Asymptotic expansion of multivariate conservative linear operators

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Abstract

In this paper we study the asymptotic expansion of the partial derivatives of a sequence of linear conservative operators. We find sufficient conditions on the sequence that guarantee that such an expansion can be derived from the asymptotic formula for the non differentiated sequence. Our results are valid even if the operators of the sequence present difficult conservative properties different from the classical preservation of the usual convexities. We use our theorems to obtain the complete asymptotic expansions and Voronovskaja formulae for the partial derivatives of multivariate versions of the Meyer-König and Zeller operators and the Bleimann, Butzer and Hahn operators.

Keywords: Asymptotic expansion; Multivariate simultaneous approximation; Conservative linear operator; Meyer-König and Zeller operator; Bleimann-Butzer and Hahn operator

1 Introduction

Given $m \in \mathbb{N} = \{1, 2, \ldots\}$, $H \subset \mathbb{R}^m$ and the linear subspaces $W_1, W_2 \subseteq \mathbb{R}^H$, a classical approach to the problem of approximating a function $f \in W_1$ consists in considering a sequence of linear positive operators, $\{L_n : W_1 \to W_2\}_{n \in \mathbb{N}}$, in order to approximate the function $f$ by means of the sequence $\{L_nf\}_{n \in \mathbb{N}}$. In the description of the way in which such convergence takes place, the asymptotic formulae for the sequence play an important role. For $r \in \mathbb{N}$, given $f \in W_1$ and $x \in H$, the expression

$$L_nf(x) = f(x) + \sum_{i=1}^{r} n^{-i} a_i(f, r, x) + o(n^{-r})$$

is called an asymptotic expansion for $\{L_n\}_{n \in \mathbb{N}}$ of order $r$. Indeed, the fact that $\lim_{n \to \infty} L_nf(x) = f(x)$ is equivalent to $L_nf(x) = f(x) + o(1)$ (i. e. an asymptotic expression of order 0). On the
other hand, the well known Voronovskaja type formulae for a sequence of operators, usually written in the form
\[ \lim_{n \to \infty} n (L_n f(x) - f(x)) = K(f, x), \]
are actually asymptotic expansions of order 1 since they can be rewritten as \( L_n f(x) = f(x) + \frac{1}{n} K(f, x) + o(n^{-1}) \).

Many important examples of sequences of linear operators (the Bernstein operators on \([0, 1]\) or on the simplex, the Baskakov operators, the Meyer-König and Zeller operators, etc.) approximate not only the function \( f \in W_1 \) but also its derivatives, i.e. we will have \( \{D^k L_n f\} \to D^k f \). In this case, we can consider as well asymptotic expansions for the sequence of the derivatives or partial derivatives in the form
\[ D^k L_n f(x) = \sum_{i=0}^{r} n^{-i} a_i(f, k, r, n) + o(n^{-r}). \]

The study of the derivatives of a linear operator is strongly related to its conservative properties. When it is preserved a given convexity it will be possible to analyze several aspects of the corresponding derivative. If the operator does not conserve the classical convexities, it is necessary to consider more difficult shape preserving properties and the study of the derivatives or partial derivatives is not always possible. This dependence with respect to the conservative properties also must be taken into account to determine the asymptotic behaviour of the derivatives of a sequence of linear operators.

In the univariate case, the problem has been investigated by Abel for several sequences of classical operators. He computes the complete asymptotic formula for the first derivative of the Kantorovich operators [2], for all the derivatives of the Jakimovski-Leviatan operators [5] and for the Durrmeyer operators [4]. Anyway, Abel carried out his calculations in a particular way for each sequence. Since the works of Sikkema [16], we can find in the literature general approximation theorems that make possible to obtain the asymptotic formula for an arbitrary sequence of linear positive operators. However, there was no result for the asymptotic study of the derivatives until the paper of López-Moreno et al. [14]. Furthermore, the operators considered by Abel present good conservative properties, all of them preserve all the classical convexities. When we work with non so well-behaved operators, as it happens in the case of the Bleimann, Butzer and Hahn operators or the Meyer-König and Zeller operators we need to handle with more general shape preserving properties like those defined in [14].

The multivariate case has been recently tackled in several papers. Lai [13], Waltz [17] and Abel [6] study the complete asymptotic expansion of the Bernstein operators on the simplex and Abel [3] obtains the asymptotic formula for the multivariate version of the Bleimann, Butzer and Hahn operators defined by Adell, de la Cal and Miguel [8]. However, they do not analyze the problem of the computation of the asymptotic expression of the partial derivatives of the considered operators.

In this paper we prove general results that allow to calculate the asymptotic formulae of the partial derivatives of sequences of multivariate linear operators even if such operators do not present classical shape preserving properties. Instead of it, we will consider convexities relative to a test function suitably chosen for each sequence of operators. Indeed, we will analyze the
operators in terms of generalized test functions instead of the usual polynomial type ones. Later, we apply our results to a generalized version of the already mentioned operators of Bleimann, Butzer and Hahn on the simplex and also to certain multivariate modification of the Meyer-König and Zeller operators. We explicitly obtain the complete asymptotic formulae for their partial derivatives. Furthermore, the technics that we use permit us to obtain more compact forms for the already known formulae for the Bleimann, Butzer and Hahn operators. Meanwhile, we also prove new expressions for these operators in terms of certain forward differences.

2 Basic notation

We will respectively denote by $|a|$ and $a$ the integer part and absolute value of $a \in \mathbb{R}$. Recall the definition of the Stirling numbers of first and second kind respectively denoted by $S^j_i$ and $\sigma^j_i$ and given by the identities

$$x^j = \sum_{i=1}^{j} S^j_i x^i \quad \text{and} \quad x^j = \sum_{i=1}^{j} \sigma^j_i x^i,$$

where $x^j = x(x-1) \ldots (x-j+1)$ is the falling factorial, being $x^0 = 1$.

Throughout the paper we will do an extensive use of the following vectorial notation: Fix $m \in \mathbb{N}$ and take $\alpha, \beta \in \mathbb{R}^m$ and $a \in \mathbb{R}$. The $i$-th component of $\alpha$ is denoted by $\alpha_i$ so that $\alpha = (\alpha_1, \ldots, \alpha_m)$. As usual, we set, whenever they make sense, $|\alpha| = \alpha_1 + \ldots + \alpha_m$, $\alpha! = \alpha_1! \cdots \alpha_m!$, $\alpha^\beta = \alpha_1^\beta_1 \cdots \alpha_m^\beta_m$, $\alpha^a = (\alpha_1^a, \ldots, \alpha_m^a)$, $a^\alpha = a^{|\alpha|}$ and $\frac{a}{\beta} = \left(\frac{\alpha_1}{\beta_1}, \ldots, \frac{\alpha_m}{\beta_m}\right)$. Notice that the bold face symbol $\lfloor \cdot \rfloor$ stands for the absolute value while $|\cdot|$ denotes the sum of the componentes of a vector. We also denote $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, $i = 1, \ldots, m$, $\mathbb{I} = (1, \ldots, 1)$ and $\mathbf{0} = (0, \ldots, 0)$, all of them in $\mathbb{R}^m$. We will say that $\alpha \leq \beta$ whenever $\alpha_i \leq \beta_i$ for all $i = 1, \ldots, m$, and then, given $\underline{\alpha}, \underline{\beta} \in \mathbb{R}^m$, we write

$$\sum_{i=\underline{\alpha}}^{\beta} A_i = \sum_{i \in \mathbb{Z}^m, \underline{\alpha} \leq i \leq \beta} A_i, \quad \sum_{i=\underline{\alpha}}^{\infty} A_i = \sum_{i \in \mathbb{Z}^m, \alpha \leq i} A_i \quad \text{and} \quad \sum_{i=\underline{\alpha}}^{\beta} A_i = \sum_{i \in \mathbb{Z}^m, \underline{\alpha} \leq i \leq \beta} A_i.$$

By $[\alpha, \beta]$ we denote the set

$$[\alpha, \beta] = \{ z \in \mathbb{R}^m : \forall i \in \{1, \ldots, m\}, \min \{\alpha_i, \beta_i\} \leq z_i \leq \max \{\alpha_i, \beta_i\}\}.$$

Besides, we will consider the following definition for the second kind Stirling numbers and binomial numbers with vectorial arguments:

$$\sigma^\beta_\alpha = \sigma^\beta_{\alpha_1} \cdots \sigma^\beta_{\alpha_m}, \quad \left(\begin{array}{c} \alpha \\ \beta \end{array}\right) = \frac{\alpha!}{\beta!(\alpha - \beta)!} \quad \text{and} \quad \left(\frac{a}{\beta}\right) = \frac{a!}{\beta!(a - |\beta|)!}.$$

By using this notation, for $r \in \mathbb{N}_0^m$ with $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$, it is immediate to prove the following version of Newton’s binomial formula:

$$(\alpha + \beta)^r = \sum_{i=0}^{r} \binom{r}{i} \alpha^i \beta^{r-i}. \quad (1)$$
Let \( f \) be a function of \( x \) and \( y \).

**Proof.** The proof is by induction on \( m \). For \( m = 1 \) the result is immediate and for \( m = 2 \) identity (2) can be written in the form \( \sum_{j=0}^{C} {\binom{C}{j} \binom{\alpha_1}{j}} = \binom{\alpha_1 + \alpha_2}{C} \), that can be obtained by applying the binomial Newton’s formula to the powers in the identity \((x + y)^{\alpha_1 + \alpha_2} = (x + y)^{\alpha_1}(x + y)^{\alpha_2}\) and equating the coefficients of the monomials in \( x, y \) of both sides of the identity.

