{k}} + m_{k\underline{i}} + x_{\underline{i}} + x_{\underline{k}} \leq m_{i\underline{i}} + m_{k\underline{k}} + x_{\underline{i}} + x_{\underline{k}} = y_i + y_k$, whence

$$m_{i\underline{k}} + m_{k\underline{i}} \leq m_{i\underline{i}} + m_{k\underline{k}}.$$

In other words, the value of the 2×2 tropical minor of M, denoted $M(i, k; \underline{i}, \underline{k})$,

$$\begin{vmatrix} m_{i\underline{i}} & m_{i\underline{k}} \\ m_{k\underline{i}} & m_{k\underline{k}} \end{vmatrix}_{trop} = \max\{m_{i\underline{i}} + m_{k\underline{k}},\ m_{i\underline{k}} + m_{k\underline{i}}\}, \tag{5}$$

is attained at the main diagonal. *One more way to put it is*

$$\operatorname{dif}(M; \underline{i}, \underline{k})_k \leq \operatorname{dif}(M; \underline{i}, \underline{k})_i. \tag{6}$$

Definition 3. *Consider $i, k \in [m]$, $i < k$, $I \in \operatorname{win}(i), K \in \operatorname{win}(k)$. We say that K* is compatible with I *if inequality (6) holds, for all $\underline{i} \in |I|$ and all $\underline{k} \in |K|$.*

Compatibility means that if $\underline{i}$ and $\underline{k}$ are fixed, then $\operatorname{dif}(M, \underline{i}, \underline{k})$ is *decreasing* on the subscripts. Then

$$[\operatorname{dif}(M; \underline{i}, \underline{k})_k, \operatorname{dif}(M; \underline{i}, \underline{k})_i]$$

is a non–empty closed interval, denoted $\mathrm{int}(M; i, k; \underline{i}, \underline{k})$. By abuse of language, some interval might actually be a half–line, if the left end point is $-\infty$. Moreover, from the inequalities in remark 2, we get the following (at most **four** different) *interval relations*

$$x_{\underline{k}} \in \mathrm{int}(M; i, k; \underline{i}, \underline{k}) + x_{\underline{i}}, \quad \underline{i} \in |I|,\ \underline{k} \in |K|. \tag{7}$$

This is trivially true for $\underline{i} = \underline{k}$, and *we will disregard this case*.

Four tropical minors of the maximum matrix M must be checked out, in order to decide compatibility of K with I. We can *erase repeated minors*. We say that (some of) these minors are *dependent* if the fact that one of them attains its value at the main diagonal follows from the fact that the rest attain their values at the main diagonals. We will *keep track only of independent minors*. We say that the minor $M(i, k; \underline{i}, \underline{i})$ is *trivial*. *Trivial minors will be disregarded.* A minor is *tropically singular* if it attains its value at both diagonals. Of course, $M(i, k; \underline{i}, \underline{i})$ is tropically singular. Now, if $\underline{i} \neq \underline{k}$, the minor $M(i, k; \underline{i}, \underline{k})$ is tropically singular if and only if the interval relation (7) reduces to the non–trivial bivariate equation

$$x_{\underline{k}} = \mathrm{dif}(M; \underline{i}, \underline{k})_k + x_{\underline{i}} \quad (= \mathrm{dif}(M; \underline{i}, \underline{k})_i + x_{\underline{i}}). \tag{8}$$

With the former notations, if K is compatible with I, notice the following:

- If $\mathrm{card}\,|I| \cup |K| = 1$, then all four minors are identical and trivial. Then (7) reduces to the empty set.
- Suppose $\mathrm{card}\,|I| \cup |K| = 2$. If $|I| = |K|$, say $|I| = \{1, 2\}$, then two minors are trivial, and the other two minors determine a point, because $\mathrm{int}(M; i, k; 1, 2) = \mathrm{int}(M; i, k; 2, 1)$. Then (7) reduces to one bivariate equation. If $|I| \neq |K|$, then (7) reduces to just one interval relation.
- If $\mathrm{card}\,|I| \cup |K| = 3$, and if, say $|I| = \{\underline{i}_1, \underline{i}_2\}$ and $|K| = \{\underline{i}_1, \underline{k}_2\}$, then one minor is trivial, and the other three minors are dependent: then $M(i, k; \underline{i}_1, \underline{k}_2)$ and $M(i, k; \underline{i}_2, \underline{i}_1)$ attain their values at the main diagonals, and this implies that the same is true for $M(i, k; \underline{i}_2, \underline{k}_2)$, by the cocycle condition. Then (7) reduces to two interval relations. If, say $|I| = \{\underline{i}_1, \underline{i}_1\}$ and $|K| = \{\underline{k}_1, \underline{k}_2\}$, it is similar.
- If $\mathrm{card}\,|I| \cup |K| = 4$, then the four minors are tropically regular and independent. The four interval relations in (7) are meaningful.