Let us suppose that (2) holds for \( m \) and take \( \alpha \in \mathbb{N}_0^{m+1} \). If we denote \( \alpha = (\alpha_1, \ldots, \alpha_m) \), we have

\[
\sum_{j \in \mathbb{N}_0^{m+1}, \ |j|=C} \binom{\alpha}{j} = \sum_{j_m+1=0}^{C} \binom{\alpha_m+1}{j_m+1} \sum_{|\alpha|-C-j_m+1} \binom{\alpha}{\alpha_m+1, \ldots, \alpha_{m+1}} = \binom{|\alpha|}{C},
\]

where we have used the induction hypotheses and the case \( m = 2 \). We have thus proved the case \( m + 1 \) which ends the proof.

We will admit that \( (\alpha_1) = 1 \) and \( (\alpha_1) = 0 \) for each \( \alpha \in \mathbb{N}_0 \). We will take into account this convention to compute binomial numbers with vectorial arguments that have some negative component.

We also denote by \( t : \mathbb{R}^m \ni z \mapsto t(z) = z \) the identity map in \( \mathbb{R}^m \) and by \( t_i : \mathbb{R}^m \ni z = (z_1, \ldots, z_m) \mapsto t_i(z) = z_i \) the \( i \)-th projection.

Given \( x \in \mathbb{R}^m \) and \( \delta \in \mathbb{R}^+ \), \( B_\delta(x) \) is the euclidean ball of radius \( \delta \) centered at \( x \) and for a subset \( H \subseteq \mathbb{R}^m \), \( \hat{H} \) is the set of all the interior points of \( H \) and \( \text{CL}(H) \) is the closure set of \( H \).

Given \( \alpha \in \mathbb{N}_0^m \), we denote by \( D^\alpha \) the partial derivative operator \( \alpha_1 \) times respect to the first variable, \( \ldots, \alpha_m \) times respect to the \( m \)-th variable. For \( H \subseteq \mathbb{R}^m \) such that \( H \subseteq \text{CL}(\hat{H}) \) and \( x \in H \), a function \( f \in \mathbb{R}^H \) is said to be differentiable with continuity of order \( \alpha \) at \( x \) if there exists \( \delta > 0 \) and \( \bar{f} \in \mathbb{R}^{B_\delta(x)} \) with \( \bar{f}_{|B_\delta(x) \cap H} = f|_{B_\delta(x) \cap H} \) such that \( D^\alpha \bar{f} \) is defined and continuous in \( B_\delta(x) \) and then we define \( D^\alpha f(x) = D^\alpha \bar{f}(x) \). Because of the conditions \( H \subseteq \text{CL}(\hat{H}) \), it is straightforward that the definition of \( D^\alpha f(x) \) is independent of the chosen function, \( \bar{f} \). Given \( r \in \mathbb{N} \), \( f \) is said to be differentiable with continuity of order \( r \) at \( x \) if it is so of order \( \beta \) for any \( \beta \in \mathbb{N}_0^m \) with \( |\beta| \leq r \). We also denote by \( C^0(H) \) or \( C(H) \) the set of all continuous functions in \( H \) and by \( C^r(H) \), \( r \in \mathbb{N} \), the set of functions differentiable with continuity of order \( r \) at \( x \), for every \( x \in H \): being \( C^\infty(H) = \cap_{r \in \mathbb{N}} C^r(H) \) and \( C^\infty(H, \mathbb{R}^m) = \{ F : H \rightarrow \mathbb{R}^m : \forall i \in \{1, \ldots, m\}, F_i \in C^\infty(H) \} \). Finally, \( \mathbb{P}_r[n] \) is the set of polynomials in the variable \( n \) of degree at most \( r \).

For \( x \in \mathbb{R}^m \), \( \alpha \in \mathbb{N}_0^m \), \( h \in \mathbb{R}^+ \) and a multivariate function \( f \), we will consider the classical finite differences \( \Delta_h^\alpha f(x) \) recursively defined by the relation \( \Delta_h^\alpha f(x) = \Delta_h^{\alpha+\alpha_1} f(x) = \Delta_h^{\alpha+\alpha_2} f(x) = \Delta_h^{\alpha+\alpha_3} f(x) = \cdots = \Delta_h^{\alpha+\alpha_m} f(x) \).
and $\Delta_0^0 f(x) = f(x)$. From this definition we can prove, whenever it makes sense, that $\Delta_h^{\alpha + \beta} f = \Delta_h^{\alpha} \Delta_h^{\beta} f$ for any $\beta \in \mathbb{N}_0^n$. From the well known property for the univariate case (see [7, Section 24.1.4.II.C]) it is easy to conclude that $\Delta_h^{\alpha} t^\beta(0) = \alpha! \sigma_\beta^\alpha h^\beta$, and then

$$\Delta_h^{\alpha} t^\beta(0) = \prod_{i=1}^m \Delta_h^{\alpha_i} t_i^{\beta_i}(0) = \prod_{i=1}^m \alpha_i! \sigma_\beta^\alpha h^\beta = \alpha! \sigma_\beta^\alpha h^\beta. \quad (3)$$

Finally, by the mean value theorem, if $f$ is differentiable enough we have

$$\Delta_h^{\alpha} f(x) = h^\alpha D^\alpha f(\xi_{x,h}), \quad (4)$$

for certain point $\xi_{x,h} \in \mathbb{R}^m$ in the domain of $f$.

3 The main result

The approximation properties of a sequence of linear operators, \{\(L_n\)\}_{n \in \mathbb{N}} , can be studied through the analysis of the behavior of the sequence on certain set of functions usually called the ‘test functions’. For instance, think about the Korovkin Theorem [12] that, in the univariate case, deduces the convergence of the sequence for any continuous function from the convergence for the test functions $1$, $t$ and $t^2$. Also consider the results of Sikkema [16] that allow to compute the asymptotic expansion for the sequence by calculating the moments $L_n(t^i)$, $i \in \mathbb{N}_0$. In both cases the test functions are obtained taking powers of the identity function $t$.

We will study the sequence of operators in terms of the powers of a more general function, $\varphi$, that we will call the ‘test function’. Joined to the selection of this new test function, different concepts of partial derivatives, forward differences and convexities will appear. In order to deal with operators that do not present classical shape preserving properties we will have to chose for every sequence of operators a suitable test function for which they are well-behaved.

In the sequel we fix $m \in \mathbb{N}$ and consider a subset $H \subseteq \mathbb{R}^m$ such that $H \subseteq \text{CL}(\hat{H})$.

In the next definition, we establish the properties that we ask $\varphi$ to verify in order to be a valid ‘test function’ in $H$. We also introduce some notation for spaces of polynomials in $\varphi$ and sets of functions bounded by $\varphi$.

**Definition 2.**

i) We denote by $\text{Dif}(H)$ the set of functions $\varphi : H \to \mathbb{R}^m$ that satisfy:

1. $\varphi \in C^\infty(H, \mathbb{R}^m)$,
2. $\forall z \in H$, $\det (\varphi'(z)) \neq 0$,
3. $\varphi : H \to \varphi(H)$ is an homeomorphism,
4. if $0$ is an accumulation point of $H$ then $\lim_{z \to 0} \varphi(z) = 0$.

ii) For each $\varphi \in \text{Dif}(H)$, $s \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}_0^n$, we set

$$\mathbb{P}_{\varphi,\alpha} = \text{span}\{\varphi^\beta : \beta \in \mathbb{N}_0^n, \beta \leq \alpha\},$$
\[ P_{\varphi, s} = \text{span}\{\varphi^\beta : \beta \in \mathbb{N}^m_0, |\beta| \leq s\}, \]

and we define \( B(H, \varphi, s) \) as the set of all the functions \( f : H \rightarrow \mathbb{R} \) such that

\[ |f(z)| \leq |p(z)| \]

holds for all \( z \in H \) and some \( p \in P_{\varphi, s} \). Besides, given \( x \in H \), \( B(H, \varphi, s, x) \) is the set of all the functions of \( B(H, \varphi, s) \) differentiable with continuity of order \( s \) at \( x \).

For the rest of the section we fix \( \varphi \in \text{Dif}(H) \). The following definition is devoted to introduce notations for partial derivatives, forward differences and integration operators relative to the test function \( \varphi \).

**Definition 3.** Let \( \alpha \in \mathbb{N}^m_0, h \in \mathbb{R}^+, x \in H \) and \( f \in \mathbb{R}^H \) be considered. We define (whenever they make sense):

i) \( D^\alpha f(x) = D^\alpha (f \circ \varphi^{-1})(\varphi(x)) \).

ii) \( \Delta^\alpha_{\varphi,h} f(x) = \Delta^\alpha_h(f \circ \varphi^{-1})(\varphi(x)) \).

iii) If \( 0 \in H \) and for all \( z \in H \) it is verified that \([0, \varphi(z)] \subseteq \varphi(H)\) then, for each \( i = 1, \ldots, m \), we define the integration operator \( S^\alpha_{\varphi,i} : C(H) \rightarrow C(H) \) by

\[ S^\alpha_{\varphi,i} f(x) = \int_{0_i}^{\varphi_i(x)} f \circ \varphi^{-1}((1 - e_i)\varphi(x) + ze_i)dz \]

and we also consider for \( \alpha \in \mathbb{N}^m_0 \) the iterations

\[ S^\alpha f(x) = (S^\alpha_{\varphi,1})^{\alpha_1} \circ \cdots \circ (S^\alpha_{\varphi,m})^{\alpha_m} (f)(x) \]

where \( (S^\alpha_{\varphi,i})^{\alpha_i} = S^\alpha_{\varphi,i} \circ \cdots \circ S^\alpha_{\varphi,i} \).

We will also follow the notation \( D^\alpha_{\varphi,f(x)} = 0 \) whenever \( 0 \not\succeq \alpha \).

If we take \( \alpha, \beta \in \mathbb{N}^m_0 \), from (3) we conclude that

\[ \Delta^\alpha_{\varphi,h} \varphi^\beta(0) = \alpha! \sigma^\alpha_{\beta} h^\beta. \quad (5) \]

From (4) it is easy to check, for a function \( f \in C^{[\alpha]}(H) \), that

\[ \Delta^\alpha_{\varphi,h} f(x) = h^\alpha D^\alpha_{\varphi,h} f(\xi_{x,h}), \quad (6) \]

for certain \( \xi_{x,h} \in H \).

As we have announced before, relative to the test function \( \varphi \) we introduce a concept of convexity that generalizes the usual one.

**Definition 4.** Given \( \alpha \in \mathbb{N}^m_0 \):

- \( f \in \mathbb{R}^H \) is said to be \( \varphi \)-convex of order \( \alpha \) if for all \( h \in \mathbb{R}^+ \) it is verified that \( \Delta^\alpha_{\varphi,h} f \geq 0 \).
• Given the subspaces $W_1, W_2 \subseteq \mathbb{R}^H$, a linear operator $L : W_1 \rightarrow W_2$ is said to be $\varphi$-convex of order $\alpha$ whenever it is verified that

$$f \varphi\text{-convex of order } \alpha \Rightarrow Lf \varphi\text{-convex of order } \alpha.$$ 

From (6), given $\alpha \in \mathbb{R}^m$, $f \in C^{[\alpha]}(H)$ is $\varphi$-convex of order $\alpha$ if and only if $D_\varphi^\alpha f \geq 0$.

Some other elementary results can be written in terms of this new notation relative to the test function $\varphi$. We summarize some of them in our next:

**Lemma 5.**

i) Given $\alpha \in \mathbb{N}^m_0$, if $f \in C(H)$ then $D_\varphi^\alpha S_\varphi^\alpha f = f$. Moreover, when $f \geq 0$ then $S_\varphi^\alpha f$ is $\varphi$-convex of order $\alpha$.

ii) (Modified Taylor’s Theorem) Given $r \in \mathbb{N}$, $x \in H$ and $f \in B(H, \varphi, r, x)$ then,

$$f(z) = \sum_{\alpha \in \mathbb{N}^m_0} \frac{D_\varphi^\alpha f(x)}{\alpha!} (\varphi(z) - \varphi(x))^\alpha + \sum_{\alpha \in \mathbb{N}^m_0 : |\alpha| = r} h_\alpha(z-x)(\varphi(z) - \varphi(x))^\alpha$$

(7)

holds for every $z \in H$, where, for each $\alpha \in \mathbb{N}^m_0$ with $|\alpha| = r$, $h_\alpha$ is a bounded function such that $\lim_{z \to 0} h_\alpha(z) = 0$.

iv) (Multivariate Leibnitz’s formula) Let $f, g \in \mathbb{R}^H$ be differentiable with continuity of order $k \in \mathbb{N}^m_0$ at $x \in \mathbb{R}^m$. Then,

$$D_\varphi^k(f \cdot g)(x) = \sum_{\alpha = 0}^{k} \binom{k}{\alpha} D_\varphi^\alpha f(x) D_\varphi^{k-\alpha} g(x).$$

(8)

**Proof.** Claim i) is immediate. For ii) we must take into account that, since $\varphi \in C^\infty(H, \mathbb{R}^m)$ is an homeomorphism, then so is $\varphi^{-1} \in C^\infty(\varphi(H), \mathbb{R}^m)$ and also that $f \circ \varphi^{-1} \in B(H, t, r, x)$. Then, it suffices to take the Taylor formula for $f \circ \varphi^{-1}$ at $\varphi(x)$ and to evaluate it at $\varphi(z)$. Claim iii) is a generalization of the well known Leibnitz’s formula that can be proved by induction on the dimension of the environment space, $m$.

From modified Taylor’s formula, for each $x \in H$ and $i \in \mathbb{N}^m_0$, it is immediate that

$$f \in \mathbb{P}_{\varphi,i} \Rightarrow f = \sum_{\alpha = 0}^{i} \frac{D_\varphi^\alpha f(x)}{\alpha!} (\varphi - \varphi(x))^\alpha.$$  

(9)

Sikkema [16] proves a result by means of which it is possible to compute the asymptotic expansion of a sequence of univariate linear positive operators. Later, Abel [6] obtains a multivariate version of such a result valid when the domain is a simplex of $\mathbb{R}^m$. We are going to prove a generalization of Abel’s theorem for an arbitrary domain $H \subseteq \mathbb{R}^m$. Moreover, we will enunciate the theorem in terms of the test function $\varphi$. 

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Theorem 6. Let \( r \) be an even number and let \( \{ L_n : W \to C^\infty(H) \}_{n \in \mathbb{N}} \) be a sequence of linear positive operators defined in the linear subspace \( W \subseteq \mathbb{R}^H \) such that \( P_{\varphi, r} \subseteq W \). Let us suppose that for certain \( x \in H \)
\[
L_n \left( |(\varphi - \varphi(x))^2| \right)(x) = O(\phi(n)^{-s}), \quad s = \frac{r}{2} + 1,
\]
where \( \phi \) is an increasing strictly positive function such that \( \lim_{n \to \infty} \phi(n) = +\infty \).

If \( f \in B(H, \varphi, r, x) \cap W \) then,
\[
L_n f(x) = \sum_{\alpha \in \mathbb{N}_0^m \atop |\alpha| \leq r} \frac{D\varphi^\alpha f(x)}{\alpha!} L_n \left( (\varphi - \varphi(x))^\alpha \right)(x) + o(\phi(n)^{-\frac{s}{2}}).
\]

Proof. Let us consider the Taylor formula (7). Since all the functions \( h_\alpha \) are bounded, for every \( \varepsilon \in \mathbb{R}^+ \) there exists \( A \in \mathbb{R}^+ \) such that
\[
|h_\alpha(z - x)| \leq \varepsilon + A|(|(\varphi(z) - \varphi(x))^2|
\]
holds for all \( z \in H \) and \( \alpha \in \mathbb{N}_0^m \) verifying \(|\alpha| = r \). It can be easily proved that for all \( \alpha \in \mathbb{N}_0^m \) with \(|\alpha| = r \),
\[
|(\varphi - \varphi(x))^\alpha| \leq |(\varphi - \varphi(x))^2|^\frac{s}{2}.
\]
Then from (7), for certain constants \( A_1, A_2 \in \mathbb{R}^+ \) we have
\[
\left| f(z) - \sum_{\alpha \in \mathbb{N}_0^m \atop |\alpha| \leq r} \frac{D\varphi^\alpha f(x)}{\alpha!} (\varphi(z) - \varphi(x))^\alpha \right| \leq \varepsilon A_1 |(\varphi - \varphi(x))^2|^\frac{s}{2} + A_2 |(\varphi - \varphi(x))^2|^\frac{s}{2} + 1.
\]

If we apply \( L_n \) and evaluate at \( x \), since \( \varepsilon \) is arbitrary, the proof follows. \( \square \)

Now let us proceed to establish our main result. We will prove that it is possible to compute the asymptotic formula for the partial derivatives, \( D\varphi^\alpha \), of a sequence of linear positive operators provided that such operators present certain conservative properties with respect to the test function \( \varphi \) and they verify some reasonable conditions on their moments. Furthermore, we prove that the asymptotic formula for these partial derivatives can be obtained from the expansion for the non differentiated sequence.

Theorem 7. Let us assume that \( 0 \in H \) and that for all \( z \in H \) we have \([0, \varphi(z)] \subseteq \varphi(H) \).