Summing up, if $1 \leq i < k \leq m$, then the conditions $|I| = \{\underline{i}_1, \underline{i}_2\}$, $I \in \mathrm{win}(i)$, $|K| = \{\underline{k}_1, \underline{k}_2\}$, $K \in \mathrm{win}(k)$ and K compatible with I provide two equations and, at most, four interval relations (some of which may actually degenerate into bivariate equations), namely

$$x_{\underline{i}_2} = \mathrm{dif}(M; \underline{i}_1, \underline{i}_2)_i + x_{\underline{i}_1}, \tag{9}$$
$$x_{\underline{k}_2} = \mathrm{dif}(M; \underline{k}_1, \underline{k}_2)_k + x_{\underline{k}_1}, \tag{10}$$
$$x_{\underline{k}_1} \in \mathrm{int}(M; i, k; \underline{i}_1, \underline{k}_1) + x_{\underline{i}_1}, \tag{11}$$
$$x_{\underline{k}_2} \in \mathrm{int}(M; i, k; \underline{i}_1, \underline{k}_2) + x_{\underline{i}_1}, \tag{12}$$
$$x_{\underline{k}_1} \in \mathrm{int}(M; i, k; \underline{i}_2, \underline{k}_1) + x_{\underline{i}_2}, \tag{13}$$
$$x_{\underline{k}_2} \in \mathrm{int}(M; i, k; \underline{i}_2, \underline{k}_2) + x_{\underline{i}_2}. \tag{14}$$

Lemma 1. *If* $1 \leq h < l \leq m$, I_h, I_l *are winning pairs,* $|I_h| = \{i, j\}$, $|I_l| = \{i, k\}$, I_l *compatible with* I_h, *then* $D = \mathrm{dif}(M; j, i)_h + \mathrm{dif}(M; i, k)_l$ *satisfies the double inequality*

$$\mathrm{dif}(M; j, k)_l \leq D \leq \mathrm{dif}(M; j, k)_h.$$

Proof. We have

$$\mathrm{dif}(M; j, i)_l + \mathrm{dif}(M; i, k)_l \leq D \leq \mathrm{dif}(M; j, i)_h + \mathrm{dif}(M; i, k)_h,$$

by the decreasing property on subscripts, so that the double inequality holds, by the cocycle condition. □

Notation: $D = D(M; j, i, k)_{h,l}$. Warning: this notation is complicated, but we will use it very little.

Lemma 2 (Reduction lemma)**.** *If* $1 \leq i < k \leq m$, $|I| = \{\underline{i}_1, \underline{i}_2\}$, $I \in \mathrm{win}(i)$, $|K| = \{\underline{k}_1, \underline{k}_2\}$, $K \in \mathrm{win}(k)$, *then* K *is compatible with* I *if and only if*

$$x_{\underline{i}_2} = \mathrm{dif}(M; \underline{i}_1, \underline{i}_2)_i + x_{\underline{i}_1}, \tag{15}$$
$$x_{\underline{k}_2} = \mathrm{dif}(M; \underline{k}_1, \underline{k}_2)_k + x_{\underline{k}_1}, \tag{16}$$
$$x_{\underline{k}_1} \in [D(M; \underline{i}_1, \underline{i}_2, \underline{k}_1)_{ik}, D(M; \underline{i}_1, \underline{k}_2, \underline{k}_1)_{ik}] + x_{\underline{i}_1}, \tag{17}$$
$$x_{\underline{k}_2} \in [D(M; \underline{i}_1, \underline{i}_2, \underline{k}_2)_{ik}, D(M; \underline{i}_1, \underline{k}_1, \underline{k}_2)_{ik}] + x_{\underline{i}_1}. \tag{18}$$

Proof. Combine (9), (10), (12) and (13) to obtain (17). Similar for (18). This proves one implication. Notice that

$$[D(M; \underline{i}_1, \underline{i}_2, \underline{k}_1)_{ik}, D(M; \underline{i}_1, \underline{k}_2, \underline{k}_1)_{ik}] \subseteq \mathrm{int}(M; i, k; \underline{i}_1, \underline{k}_1),$$
$$[D(M; \underline{i}_1, \underline{i}_2, \underline{k}_2)_{ik}, D(M; \underline{i}_1, \underline{k}_1, \underline{k}_2)_{ik}] \subseteq \mathrm{int}(M; i, k; \underline{i}_1, \underline{k}_2),$$

by lemma 1, proving that (15)–(18) imply (9)–(12). Moreover, (15) and (17) imply (13). Similar for (14). □