Let \( \{ L_n : W \to C^\infty(H) \}_{n \in \mathbb{N}} \) be a sequence of linear operators defined on the linear subspace \( W \subseteq \mathbb{R}^H \) such that \( \{ \varphi^\alpha : \alpha \in \mathbb{N}_0^m \} \subseteq W \). Let us denote
\[
V_{n, \alpha, \beta}(z) = \left[ D\varphi^\beta L_n \left( (\varphi - \varphi(z))^\alpha \right) \right](z)
\]
for every \( \alpha, \beta \in \mathbb{N}_0^m \), \( n \in \mathbb{N} \) and \( z \in H \). Let us suppose that for all \( \alpha \in \mathbb{N}_0^m \) and \( z \in H \) the following conditions are satisfied:
We are going to divide the proof into three steps:

i) \( \forall \beta \in \mathbb{N}_0^m, \forall n \in \mathbb{N}, \forall i \in \{1, \ldots, m\} \) such that \( \alpha_i > 0 \),

\[
D^\varphi_{\alpha}V_{n,\alpha,\beta}(z) = V_{n,\alpha,\beta+\epsilon_i}(z) - \alpha_i V_{n,\alpha-\epsilon_i,\beta}(z).
\]

ii) \( \forall n \in \mathbb{N} \), \( L_n \) is \( \varphi \)-convex of order \( \alpha \).

iii) There exists \( \phi \in \mathbb{P}_1[n] - \mathbb{P}_0[n] \) non depending on \( \alpha \), such that

\[
\phi(n)^aV_{n,\alpha,\beta}(z) \in \mathbb{P}_{\left\lfloor \frac{a+\beta}{2} \right\rfloor}[n].
\]

Then, given \( x \in H \), an even number \( r \in \mathbb{N}_0 \), \( k \in \mathbb{N}_0^m \) and \( f \in W \cap C[k](H) \) such that \( D^\varphi_{\alpha}f \in B(H, \varphi, r, x) \) and \( S^k\varphi D^k f \in W \) we have

\[
D^k\varphi L_n f(x) = P_{f,k,r,n}(x) + o(\phi(n)^{-\frac{r}{2}}),
\]

where \( P_{f,k,r,n}(x) \) is a polynomial in \( \phi(n)^{-\frac{r}{2}} \) of degree \( \frac{r}{2} \). Moreover, given \( s \in \mathbb{N}_0^m \), if in addition we have that \( f \in C[k+s](H) \), \( D^k\varphi f \in B(H, \varphi, r, x) \) and \( S^k\varphi D^{k+s} f \in W \), then

\[
D^k\varphi P_{f,k,r,n}(x) = P_{f,k,r+s,n}(x).
\]

**Proof.** We are going to divide the proof into three steps:

1) As a consequence of the shape preserving properties (hypotheses ii)) of the operators \( L_n \), it is clear that they preserve the spaces \( \mathbb{P}_\varphi,\alpha \) (i.e. \( L_n(\mathbb{P}_\varphi,\alpha) \subseteq \mathbb{P}_\varphi,\alpha \) for all \( n \in \mathbb{N} \) and \( \alpha \in \mathbb{N}_0^m \)) and then it is not difficult to prove that

\[
\begin{cases}
V_{n,\alpha,\beta} = 0, & \text{whenever } \beta \notin \alpha, \\
V_{n,\alpha,\beta} \in \mathbb{P}_\varphi,\alpha-\beta, & \text{whenever } \beta \leq \alpha.
\end{cases}
\]  

Anyway, \( V_{n,\alpha,\beta} \in C^\infty(H) \). Furthermore, from i) and iii) it is immediate to conclude by induction that for each \( z \in H \),

\[
\phi(n)^aV_{n,\alpha,\beta}(z) \in \mathbb{P}_{\left\lfloor \frac{a+\beta}{2} \right\rfloor}[n].
\]

2) Let us define the operators \( \tilde{L}_n : S^{-k}_\varphi(W) \to C^\infty(H) \) by

\[
\tilde{L}_n = D^k\varphi L_n S^k\varphi,
\]

where \( S^{-k}_\varphi(W) = \{g \in C(H) : S^k\varphi g \in W\} \). In this case, it is immediate that \( \tilde{L}_n \) is a linear positive operator and that \( \{\varphi^\alpha : \alpha \in \mathbb{N}_0^m\} \subseteq S^{-k}_\varphi(W) \). Furthermore, we have:

a) If \( g \in C[k](H) \cap W \) and \( D^k\varphi g \in S^{-k}_\varphi(W) \), then,

\[
\tilde{L}_n(D^k\varphi g) - D^k\varphi L_n g = D^k\varphi L_n \left(S^k\varphi D^k\varphi g - g\right).
\]

We know that

\[
D^k\varphi \left(S^k\varphi D^k\varphi g - g\right) = D^k\varphi g - D^k\varphi g = 0
\]

and, since \( L_n \) is convex of order \( k \), we have \( D^k\varphi L_n (S^k\varphi D^k\varphi g - g) = 0 \) and therefore

\[
\tilde{L}_n(D^k\varphi g) = D^k\varphi L_n g.
\]
b) From a), for each \( \alpha \in \mathbb{N}_0^n \),

\[
\tilde{L}_n ((\varphi - \varphi(x))^\alpha)(x) = \frac{\alpha!}{(\alpha + k)!} \tilde{L}_n \left( D^k_\varphi (\varphi - \varphi(x))^{k+\alpha} \right)(x) = \frac{\alpha!}{(\alpha + k)!} V_{n,\alpha+k,k}(x).
\]

c) From b) and 1),

\[
\phi(n)^{\alpha+k} \tilde{L}_n ((\varphi - \varphi(x))^\alpha)(x) = \frac{\alpha! \phi(n)^{\alpha+k}}{(\alpha + k)!} V_{n,\alpha+k,k}(x) \in \mathbb{P}_{\left[\frac{\alpha+2k}{2}\right]}[n]
\]

and then,

\[
\tilde{L}_n ((\varphi - \varphi(x))^\alpha)(x) = O(\phi(n)^{-((\alpha+k) - \left(\frac{\alpha+2k}{2}\right))}) = O(\phi(n)^{-\left(\frac{\alpha+1}{2}\right)}),
\]

where we have used that \( \mathbb{P}_{\left[\frac{\alpha+2k}{2}\right]}[n] = \mathbb{P}_{\left[\frac{\alpha+2k}{2}\right]}[\phi(n)] \). Furthermore, we can also conclude that

\[
\tilde{L}_n ((\varphi - \varphi(x))^\alpha)(x), V_{n,\alpha+k,k}(x) \in \mathbb{P}_{\left[\alpha+k\right]}[\phi(n)^{-1}].
\]

From c), it is clear that the hypotheses of Theorem 6 are verified by the sequence \( \{\tilde{L}_n\}_{n \in \mathbb{N}} \) and the function \( D^k_\varphi f \in S_{\varphi}^{-k}(W) \) so that

\[
\tilde{L}_n(D^k_\varphi f)(x) = \sum_{i \in \mathbb{N}_0^n, |i| \leq r} \frac{D^{i+k}_f(x)}{i!} \tilde{L}_n ((\varphi - \varphi(x))^i)(x) + o(\phi(n)^{-\frac{7}{2}}).
\]

If we consider the projection \( \Pi : \bigcup_{s \in \mathbb{N}} \mathbb{P}_{s}[\phi(n)^{-1}] \rightarrow \mathbb{P}_{\frac{7}{2}}[\phi(n)^{-1}] \) given by

\[
\Pi \left( \sum_{i=0}^s a_i \phi(n)^{-i} \right) = \sum_{i=0}^\frac{7}{2} a_i \phi(n)^{-i},
\]

taking into account the remarks a) and b), we obtain the formula

\[
D^k_\varphi L_n f(x) = \sum_{i \in \mathbb{N}_0^n, |i| \leq r} \frac{D^{i+k}_\varphi f(x)}{(i+k)!} \Pi (V_{n,i+k,k}(x)) + o(\phi(n)^{-\frac{7}{2}})
\]

which proves (10). From the properties proved for \( V_{n,\alpha,\beta} \) we can also assert that \( P_{f,k,r,n}(x) \) is a polynomial in \( \phi(n)^{-1} \) of degree \( \frac{7}{2} \).

3) Let us suppose that \( f \in C^{k+s}(H) \), \( D^k_\varphi f \in B(H, \varphi, r, x) \) and that \( S^{k+s}_\varphi D^{k+s}_\varphi f \in W \). Under these assumptions we know that we can proceed as in 2) for the partial derivative of order \( k+s \), so

\[
D^{k+s}_\varphi L_n f(x) = P_{f,k+s,r,n}(x) + o(\phi(n)^{-\frac{7}{2}})
\]
with $P_{f,k+s,r,n}(x)$ defined by (12). Since the function $f$ is differentiable enough, by (12) we can also consider $P_{f,k+i,r,n}$, $0 \leq i \leq s$. We will see that all these expansions can be obtained as partial derivatives of $P_{f,k,r,n}$.