Corollary 1 (Reduction corollary)**.** *Assume that* $1 \le i < k \le m$*,* $|I| = \{\underline{i}_1, \underline{i}_2\}$*,* $I \in \mathrm{win}(i)$*,* $|K| = \{\underline{k}_1, \underline{k}_2\}$*,* $K \in \mathrm{win}(k)$ *and* K *is compatible with* I*.*

- *If* $\mathrm{card}\,|I| \cup |K| = 1$*, then the relations (15)–(18) reduce to one bivariate equation.*
- *If* $\mathrm{card}\,|I| \cup |K| = 2$*, and* $|I| = |K|$*, then the relations (15)–(18) reduce to two bivariate equations.*
- *If* $\mathrm{card}\,|I| \cup |K| = 2$*,* $|I| \neq |K|$ *and* $\mathrm{card}\,|I| = \mathrm{card}\,|K| = 1$*, then the relations (15)–(18) reduce to one interval relation.*
- *If* $\mathrm{card}\,|I| \cup |K| = 2$*,* $|I| \neq |K|$ *and* $\mathrm{card}\,|I| \neq 1$ *or* $\mathrm{card}\,|K| \neq 1$*, then the relations (15)–(18) reduce to one bivariate equation and one interval relation.*
- *If* $\mathrm{card}\,|I| \cup |K| = 3$ *and* $\mathrm{card}\,|I| = 1$ *or* $\mathrm{card}\,|K| = 1$*, then the relations (15)–(18) reduce to one bivariate equation and two interval relations.*
- *If* $\mathrm{card}\,|I| \cup |K| = 3$ *and* $\mathrm{card}\,|I| \neq 1 \neq \mathrm{card}\,|K|$*, then the relations (15)–(18) reduce to two bivariate equations and two interval relations.* □

In addition to the expressions in the former corollary, a solution x to (1) arising from I and from K satisfies the following *half–line relation*:

$$x_j \le (y_i - m_{ij}) \oplus' (y_k - m_{kj}), \tag{19}$$

for all $j \notin |I| \cup |K| \cup F(i) \cup F(k)$. This follows from remark 2.

Definition 4. *Let* $I = (I_1, \ldots, I_m)$ *be an* m*–tuple with* $I_h \in \mathrm{win}(h)$*, for all* $h \in [m]$*. We say that* I *is a* win sequence *if* I_h *is compatible with* I_i*, for all* $1 \le i < h \le m$*.*

For a win sequence $I = (I_1, \ldots, I_m)$, write

$$|I| = \bigcup_{h=1}^{m} |I_h| \,.$$

Given $i, j \in |I|$, write $i \sim j$ if there exist $k, l \in [m]$ such that $i \in |I_k|$, $j \in |I_l|$ and $|I_k| \cap |I_l| \neq \emptyset$. Closing up under transitivity, we obtain an equivalence relation on $|I|$.

Definition 5. *Let* $I = (I_1, \ldots, I_m)$ *be a win sequence.*

1. *An index* $i \in [n]$ *is* free *in* I *if* $i \notin |I|$*.*
2. *An equivalence class for the relation above is called a* cycle *in* I*.*

Consider a win sequence $I = (I_1, \ldots, I_m)$. Let c denote the number of cycles in I. We have $1 \leq c \leq \operatorname{card}|I| \leq \min\{2m, n\}$. After relabeling columns, we can suppose that the cycles in I are

$$\begin{aligned} C_1 &= [k_1], \\ C_2 &= [k_2] \setminus [k_1], \\ &\ldots \\ C_c &= [k_c] \setminus [k_{c-1}], \end{aligned}$$

for some $1 \leq c \leq \operatorname{card}|I|$, some $1 \leq k_1 < \cdots < k_c \leq n$ and that $[n] \setminus [k_c]$ are the free indices. Write

$$F = \bigcup_{h=1}^{m} F(h).$$

Theorem 1. *Each win sequence $I = (I_1, \ldots, I_m)$ provides a convex set, sol_I, of solutions to the system (1). The set sol_I consists of all the solutions x arising from I_h, for all $h \in [m]$. Moreover,*

$$\dim(\operatorname{sol}_I) \leq n - \operatorname{card}|I| + c.$$

All solutions to (1) are obtained this way.

Proof. The last statement follows from remark 2. Convexity follows from the fact that sol_I will be the solution set of a system of classical linear inequalities. In order to prove the rest, we reason by induction on the number m of rows of A.