Let us compute $D^j_{\varphi} P_{f,k,r,n}(x)$, $j = 1, \ldots, m$, by differentiating in (12):

$$D^j_{\varphi} P_{f,k,r,n}(x) = \sum_{i \in \mathbb{N}_{0}^m, |i| \leq r} \frac{D_{\varphi}^{i+k} f(x)}{(i+k)!} \Pi (V_{n,i,k,k}(x))$$

$$+ \sum_{i \in \mathbb{N}_{0}^m, |i| \leq r} \frac{D_{\varphi}^{i+k} f(x)}{(i+k)!} \Pi (D_{\varphi}^e V_{n,i,k,k}(x))$$

$$= \sum_{i \in \mathbb{N}_{0}^m, |i| \leq r+1} \frac{D_{\varphi}^{i+k} f(x)}{(i+k)!} \Pi (V_{n,i+k-\epsilon,\epsilon,k}(x))$$

$$+ \sum_{i \in \mathbb{N}_{0}^m, |i| \leq r+1} \frac{D_{\varphi}^{i+k} f(x)}{(i+k)!} \Pi (D_{\varphi}^e V_{n,i,k,k}(x))$$

$$- \sum_{i \in \mathbb{N}_{0}^m, |i| = r+1} \frac{D_{\varphi}^{i+k} f(x)}{(i+k)!} \Pi (D_{\varphi}^e V_{n,i,k,k}(x)),$$

where we have used that because of (11) it is immediate that $D^j_{\varphi} V_{n,i+k,k}(x) = 0$ whenever $i_j = 0$. From 1) and c) we know that $\phi(n)^{i+k} V_{n,i+k,k}(x) \in \mathbb{P} \left[ \frac{|i+2k|}{2} \right] \left[ \phi(n) \right]$ so in the last sum, since $|i| = r+1$, we have that the lower power of $\phi(n)^{-1}$ in $V_{n,i+k,k}(x)$ has degree $|i| + |k| - \left[ \frac{|i+2k|}{2} \right] = \frac{s}{2} + 1$ and then $\Pi (V_{n,i+k,k}(x)) = 0$ and in the same manner $\Pi (D^e_{\varphi} V_{n,i+k,k}(x)) = 0$. Thus, we can write

$$D^j_{\varphi} P_{f,k,r,n}(x) = \sum_{i \in \mathbb{N}_{0}^m, |i| \leq r+1} \frac{D_{\varphi}^{i+k} f(x)}{(i+k)!} \Pi ((i_j + k_j) V_{n,i+k-\epsilon,\epsilon,k}(x) + D^e_{\varphi} V_{n,i,k,k}(x))$$

$$= \text{(using i)} = \sum_{i \in \mathbb{N}_{0}^m, |i| \leq r+1} \frac{D_{\varphi}^{i+k} f(x)}{(i+k)!} \Pi (V_{n,i+k,\epsilon,\epsilon,k}(x))$$

$$= \sum_{i \in \mathbb{N}_{0}^m, |i| \leq r} \frac{D_{\varphi}^{i+k+\epsilon} f(x)}{(i+k+\epsilon)!} \Pi (V_{n,i+k+\epsilon,\epsilon,k}(x)) = P_{f,k,\epsilon,\epsilon,r,n}(x).$$

Now we can proceed by induction to prove that $D^e_{\varphi} P_{f,k,r,n}(x) = P_{f,k+s,r,n}(x)$ which completes the proof. \hfill \Box

An immediate consequence of the preceding result is the fact that, for a suitable function $f$, $P_{f,k,r,n}(x) = D^e_{\varphi} P_{f,0,r,n}(x)$, i.e. the expansion for the partial derivative of order $k$ is the partial derivative of the expansion of the sequence.
Notice that conditions $i)$, $ii)$ and $iii)$ of Theorem 7 are quite natural. In fact, many important examples of sequences of linear operators preserve all the convexities and their moments have good properties as polynomials in $n$, that is to say, they verify points $ii)$ and $iii)$. On the other hand, it is easy to check that point $i)$ is satisfied by any interpolatory type operator.

If the hypotheses of Theorem 7 hold for a sequence of operators $\{L_n\}_{n \in \mathbb{N}}$ with certain test function $\varphi$, we have proved two things: First, that, given $k \in \mathbb{N}_0^m$, we can compute the asymptotic expansion for the $D^k_\varphi$ partial derivative of the sequence and, second, that the expansion for such a partial derivative can be obtained by applying the differential operator $D^k_\varphi$. Now we are going to show that if the conditions of Theorem 7 are verified for a function $\varphi$, we can obtain the theses of the theorem for any other test function $\varphi_1$ and we can also apply the $D^k_\varphi$ operator in the asymptotic formulae. This is useful when the operators of the sequence present good properties with respect to a given test function $\varphi$ but we are interested in computing the partial derivatives with respect to another one, $\varphi_1$. For example, when we want to study the usual derivatives that are obtained by taking $\varphi_1 = t$.

With the purpose of proving the mentioned result we will need a generalized version of the Faà di Bruno’s derivation formula for a composite function. In [7, Section 24.1.2.II.C] we find the classical univariate expression of Faà di Bruno’s formula that expresses the derivative of a composition of two functions in terms of the derivatives of each one. In [11], Constantine and Savits study the case of a composite function with a vector argument. We will make use of this last result in the following form:

**Lemma 8.** Let $g : H_1 \subseteq \mathbb{R}^m \rightarrow H_2 \subseteq \mathbb{R}^m$ and $f : H_2 \rightarrow \mathbb{R}$ be considered. Let us suppose that $g$ and $f$ are differentiable enough at $x$ and $g(x)$ respectively. Then, we have 

$$D^\alpha (f \circ g)(x) = \sum_{i \in \mathbb{N}_0^m, |i| \leq |\alpha|} \frac{D^i f(g(x))}{i!} D^\alpha ((g - g(x))^i)(x).$$

**Proof.** Constantine and Savits prove [11, Theorem 2.1] the following explicit expression for $D^\alpha (f \circ g)(x)$,

$$D^\alpha (f \circ g)(x) = \sum_{i \in \mathbb{N}_0^m, |i| \leq |\alpha|} D^i f(g(x)) \sum_{s=1}^{|\alpha|} \sum_{k_1, \ldots, k_s \in \mathbb{N}_0^m} \sum_{l_1, \ldots, l_s \in \mathbb{N}_0^m} \sum_{0 < l_1 < \cdots < l_s} \prod_{j=1}^s \frac{(D^{l_j} g(x))^{k_j}}{k_j! l_j!} \prod_{i=p}^{s} k_p = i \sum_{p=1}^{s} |k_p| l_p = \alpha$$

where $D^{l_j} g(x) = (D^{l_j} g_1(x), \ldots, D^{l_j} g_m(x))$ and $l_i < l_j$ whenever $|l_i| < |l_j|$ or $|l_i| = |l_j|$ and $l_i$ precedes $l_j$ with the lexicographical order. We only need to use this formula to compute $D^\alpha ((g - g(x))^i)(x)$ as the composition of $f_1 = t^i$ and $g_1 = g - g(x)$. Then, it is immediate to realize that the identity of Constantine and Savits is equivalent to the one of the lemma. \qed

It follows easily that the version of the multivariate Faà di Bruno formula given in the lemma above allows us to find a relation between the partial derivatives of a function $f$ with respect to two different test functions.
Lemma 9. Given \( x \in H, \alpha \in \mathbb{N}_0^m, f \in \mathbb{R}^H \) differentiable with continuity of order \(|\alpha|\) at \( x \) and \( \varphi_1, \varphi_2 \in \text{Dif}(H) \), we have

\[
D_{\varphi_1}^\alpha f(x) = \sum_{i \in \mathbb{N}_0^m, |i| \leq |\alpha|} \frac{D_{\varphi_2}^i f(x)}{i!} D^\alpha h_x^i(x),
\]

where \( h_x(z) = \varphi_2 \circ \varphi_1^{-1}(z) - \varphi_2(x) \).

Theorem 10. Let us assume that conditions i), ii) and iii) of Theorem 7 are verified for the test function \( \varphi \) and let us take another test function \( \varphi_1 \in \text{Dif}(H) \). Given an even number \( r \in \mathbb{N}, k \in \mathbb{N}_0^m \) and \( f \in W \cap C^{|k|}(H) \) such that \( D^i f \in B(H, \varphi, r, x) \) and \( S^i \varphi D^i f \in W \) for all \( i \in \mathbb{N}_0^m \) with \(|i| \leq |k|\), we have

\[
D^k_{\varphi_1} L_n f(x) = P^{\varphi_1}_{f,k,r,n}(x) + o(\phi(n)^{-\frac{r}{2}}),
\]

where \( P^{\varphi_1}_{f,k,r,n}(x) \) is a polynomial in \( \phi(n)^{-1} \) of degree \( \frac{r}{2} \) that verifies

\[
P^{\varphi_1}_{f,k,r,n}(x) = D^k_{\varphi_1} P_{f,0,r,n}(x),
\]

being \( P_{f,0,r,n}(x) \) the one obtained in Theorem 7 with the test function \( \varphi \). Accordingly, given \( s_1, s_2 \in \mathbb{N}_0^m \) such that \( s_1 + s_2 \leq k \), it is verified that

\[
D^{s_1}_{\varphi_1} P^{\varphi_1}_{f,s_2,r,n}(x) = P^{\varphi_1}_{f,s_1+s_2,r,n}(x).
\]

Proof. Application of Lemma 9 for \( \varphi_2 = \varphi \) yields

\[
D^k_{\varphi_1} L_n f(x) = \sum_{i \in \mathbb{N}_0^m, |i| \leq |k|} \frac{D^i L_n f(x)}{i!} D^k h_x^i(x)
\]

\[
= \sum_{i \in \mathbb{N}_0^m, |i| \leq |k|} \frac{P_{f,i,r,n}(x)}{i!} D^k h_x^i(x) + o(\phi(n)^{-\frac{r}{2}})
\]

\[
= \sum_{i \in \mathbb{N}_0^m, |i| \leq |k|} \frac{D^i P_{f,0,r,n}(x)}{i!} D^k h_x^i(x) + o(\phi(n)^{-\frac{r}{2}})
\]

\[
= D^k_{\varphi_1} P_{f,0,r,n}(x) + o(\phi(n)^{-\frac{r}{2}}),
\]

where we have used that Theorem 7 guarantees that for all \( i \in \mathbb{N}_0^m \) with \(|i| \leq |k|\), \( P_{f,i,r,n}(x) = D^i_{\varphi_1} P_{f,0,r,n}(x) \).

\[\Box\]

4 The multivariate Meyer-König and Zeller operators

The Meyer-König and Zeller operators, in the slight modification of Cheney and Sharma [10] were introduced in 1964. Since then, their asymptotic behavior has been studied by several
If and from Leibnitz’s formula (8), Lemma 11.

Then, the power series that defines $M_n$ may not converge so we will consider the subspace

$$\mathcal{D}_M = \{ f \in \mathbb{R}^H : \forall x \in H, \forall n \in \mathbb{N}, |M_nf(x)| < +\infty \}.$$ 

Then, $M_n : \mathcal{D}_M \to C^\infty(H)$ is a linear positive operator. Notice that $H$ is a non compact set so in $\mathcal{D}_M$ we can find non bounded continuous functions.

With the purpose of applying our results, first we are going to choose a test function in $H$ for which the $M_n$ operators present suitable properties. Take $\varphi = \frac{1}{1 + |\mu|} : \mathcal{D}_M \to (\mathbb{R}_0^+)^m$. It is immediate that $\varphi \in \text{Dif}(H)$ and for every $\alpha \in \mathbb{N}_0^m$ we have $\varphi^\alpha \in \mathcal{D}_M$. The inverse function of $\varphi$ is $\varphi^{-1} = \frac{1}{1 + |\mu|} : (\mathbb{R}_0^+)^m \to \mathcal{D}_M$.

We start proving that there exists a relationship between the partial derivatives with respect to $\varphi$ and the usual ones.

Lemma 11. Let $\alpha \in \mathbb{N}_0^m$, $x \in H$ and a function $f \in \mathbb{R}^H$ differentiable with continuity of order $|\alpha|$ at $x$. The following assertion holds true:

$$D^\alpha \varphi f(x) = (|x| - 1)^{\alpha} \sum_{i \in \mathbb{N}_0^m, |i| \leq |\alpha|}^\infty D^i f(x) \left(\frac{|\alpha| - 1}{|i| - 1}\right) \sum_{\mu = 0}^i (-1)^\mu x^\mu \left(\frac{\alpha}{\mu}\right) \frac{|\alpha - \mu|!}{(i - \mu)!}.$$ 

Proof. If $\alpha = 0$ the result is immediate so let us suppose that $\alpha \neq 0$. By means of Lemma 9 we have

$$D^\alpha \varphi f(x) = \sum_{i \in \mathbb{N}_0^m, |i| \leq |\alpha|} \frac{D^i f(x)}{i!} D^\alpha \left((\varphi^{-1} - x)^i\right) (\varphi(x)) \quad (13)$$

and from Leibnitz’s formula (8),

$$D^\alpha \left((\varphi^{-1} - x)^i\right) (\varphi(x)) = D^\alpha \left[ \left(\frac{t}{1 + |t|} - x\right)^i\right] (\varphi(x)) = D^\alpha \left[ (t - x(1 + |t|))^i (1 + |t|)^{-i}\right] (\varphi(x)) = \sum_{j=0}^\alpha \binom{\alpha}{j} D^j \left[ (t - x(1 + |t|))^i\right] (\varphi(x)) D^{\alpha-j}(1 + |t|)^{-i}(\varphi(x)). \quad (14)$$
It is immediate that \( D^j (t - x(1 + |t|)^i) = 0 \) whenever \(|j| \neq |i|\). Under the assumption that \(|j| = |i|\) we have

\[
D^\alpha - j (1 + |t|)^{-i} (\varphi(x)) = (-1)^{|i|-j} \frac{(|\alpha| - 1)!}{(|i| - 1)!} D^\alpha (1 - |x|)^\alpha,
\]

which remains valid even if \(|i| = 0\) due to the convention on binomial numbers with negative arguments adopted in page 4. From (1)

\[
D^j (t - x(1 + |t|)^i) = \sum_{\mu=0}^{i} \binom{i}{\mu} (-1)^{i-\mu} x^{i-\mu} D^j [t^\mu (1 + |t|)^{i-\mu}]
\]

\[
= \sum_{\mu=0}^{i} (-1)^{i-\mu} x^{i-\mu} \binom{j}{\mu} \mu! |i - \mu|!,
\]

where we have used again Leibniz's formula and the hypotheses \(|j| = |i|\) to compute

\[
D^j [t^\mu (1 + |t|)^{i-\mu}] = \sum_{\gamma=0}^{j} \binom{j}{\gamma} D^\gamma t^\mu D^{j-\gamma} (1 + |t|)^{i-\mu}
\]

\[
= \sum_{\gamma=0}^{j} \frac{j!}{(\gamma - \mu)!} \frac{|i - \mu|!}{|i - \mu + \gamma - j|!} (1 + |t|)^{i-\mu+j-\gamma} = \binom{j}{\mu} \mu! |i - \mu|!.
\]

Therefore, from (14), (15) and (16) we only need to modify the order of the sums to write

\[
D^\alpha ((\varphi^{-1} - x)^i) (\varphi(x)) = \frac{(|\alpha| - 1)!}{(|i| - 1)!} (|x| - 1)^\alpha 
\]

\[
\times \sum_{\mu=0}^{i} \binom{i}{\mu} (-1)^{i-\mu} x^{i-\mu} \mu! |i - \mu|! \sum_{\substack{\alpha=0 \\ |j|=|i|}}^{\alpha} \binom{\alpha}{\mu} \binom{j}{\mu}.
\]

Finally, we use Lemma 1 to get the chain of identities,

\[
\sum_{\substack{\alpha=0 \\ |j|=|i|}}^{\alpha} \binom{\alpha}{j} \binom{j}{\mu} = \binom{\alpha}{\mu} \sum_{\substack{\alpha=0 \\ |j|=|\alpha-i|}}^{\alpha-\mu} \binom{\alpha-\mu}{j} \binom{|\alpha|}{|\alpha-i|},
\]

which, joined to the fact that \(|\alpha - i|!\binom{i}{\mu} \mu! |i - \mu|! \binom{|\alpha|}{|\alpha-i|} = i! \binom{\alpha-\mu}{i}! (i-\mu)!\), allows us to simplify (17) that used in (13) ends de proof.

In order to apply our results, we need to prove that the \(M_n\) operators present good preserving properties respect to some test function. With this purpose, we are going to prove that their partial derivatives have good representations in terms of forward differences relatives to the chosen test function \(\varphi = \frac{t}{1-|t|}\). By means of such representations we will find the properties that we are looking for. We will begin studying partial derivatives of the first order and then we will extend the result to any order.
Proposition 12. Given \( n \in \mathbb{N}, f \in \mathcal{D}_M \) and \( i \in \{1, \ldots, m\} \), we have

\[
D^{e_i}_\varphi M_n f = \frac{1}{1 + |\varphi|} M_n \left( (n(|\varphi| + 1) + 1) \Delta^{e_i}_{\varphi, \frac{1}{n}} f \right).
\]

Proof. We know that

\[
M_n(f) \circ \varphi^{-1} = (1 + |t|)^{(n+1)} \sum_{k=0}^\infty \binom{n+|k|}{k} \left( \frac{t}{1 + |t|} \right)^k f \left( \frac{k}{n + |k|} \right).
\]

Then, since

\[
D^{e_i} \left( M_n(f) \circ \varphi^{-1} \right) = -(n+1)(1 + |t|)^{(n+2)} \sum_{k=0}^\infty \binom{n+|k|}{k} \left( \frac{t}{1 + |t|} \right)^k f \left( \frac{k}{n + |k|} \right)
\]

\[
-(1 + |t|)^{(n+2)} \sum_{k=0}^\infty \binom{n+|k|}{k} |k| \left( \frac{t}{1 + |t|} \right)^k f \left( \frac{k}{n + |k|} \right)
\]

\[
+(1 + |t|)^{(n+2)} \sum_{k=0}^\infty \binom{n+|k|}{k} k_i \left( \frac{t}{1 + |t|} \right)^{k-e_i} f \left( \frac{k}{n + |k|} \right).
\]

Make the change of index \( k_i = k_i - 1 \) in the last sum and use the identity \( \binom{n+|k+1|}{k+1} = (n+|k|+1)(\binom{n+|k|}{k}+1) \) to obtain

\[
D^{e_i} \left( M_n(f) \circ \varphi^{-1} \right) = -(1 + |t|)^{(n+2)} \sum_{k=0}^\infty \binom{n+|k|}{k} (n+1+|k|) \left( \frac{t}{1 + |t|} \right)^k f \left( \frac{k}{n + |k|} \right)
\]

\[
+(1 + |t|)^{(n+2)} \sum_{k=0}^\infty \binom{n+|k|}{k} (n+1+|k|) \left( \frac{t}{1 + |t|} \right)^k f \left( \frac{k+e_i}{n + |k|+e_i} \right)
\]

and finally, since \( |k| = n|\varphi||\frac{k}{n+|k|}| \) and \( \Delta^{e_i}_{\varphi, \frac{1}{n}} f \left( \frac{k}{n+|k|} \right) = f \left( \frac{k+e_i}{n+|k|+e_i} \right) - f \left( \frac{k}{n+|k|} \right) \), it only remains to compute the composition \( D^{e_i} \left( \left( M_n(f) \circ \varphi^{-1} \right) \right) \circ \varphi \) to conclude the proof.