1. Suppose $m = 1$ and consider a winning pair $I \in \operatorname{win}(1)$.

   If $\operatorname{card}|I| = 2$, say $I = (1, 2)$, then a solution $x \in \operatorname{sol}_I$ satisfies

   $$\begin{aligned} &m_{11} + x_1 = y_1, \\ &m_{12} + x_2 = y_1, \\ &m_{1j} + x_j \leq y_1, \ j \geq 3, \ j \notin F(1). \end{aligned}$$

   Each time we fix $x_1 \in \mathbb{T}$, we obtain a fixed value for y_1 and the bivariate equation

   $$x_2 = \operatorname{dif}(M; 1, 2)_1 + x_1.$$

   Moreover, for each $j \geq 3, \ j \notin F(1)$

   $$x_j \leq y_1 - m_{1j} = \operatorname{dif}(M; 1, j)_1 + x_1.$$

   The dimension of sol_I is $n - 1$. Here, $c = 1$, $C_1 = [2]$, $[n] \setminus [2]$ are the free indices, $S_1 = \{x_1 - x_2 + \operatorname{dif}(M; 1, 2)_1 = 0\}$ and T_1 is empty (see notations in p. 11 below).

If card $|I|=1$, say $I=(1,1)$, then a solution $x \in \mathrm{sol}_I$ satisfies

$$\begin{aligned} m_{11}+x_1 &= y_1, \\ m_{1j}+x_j &\leq y_1, \; j \geq 2, \; j \notin F(1). \end{aligned}$$

Each time we fix $x_1 \in \mathbb{T}$, we obtain a fixed value for y_1 and, for each $j \geq 2, \; j \notin F(1)$

$$x_j \leq y_1 - m_{1j} = \mathrm{dif}(M;1,j)_1 + x_1.$$

In this case, sol_I has dimension n. Here, $c=1$, $C_1=[1]$, $[n] \setminus [1]$ free, S_1 and T_1 are empty.

2. Suppose $m \geq 2$ and consider a win sequence $I=(I_1,\ldots,I_m)$. Write $I'=(I_1,\ldots,I_{m-1})$. We will add a prime symbol to c, S_i etc., to denote that these correspond to I'. By induction hypothesis, the theorem is true for I'. In particular,

$$\dim(\mathrm{sol}_{I'}) \leq n - \mathrm{card}\,|I'| + c'.$$

We have the following:

(a) For each $i \in [c']$, a system S'_i of, at most, $k'_i - k'_{i-1} - 1$ *bivariate equations, the coefficients of the variables being* 1 *and* -1; it is a collection of equations as in (4). For instance, for $i=1$, S'_1 will be

$$x_{\underline{i}_1} - x_{\underline{i}_2} + \mathrm{dif}(M;\underline{i}_1,\underline{i}_2)_l = 0, \quad |I_l| = \{\underline{i}_1,\underline{i}_2\} \subseteq [k'_1], \; \underline{i}_1 \neq \underline{i}_2. \quad (20)$$

S'_i might be incompatible, for some i.

(b) For each $i \in [c']$, a system T'_i of, at most, $m-3$ *interval relations*; it is the collection of relations as in (17) with $\underline{i}_1 \in C'_i$ fixed and $\underline{k}_1$ running in C'_i, with the condition that $\underline{i}_1, \underline{k}_1$ do not form a winning pair. The system T'_i might be incompatible, for some i. Moreover, T'_i can be converted into a system of, at most, $2(m-3)$ bivariate linear inequalities, the coefficients of the variables being 1 and -1.

(c) For each index j free in I' and $j \notin F'$, we have a *half–line relation*:

$$x_j \leq {\bigoplus_{h=1}^{m-1}}' (y_h - m_{hj}).$$

Any member of $\mathrm{sol}_{I'}$ satisfies the three items above. Of course, $\mathrm{sol}_I \subseteq \mathrm{sol}_{I'}$. Several cases arise.

(a) If $|I|=|I'|$. Then $c=c'$.

If, say $|I_m| = \{1, 2\}$, then a solution $x \in \mathrm{sol}_{I'}$ belongs to sol_I only if x satisfies

$$\begin{aligned}
&m_{m1} + x_1 = y_m, \\
&m_{m2} + x_2 = y_m, \\
&m_{mj} + x_j \leq y_m, \ j \notin [2] \cup F(m).
\end{aligned}$$

A new equation involving x_1, x_2 is added to S'_1 to form S_1. New interval relations are added to T'_1 to form T_1 and new half–line relations on x_j are added, for each $j \notin [2] \cup F(m)$. This can only make de dimension decrease.

If, say $I_m = (1, 1)$, then a solution $x \in \mathrm{sol}_{I'}$ belongs to sol_I only if x satisfies

$$m_{mj} + x_j \leq y_m, \ 1 \neq j \notin F(m).$$

A new half–line relation on x_j is introduced, for each $1 \neq j \notin F(m)$. Here $\dim(\mathrm{sol}_I) = \dim(\mathrm{sol}_{I'})$.

(b) If $\operatorname{card}|I| = \operatorname{card}|I'| + 1$.

Suppose, in addition, that $c = c'$. If, say $|I_m| = \{1, n\}$, with $1 \neq n$ free index in I', then a solution $x \in \mathrm{sol}_{I'}$ belongs to sol_I only if x satisfies

$$\begin{aligned}
&m_{m1} + x_1 = y_m, \\
&m_{mn} + x_n = y_m, \\
&m_{mj} + x_j \leq y_m, \ j \neq 1, n, \ j \notin F(m).
\end{aligned}$$

Here, $C'_1 \cup \{n\} = C_1$. One equation, involving x_1, x_n is added to S'_1 to form S_1. New interval relations are added to T'_1 to form T_1. A new half–line relation on x_j is introduced, for $j \neq 1, n, j \notin F(m)$.

Suppose now that $c = c' + 1$. If, say $I_m = (n, n)$, with n free index in I', then a solution $x \in \mathrm{sol}_{I'}$ belongs to sol_I only if x satisfies

$$\begin{aligned}
&m_{nn} + x_n = y_n, \\
&m_{nj} + x_j \leq y_n, \ j \neq n, \ j \notin F(n).
\end{aligned}$$

The new cycle is $C_c = \{n\}$ and S_c, T_c are empty. A new half–line relation on x_j is introduced, for $j \neq n$, $j \notin F(m)$. Here $\dim(\mathrm{sol}_I) = \dim(\mathrm{sol}_{I'})$.

(c) If $\operatorname{card}|I| = \operatorname{card}|I'| + 2$, then $c = c' + 1$. If, say $|I_m| = \{n-1, n\}$, with $n-1, n$ free indices in I', then a solution $x \in \mathrm{sol}_{I'}$ belongs to sol_I

only if x satisfies

$$\begin{aligned} m_{m,n-1} + x_{n-1} &= y_m, \\ m_{mn} + x_n &= y_m, \\ m_{mj} + x_j &\leq y_m, \; j \neq n-1, n, \; j \notin F(m). \end{aligned}$$

The new cycle is $C_c = \{n-1, n\}$, S_c contains just one equation (involving x_{n-1} and x_n) and T_c is empty. A new half–line relation on x_j is introduced, for $j \neq n-1, n, j \notin F(m)$.

We see that, in all the cases, $\dim(\mathrm{sol}_I) \leq \dim(\mathrm{sol}_{I'})$ and the dimension formula holds for I.

□

3 The algorithm

If no win sequences exist, then the only solution to the system (1) is trivial. The number p of win sequences is no bigger that r^m, where $r = \max\{\lceil \frac{n}{2} \rceil^2, n\}$. Even if some win sequence does exist, it may happen that the only solution to the system (1) is trivial.

More notations:

- If $C \in \mathcal{M}_{m\times(n+1)}(\mathbb{R})$, let $C' \in \mathcal{M}_{m\times n}(\mathbb{R})$ be obtained from C by deleting the last column.
- A *plus–half–line* means an inequality of the form $x_j - x_l + a \leq 0$, for some $1 \leq j < l \leq n$ and $a \in \mathbb{R}$; it will be encoded by the triple (j, l, a).
- A *minus–half–line* means an inequality of the form $-x_j + x_l + a \leq 0$, for some $1 \leq j < l \leq n$ and $a \in \mathbb{R}$; it will also be encoded by the triple (j, l, a).
- An *interval* means two inequalities $x_j + a \leq x_l \leq x_j + b$, for some $1 \leq j < l \leq n$ and $a < b \in \mathbb{R}$; it will be encoded by the tuple (j, l, a, b).

Definition 6. *Let $S = (s_{ij}) \in \mathcal{M}_{m\times(n+1)}(\mathbb{R})$.*

1. *The matrix S is* special *if each row of S' is a permutation of the n–vector $(1, -1, 0, \ldots, 0)$.*
2. *The special matrix S is* super–special *if the first non–zero entry of each row is 1.*
3. *The special matrix S is* sub–special *if*

(a) *all rows in S are different and different from rows in $-S$,*

(b) *all rows in S' are different,*

(c) *if* $\mathrm{row}(S', i) = -\mathrm{row}(S', k)$, *for some $i < k$, then $k = i + 1$,*

$$s_{i,n+1} < -s_{i+1,n+1}$$

and $\mathrm{row}(S', i) = (\overbrace{0, \ldots, 0}^{j-1}, 1, \overbrace{0, \ldots, 0}^{l-1}, -1, 0, \ldots, 0)$, *for some $j, l \in [n]$,*

(d) *if* $\mathrm{row}(S', i) \neq -\mathrm{row}(S', i+1)$, *for some i, then* $\min\{j : s_{ij} \neq 0\} \leq \min\{j : s_{i+1,j} \neq 0\}$.

Notice that condition (3c) in definition 6 corresponds to the pair of inequalities

$$x_j - x_l + s_{i,n+1} \leq 0,$$
$$-x_j + x_l + s_{i+1,n+1} \leq 0,$$

meaning that $x_l \in [s_{i,n+1}, -s_{i+1,n+1}] + x_j$ is an interval relation.