Theorem 13. Given \( n \in \mathbb{N}, i \in \mathbb{N}_0^m \) and \( f \in \mathcal{D}_M \), the following identity holds,

\[
D^i_\varphi M_n f = \frac{1}{(1 + |\varphi|)^i} M_n \left( (n(|\varphi| + 1) + |i|) \Delta^{1}_{\varphi, \frac{1}{n}} f \right).
\]

Proof. We will proceed by induction on \( |i| \) so that let us suppose that the result is true for \( |i| \) and prove it for \( |i| + 1 \). We have

\[
D^i M_n f = D^{e_i}_\varphi \left( \frac{1}{(1 + |\varphi|)^i} M_n \left( (n(|\varphi| + 1) + |i|) \Delta^{1}_{\varphi, \frac{1}{n}} f \right) \right)
\]
Let
\[ \|j\| = \frac{|\|j\|-1|}{\|\|j\|\|} = \left\{ \begin{array}{ll}
-1 & \text{if } |\|j\|\| = 1 \\
\|\|j\|\| & \text{if } |\|j\|\| > 1
\end{array} \right. \]
where we assume that \( \|j\| \) is \( \|\|j\|\| \)-convex of order \( \|\|j\|\| + 1 \).

Now we only need to use this identity in (19) to end the proof.

On the other hand, for any two functions, \( g_1, g_2 \in D_M \), it is easy to check that
\[
\Delta_{\varphi, \frac{1}{n}}^e (g_1 \cdot g_2) (z) = g_1(z)\Delta_{\varphi, \frac{1}{n}}^e g_2(z) + g_2(\varphi^{-1}(\varphi(z) + \frac{e_j}{n}))\Delta_{\varphi, \frac{1}{n}}^e g_1(z), \quad z \in H,
\]
and therefore,
\[
\Delta_{\varphi, \frac{1}{n}}^e \left[ (n(|\varphi| + 1) + |i|)\Delta_{\varphi, \frac{1}{n}}^i f \right] = (n(|\varphi| + 1) + |i| + 1)\Delta_{\varphi, \frac{1}{n}}^{i+e_j} f
\]
\[
+ \left[ (n(|\varphi| + 1) + |i| + 1)|i| - (n(|\varphi| + 1) + |i|)|i| \right] \Delta_{\varphi, \frac{1}{n}}^i f
\]
\[
= (n(|\varphi| + 1) + |i| + 1)|i|\Delta_{\varphi, \frac{1}{n}}^{i+e_j} f + (n(|\varphi| + 1) + |i|)|i-1|\Delta_{\varphi, \frac{1}{n}}^i f.
\]

Now we only need to use this identity in (19) to end the proof. \qed

As we said, we use this last result to establish an immediate consequence certain conservative properties of \( M_n \) with respect to the test function \( \varphi \) in the following:

**Corollary 14.** Given \( f \in D_M, n \in \mathbb{N} \) and \( k \in \mathbb{N}_0^m \) the following claims hold:

i) If \( f \in \mathbb{P}_{\varphi, k} \) then \( M_n f \in \mathbb{P}_{\varphi, k} \) and the degree is preserved.

ii) If \( f \) is \( \varphi \)-convex of order \( k \) then \( M_n f \) is \( \varphi \)-convex of order \( k \).

To apply Theorem 7 it is also necessary to check hypotheses iii). For this purpose we will study the moments of \( M_n \) relative to \( \varphi \) and we do so by using a notation similar to that of the theorem, i.e. given \( \alpha \in \mathbb{N}_0^m \), \( n \in \mathbb{N} \) and \( x \in H \), we set
\[
V_{n, \alpha}(x) = M_n ( (\varphi - \varphi(x))^\alpha ) (x).
\]

We will study the behavior of \( V_{n, \alpha}(x) \) as a polynomial in \( n^{-1} \). As we can see in [16, eq. (19), (23) and (26)] for the univariate case, we will do so by computing a recurrence formula for the moments. Let us obtain such a recurrence relation for the \( M_n \) operators.

**Lemma 15.** Let \( i \in \{1, \ldots, m\} \) and \( \alpha \in \mathbb{N}_0^m \) be given. Then,
\[
n(1 - t_i)V_{n, \alpha + e_i} - nt_i \sum_{j=1}^{m} V_{n, \alpha + e_j} = \frac{t_i D_{\varphi} V_{n, \alpha}}{1 - |t|} + t_i V_{n, \alpha} + \frac{\alpha_i t_i V_{n, \alpha - e_i}}{1 - |t|},
\]
where we assume that \( V_{n, \beta} = 0 \) whenever \( 0 \leq \beta \).

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Proof. For those point $x \in H$ such that $t_i(x) = 0$ it is easy to prove that $V_{n, \alpha + \epsilon_i}(x) = 0$ and the result is immediate. Therefore we only have to consider the points $x \in H$ such that $t_i(x) \neq 0$ so that in the sequel we admit that $t_i \neq 0$.

We have

$$n^\alpha V_{n, \alpha} = (1 - |t|)^{n+1} \sum_{k=0}^{\infty} (k - n\alpha)^\alpha \binom{n + |k|}{k} t^k$$

and then

$$n^\alpha V_{n, \alpha} \circ \varphi^{-1} = (1 + |t|)^{-(n+1)} \sum_{k=0}^{\infty} (k - nt)^\alpha \binom{n + |k|}{k} \left( \frac{t}{1 + |t|} \right)^k.$$

Hence, if $\alpha_i \neq 0$,

$$n^\alpha D^{\epsilon_i} (V_{n, \alpha} \circ \varphi^{-1}) = -(n+1)(1 + |t|)^{-(n+2)} \sum_{k=0}^{\infty} (k - nt)^\alpha \binom{n + |k|}{k} \left( \frac{t}{1 + |t|} \right)^k$$

$$- (1 + |t|)^{-(n+1)} \sum_{k=0}^{\infty} \alpha_i n (k - nt)^{\alpha - \epsilon_i} \binom{n + |k|}{k} \left( \frac{t}{1 + |t|} \right)^k$$

$$+(1 + |t|)^{-(n+1)} \sum_{k=0}^{\infty} (k - nt)^\alpha \binom{n + |k|}{k} \left[ - \frac{|k| - k_i}{1 + |t|} \left( \frac{t}{1 + |t|} \right)^k \right]$$

$$+ k_i \frac{1 + |t| - t_i}{(1 + |t|)^2} \left( \frac{t}{1 + |t|} \right)^{k-\epsilon_i},$$

where we have used for the last sum a slight modification of identity (18). In this last sum it is also clear that $|k| - k_i = \sum_{j=1}^{m} (k_j - nt_j + nt_j)$ and $k_i = k_i - nt_i + nt_i$ so we can write

$$n^\alpha D^{\epsilon_i} (V_{n, \alpha} \circ \varphi^{-1}) = -(n+1)(1 + |t|)^{-(n+2)} \sum_{k=0}^{\infty} (k - nt)^\alpha \binom{n + |k|}{k} \left( \frac{t}{1 + |t|} \right)^k$$

$$- \alpha_i n(1 + |t|)^{-(n+1)} \sum_{k=0}^{\infty} (k - nt)^{\alpha - \epsilon_i} \binom{n + |k|}{k} \left( \frac{t}{1 + |t|} \right)^k$$

$$- \sum_{j=1}^{m} (1 + |t|)^{-(n+2)} \sum_{k=0}^{\infty} (k - nt)^{\alpha + \epsilon_j} \binom{n + |k|}{k} \left( \frac{t}{1 + |t|} \right)^k$$

$$- \sum_{j=1}^{m} (1 + |t|)^{-(n+2)} nt_j \sum_{k=0}^{\infty} (k - nt)^{\alpha + \epsilon_j} \binom{n + |k|}{k} \left( \frac{t}{1 + |t|} \right)^k$$

$$+(1 + |t|)^{-(n+2)} n(1 + |t| - t_i) \sum_{k=0}^{\infty} (k - nt)^\alpha \binom{n + |k|}{k} \left( \frac{t}{1 + |t|} \right)^k$$

$$+(1 + |t|)^{-(n+2)} \frac{1 + |t| - t_i}{t_i} \sum_{k=0}^{\infty} (k - nt)^{\alpha + \epsilon_i} \binom{n + |k|}{k} \left( \frac{t}{1 + |t|} \right)^k.$$
We can cancel out the fifth sum with the fourth one and the factor $n$ of the first one. If we compose with $\varphi$ we easily obtain the result for every $x \in H$ such that $t_i(x) = x_i \neq 0$.

If $\alpha_i = 0$, the second sum in (20) does not appear and the proof is the same. □

From the recurrence formula that we have just proved we are going to deduce the degree of $V_{n,\alpha}(x)$ as a polynomial in $n^{-1}$.

**Theorem 16.** For each $\alpha \in \mathbb{N}_0^m$ and $x \in H$, $n^\alpha V_{n,\alpha}(x) \in \mathbb{P}_{\left\lfloor |\alpha| \right\rfloor} [n]$.

**Proof.** The proof is by induction on $|\alpha|$. When $|\alpha| = 0$ the result is obvious. Let us suppose that the property holds for every $\beta \in \mathbb{N}_0^m$ such that $|\beta| \leq |\alpha|$ and prove it for $\alpha + e_i$, $i = 1, \ldots, m$ (i.e. for $|\alpha| + 1$).