Example: the following matrix S is sub–special but not super–special

$$\begin{pmatrix} 1 & 0 & -1 & 0 & 3 \\ -1 & 0 & 1 & 0 & -8 \\ 0 & -1 & 1 & 0 & -4 \\ 0 & 0 & 1 & -1 & 0 \end{pmatrix}.$$

The solutions to the classical linear system of inequalities $S[x, 1] \leq 0$ are encoded in the plus–half–line $(3, 4, 0)$, the minus–half–line $(2, 3, -4)$ and the interval $(1, 3, 3, 8)$.

An algorithm must find first all win sequences. Then, for each win sequence I, the algorithm must find matrices $C_I, D_I \in \mathcal{M}_{m\times(n+1)}(\mathbb{R})$ and must solve

$$C_I[x, 1] = 0, \qquad D_I[x, 1] \leq 0. \tag{21}$$

These are systems of linear equations and inequalities, where classical matrix operations are used. The matrices C_I and D_I are special. The set of solutions to (21), denoted $\mathrm{sol}_I = \mathrm{sol}_{C_I} \cap \mathrm{sol}_{D_I}$, is convex, possibly empty. The non–trivial solutions to (1) is the union of sol_I, as I runs over all win sequences.

ALGORITHM

- STEP 1: compute the matrices $\mathbf{A}, \mathbf{B}$ and M. Replace A and B by $\mathbf{A}$ and $\mathbf{B}$.

- STEP 2: Compute all winning pairs, for all $i \in [m]$. Store them in a tridimensional array W (r rows, 2 columns, m pages). In page i we store all members of $\mathrm{win}(i)$. Blanks are padded with zeros.
- STEP 3: Compute all win sequences. Store them in a tridimensional array WS (m rows, 2 columns, p pages), with $0 \le p \le r^m$. No entry of WS is zero. If WS is empty, then the only solution to system (1) is trivial, RETURN.
- FOR each win sequence I
  - STEP 4: Compute the special matrices C_I and D_I.
  - STEP 5: By elementary row transformations, work on D_I to obtain special matrices E_I, N_I such that

    $$D_I[x, 1] \le 0 \;\Leftrightarrow\; E_I[x, 1] = 0 \text{ and } N_I[x, 1] \le 0$$

    and

    $$2 \operatorname{card} \operatorname{rows}(E_I) + \operatorname{card} \operatorname{rows}(N_I) \le \operatorname{card} \operatorname{rows}(D_I).$$

    Either matrix E_I or N_I could be empty. By elementary row transformations, work on N_I to make it sub–special. Solve the classical linear system of inequalities $N_I[x, 1] \le 0$, by backward substitution. The solution set, denoted sol_{N_I}, is expressed in terms of half–lines (plus and minus) and intervals. If sol_{N_I} is empty, go to work with the next win sequence.
  - STEP 6: Concatenate the matrices C_I and E_I into a matrix, which we can denote again by C_I. By elementary row transformations, work on C_I to make it super–special and upper triangular. Solve the classical linear system $C_I[x, 1] = 0$, by backward substitution. The solution set is denoted sol_{C_I}; if it is empty, go to work with the next win sequence.
  - STEP 7: Substitute sol_{C_I} into sol_{N_I} to obtain a new system of linear inequalities, which we can denote again by $D_I[x, 1] \le 0$.
  - STEP 8: By elementary row transformations, work on D_I to obtain special matrices E_I, N_I such that

    $$D_I[x, 1] \le 0 \;\Leftrightarrow\; E_I[x, 1] = 0 \text{ and } N_I[x, 1] \le 0$$

    and

    $$2 \operatorname{card} \operatorname{rows}(E_I) + \operatorname{card} \operatorname{rows}(N_I) \le \operatorname{card} \operatorname{rows}(D_I).$$

    Either matrix E_I or N_I could be empty. By elementary row transformations, work on N_I to make it sub–special. Solve the classical linear system of inequalities $N_I[x, 1] \le 0$, by backward substitution. The solution set, denoted sol_{N_I}, is expressed in terms of half–lines (plus and minus) and intervals. If sol_{N_I} is empty, go to work with the next win sequence. Otherwise, if E_I is empty, then

$$\mathrm{sol}_I = \mathrm{sol}_{C_I} \cap \mathrm{sol}_{N_I}\,.$$

Otherwise, GOTO STEP 6.

- ENDFOR

  All the solutions to (1) are $\bigcup_{I \in WS} \mathrm{sol}_I$.

We have programmed the former algorithm to solve system (1). Working on $\mathbb{Q} \cup \{-\infty\}$, let us compute the complexity of it. The arithmetic complexity counts the number of arithmetic operations ($+, -, \max, \min, <, =$ and $>$, in our situation) in the worst possible case.

Our programme is divided into two parts. In the first part, we determine all the win sequences. Say we get p win sequences. The arithmetical complexity of this part is

$$O(m^2 n^3 p).$$

In the second part, we compute the matrices C_I, D_I and all the solutions (if any), for each win sequence I. The arithmetic complexity of the second part is

$$O(m(m^2 + n)p).$$

Since the maximum number of winning pairs is $r = \max\{\lceil \frac{n}{2} \rceil^2, n\}$, then $p \leq r^m$, where r is $O(n^2)$. This gives an exponential arithmetical complexity! But, let us take a closer look. Clearly, the bigger n, the more winning pairs we have, for each $i \in [m]$. On the other hand, *the bigger m, the fewer win sequences we have, in probability*. Indeed, given winning pairs $I \in \mathrm{win}(i)$, $K \in \mathrm{win}(k)$ with $1 \leq i < k \leq m$, let us define the probability of K being compatible with I as $1/2$ (since this a yes/no event). Thus, given any sequence of pairs $I = (I_1, \ldots, I_m)$, the probability of I being a win sequence is, roughly,

$$\frac{1}{2^{\binom{m}{2}}} \sim \frac{1}{2^{m^2}}.$$

This proves that if m is big, then we expect p small. In particular, the worst case (p big) is unlikely to happen. With this in mind, an average complexity for the first part is

$$O(m^2 n^{3+2m}/2^{m^2}) = O(m^2 2^{(3+2m)\log_2 n - m^2})$$

and it will be, at most polynomial, if $\log_2 n \leq \frac{m^2}{3+2m}$. For the second part we get two terms:

$$O(m^3 n^{2m}/2^{m^2}) = O(m^3 2^{2m \log_2 n - m^2})$$

and it will be, at most polynomial, if $\log_2 n \leq \frac{m}{2}$, and

$$O(mn^{1+2m}/2^{m^2}) = O(m2^{(1+2m)\log_2 n - m^2})$$

and it will be, at most polynomial, if $\log_2 n \leq \frac{m^2}{1+2m}$.

Notice that we could define a much finer probability (based on each of the four tropical minor attaining (or not) its value on the main diagonal; see expression 5) so that the probability of I being a win sequence would be smaller.

4 Some examples

Example 1. *Given*

$$A = \begin{bmatrix} 1 & 3 & -\infty \\ 5 & 0 & -\infty \\ -\infty & 3 & -\infty \end{bmatrix}, \quad B = \begin{bmatrix} -\infty & -\infty & 3 \\ 5 & 0 & 2 \\ 3 & -\infty & 2 \end{bmatrix},$$

we get

$$M = \begin{bmatrix} 1 & 3 & 3 \\ 5 & 0 & 2 \\ 3 & 3 & 2 \end{bmatrix}.$$

The only win sequence is $I = ((2,3),(1,1),(2,1))$.

The solutions arising from I are

$$x = \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix}.$$

Example 2. *Given*

$$A = \begin{bmatrix} 3 & 7 & -1 & -\infty \\ 6 & 7 & -\infty & -\infty \\ 1 & 0 & 1 & -\infty \end{bmatrix}, \quad B = \begin{bmatrix} -\infty & -\infty & -\infty & 8 \\ -\infty & -\infty & 5 & 1 \\ 1 & 0 & 1 & 2 \end{bmatrix},$$

we get

$$M = \begin{bmatrix} 3 & 7 & -1 & 8 \\ 6 & 7 & 5 & 1 \\ 1 & 0 & 1 & 2 \end{bmatrix}.$$

The win sequences are $I = ((1,4),(1,3),(3,3))$ *and* $J = ((2,4),(1,3),(3,3))$.

The solutions arising from I are

$$x = \begin{bmatrix} x_4 + 5 \\ x_2 \\ x_4 + 6 \\ x_4 \end{bmatrix}, \quad \text{s.t. } x_2 - x_4 - 1 \leq 0.$$

The solutions arising from J are

$$x = \begin{bmatrix} x_3 - 1 \\ x_4 + 1 \\ x_3 \\ x_4 \end{bmatrix}, \quad \text{s.t. } x_3 - 6 \leq x_4 \leq x_3 - 3.$$

Example 3. *(From [9]) Given*

$$A = \begin{bmatrix} -\infty & -\infty & -\infty & 0 & 4 & 2 & 6 \\ -\infty & 5 & 6 & -\infty & -\infty & -\infty & 2 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 1 & 5 & -\infty & -\infty & -\infty & -\infty \\ 3 & -\infty & -\infty & 0 & 2 & 4 & -\infty \end{bmatrix},$$

we get

$$M = \begin{bmatrix} 0 & 1 & 5 & 0 & 4 & 2 & 6 \\ 3 & 5 & 6 & 0 & 2 & 4 & 2 \end{bmatrix}.$$

The win sequences are $I_1 = ((4,1),(2,1))$, $I_2 = ((4,3),(2,1))$, $I_3 = ((5,1),(2,1))$, $I_4 = ((5,3),(2,1))$, $I_5 = ((6,1),(2,1))$, $I_6 = ((6,3),(2,1))$, $I_7 = ((7,1),(2,1))$ and $I_8 = ((7,3),(2,1))$.

The solutions arising from I_1 are

$$x = \begin{bmatrix} x_4 \\ x_4 - 2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix}, \quad \text{s.t.} \quad \begin{aligned} x_3 - x_4 + 5 &\leq 0, \\ -x_4 + x_5 + 4 &\leq 0, \\ -x_4 + x_6 + 2 &\leq 0, \\ -x_4 + x_7 + 6 &\leq 0. \end{aligned}$$

The solutions arising from I_2 are

$$x = \begin{bmatrix} x_2 + 2 \\ x_2 \\ x_4 - 5 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix}, \quad \text{s.t.} \quad \begin{aligned} x_2 + 2 \leq x_4 &\leq x_2 + 4, \\ -x_2 + x_5 - 3 &\leq 0, \\ -x_2 + x_6 - 1 &\leq 0, \\ -x_2 + x_7 - 3 &\leq 0, \\ -x_4 + x_5 + 4 &\leq 0, \\ -x_4 + x_6 + 2 &\leq 0, \\ -x_4 + x_7 + 6 &\leq 0. \end{aligned}$$

The solutions arising from I_3 are

$$x = \begin{bmatrix} x_5+4 \\ x_5+2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix}, \quad \textit{s.t.} \quad \begin{aligned} x_3 - x_5 + 1 &\le 0, \\ x_4 - x_5 - 4 &\le 0, \\ -x_5 + x_6 - 2 &\le 0, \\ -x_5 + x_7 + 2 &\le 0. \end{aligned}$$

The solutions arising from I_4 are

$$x = \begin{bmatrix} x_2+2 \\ x_2 \\ x_5-1 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix}, \quad \textit{s.t.} \quad \begin{aligned} x_2 - 2 \le x_5 &\le x_2, \\ x_4 - x_5 - 4 &\le 0, \\ -x_2 + x_4 - 5 &\le 0, \\ -x_2 + x_6 - 1 &\le 0, \\ -x_2 + x_7 - 3 &\le 0, \\ -x_5 + x_6 - 2 &\le 0, \\ -x_5 + x_7 + 2 &\le 0. \end{aligned}$$

The solutions arising from I_5 are

$$x = \begin{bmatrix} x_6+2 \\ x_6 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix}, \quad \textit{s.t.} \quad \begin{aligned} x_3 - x_6 + 3 &\le 0, \\ x_4 - x_6 - 2 &\le 0, \\ x_5 - x_6 + 2 &\le 0, \\ -x_6 + x_7 + 4 &\le 0. \end{aligned}$$

The solutions arising from I_6 are

$$x = \begin{bmatrix} x_2+2 \\ x_2 \\ x_6-3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix}, \quad \textit{s.t.} \quad \begin{aligned} x_2 \le x_6 &\le x_2 + 1, \\ x_4 - x_6 - 2 &\le 0, \\ x_5 - x_6 + 2 &\le 0, \\ -x_2 + x_4 - 5 &\le 0, \\ -x_2 + x_5 - 3 &\le 0, \\ -x_2 + x_7 - 3 &\le 0, \\ -x_6 + x_7 + 4 &\le 0. \end{aligned}$$

The solutions arising from I_7 are

$$x = \begin{bmatrix} x_7+6 \\ x_7+4 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix}, \quad \textit{s.t.} \quad \begin{array}{r} x_3 - x_7 - 1 \leq 0, \\ x_4 - x_7 - 6 \leq 0, \\ x_5 - x_7 - 2 \leq 0, \\ x_6 - x_7 - 4 \leq 0. \end{array}$$

The solutions arising from I_8 are

$$x = \begin{bmatrix} x_2+2 \\ x_2 \\ x_7+1 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix}, \quad \textit{s.t.} \quad \begin{array}{r} x_2 - 4 \leq x_7 \leq x_2 - 2, \\ x_4 - x_7 - 6 \leq 0, \\ x_5 - x_7 - 2 \leq 0, \\ x_6 - x_7 - 4 \leq 0, \\ -x_2 + x_4 - 5 \leq 0, \\ -x_2 + x_5 - 3 \leq 0, \\ -x_2 + x_6 - 1 \leq 0. \end{array}$$

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