Consider the matrix

$$A(x) = \begin{pmatrix}
1 - x_1 & -x_1 & \ldots & -x_1 \\
-2 & 1 - x_2 & \ldots & -x_2 \\
\vdots & \vdots & \ddots & \vdots \\
-x_m & -x_m & \ldots & 1 - x_m
\end{pmatrix}.$$ 

It is easy to check that $\det(A(x)) = (1 - |x|) \neq 0$. Lemma 15 can be written by the matricial equation

$$n^{\left\lfloor |\alpha| \right\rfloor + 1} \begin{pmatrix}
V_{n,\alpha + e_1}(x) \\
V_{n,\alpha + e_2}(x) \\
\vdots \\
V_{n,\alpha - e_m}(x)
\end{pmatrix} = n^\alpha \begin{pmatrix}
\frac{x_1 D_1^{\alpha_1} V_{n,\alpha}(x)}{1 - |x|} + x_1 V_{n,\alpha}(x) + \frac{\alpha_1 x_1 V_{n,\alpha - e_1}(x)}{1 - |x|} \\
\frac{x_2 D_2^{\alpha_2} V_{n,\alpha}(x)}{1 - |x|} + x_2 V_{n,\alpha}(x) + \frac{\alpha_2 x_2 V_{n,\alpha - e_2}(x)}{1 - |x|} \\
\vdots \\
\frac{x_m D_m^{\alpha_m} V_{n,\alpha}(x)}{1 - |x|} + x_m V_{n,\alpha}(x) + \frac{\alpha_m x_m V_{n,\alpha - e_m}(x)}{1 - |x|}
\end{pmatrix}.$$ 

By the induction hypotheses, all the entries of the column of the right hand side, when they are multiplied by $n^\alpha$, have degree in $n$ at most $\left\lfloor \frac{|\alpha| + 1}{2} \right\rfloor$. Since $A(x)$ is a non singular matrix, $n^{\left\lfloor |\alpha| + 1 \right\rfloor} V_{n,\alpha + e_i}(x)$, $i = 1, \ldots, m$, can be computed as a linear combination of such expressions and thus it also has degree $\left\lfloor \frac{|\alpha| + 1}{2} \right\rfloor$ in $n$. □

Theorem 16 shows immediately that

$$V_{n,\alpha}(x) = O\left(n^{\left\lfloor \frac{|\alpha|}{2} \right\rfloor - |\alpha|}\right) = O\left(n^{-\left\lfloor \frac{|\alpha| + 1}{2} \right\rfloor}\right).$$

(20)

From Corollary 14 and Theorem 16 we deduce the properties of the operators $M_n$ that will make possible to apply our results of Section 3. In fact, (20) guarantees that $M_n$ verifies the conditions of Theorem 6. Therefore, we begin with the computation of the asymptotic formula for the sequence $\{M_n\}_{n \in \mathbb{N}}$ by means of such a result.
Theorem 17. Let \( r \in \mathbb{N} \) be an even number, let \( x \in H \) and \( f \in \mathcal{D}_M \) such that \( f \in B(H, \varphi, r, x) \). We have
\[
M_n f(x) = P_{f,0,r,n}(x) + o(n^{-\frac{r}{2}})
\]
and \( P_{f,0,r,n}(x) \) can be explicitly written in the following two forms:
\[
P_{f,0,r,n}(x) = \sum_{\delta=0}^{\frac{r}{2}} n^{-\delta} \sum_{\alpha \in \mathbb{N}_0^m} \frac{D_\varphi^\delta f(x)}{\alpha!} \sum_{\beta=0}^{\alpha} \varphi(x)^{\alpha-\beta} H(\alpha, \delta, \beta) \tag{21}
\]
and
\[
P_{f,0,r,n}(x) = \sum_{\delta=0}^{\frac{r}{2}} n^{-\delta} \sum_{\alpha \in \mathbb{N}_0^m} \frac{(-1)^\alpha}{\alpha!} \sum_{i \in \mathbb{N}_0^m} D_i f(x) \left( \frac{|i| - 1}{i} \right) \times \sum_{\mu=0}^{i} (-1)^\mu \left( \frac{\alpha - \mu}{\mu} \right) \frac{|\alpha - \mu|!}{(i - \mu)!} \sum_{\beta=0}^{\alpha} x^{\alpha-\beta+i-\mu} (1 - |x|)^\beta H(\alpha, \delta, \beta), \tag{22}
\]
where
\[
H(\alpha, \delta, \beta) = \sum_{i=\beta, |i| \geq \delta}^{\alpha} \binom{\alpha}{i} (-1)^{\alpha-i} \varphi^i \sum_{j=|i| - \delta}^{i-\beta} S_j^{i-\beta} \left( \frac{j}{|i| - \delta} \right) |i - \beta|^{j+\delta - |i|}.
\]

Proof. Given \( i \in \mathbb{N}_0^m \) we know that \( M_n(\varphi^i) \in \mathbb{P}_{\varphi,i} \) so that by Taylor’s formula (9) we have
\[
M_n(\varphi^i) = \sum_{p=0}^{i} \frac{D_\varphi^p M_n(\varphi^i)(0)}{p!} \varphi^p.
\]
From Theorem 13 and identity (5), since for every \( g \in \mathcal{D}_M \) we have \( M_ng(0) = g(0) \),
\[
D_\varphi^p M_n(\varphi^i)(0) = (n + |p|)^{|p|} \varphi^{|p|} \frac{\Delta_p^{|p|}}{n^{|p|}} \varphi^i(0) = \frac{(n + |p|)^{|p|} p! \sigma^p}{n^{|p|}},
\]
and from the definition of the first kind Stirling numbers,
\[
(n + |p|)^{|p|} = \sum_{j=0}^{|p|} S_j^{|p|} (n + |p|)^j = \sum_{j=0}^{|p|} S_j^{|p|} \sum_{\beta=0}^{j} \binom{j}{\beta} |p|^{|\beta|} n^{|\beta|}.
\]
With this all at hand we conclude that
\[
M_n(\varphi^i) = \sum_{p=0}^{i} \frac{\sigma^p}{n^{|p|} \varphi^{|p|}} \sum_{j=0}^{|p|} S_j^{|p|} \sum_{\beta=0}^{j} \binom{j}{\beta} |p|^{|\beta|} n^{|\beta|}.
\]
Let the partial derivatives of \( f \) calculate the partial derivatives with respect to any other test function \( \varphi \) with respect to the test function \( \varphi \) and accordingly, from Theorem 10, we know that we can also study the usual partial derivatives. In our next result we analyze the partial derivatives of \( f \) both usual and with respect to \( \varphi \).

Theorem 18. Let \( r \in \mathbb{N} \) be an even number, let \( k \in \mathbb{N}_0^m \), \( x \in H \) and \( f \in C^{|k|}(H) \) such that \( D^k f \in B(H, \varphi, r, x) \) for all \( i \in \mathbb{N}_0^m \) with \(|i| \leq |k|\). Then,

\[
D^k_\varphi M_n f(x) = D^k_\varphi P_{f,0,r,n}(x) + o(n^{-\frac{r}{2}}), \quad (25)
\]

\[
D^k M_n f(x) = D^k P_{f,0,r,n}(x) + o(n^{-\frac{r}{2}}), \quad (26)
\]

where \( P_{f,0,r,n}(x) \) is defined as in Theorem 7 for the sequence \( \{M_n\}_{n \in \mathbb{N}} \). Furthermore, the above partial derivatives of \( P_{f,0,r,n} \) can be explicitly expressed in the following forms:

\[
D^k_\varphi P_{f,0,r,n}(x) = k! \sum_{\delta=0}^{\frac{r}{2}} n^{-\delta} \sum_{\substack{\alpha \in \mathbb{N}_0^m \\delta \leq |\alpha| \leq 2\delta \\delta \leq |\alpha|}} \sum_{\gamma \geq 0} D^\alpha_\varphi f(x) \frac{\alpha! \gamma!}{\alpha! \gamma!} \sum_{\beta=0}^{\alpha+\gamma-k} \left( \frac{\alpha - \beta}{k - \gamma} \right) \varphi(x)^{\alpha - \beta + \gamma - k} H(\alpha, \delta, \beta),
\]
\[ D^k P_{f,0,r,n}(x) = \sum_{\delta=0}^{\gamma} n^{-\delta} \sum_{i=0}^{\gamma} \sum_{|\gamma_1|=0}^{k} \binom{k}{\gamma_1} D^{i+\gamma_1} f(x) \sum_{\alpha, \delta, \beta} \frac{1}{\alpha! \delta! \beta!} \left( \frac{|\alpha|}{|i|-1} \right) \times \]
\[ \times \sum_{\mu=0}^{i} (-1)^{i-\mu+\alpha} \left( \frac{\alpha-\mu+i}{\mu!} \prod_{\beta=0}^{\gamma_2} \frac{H(\alpha, \delta, \beta)(-1)^\beta \times}{|\beta| \leq \delta} \right) \]
\[ \times \sum_{\gamma_2=\beta}^{k-\gamma_1+\beta+\alpha} \binom{k-\gamma_1}{\gamma_2} \frac{(\mu+\alpha-\beta)!}{(\mu+\alpha-\gamma_2)!} \frac{\alpha-\beta-2}{\gamma_1+\gamma_2-k!} \frac{|i|}{|\gamma_1+\gamma_2-k|!} \]

where \( H(\alpha, \delta, \beta) \) is defined as in Theorem 17.

**Proof.** Take the space \( W = \cup_{j \in \mathbb{N}} B(H, \varphi, s) \). It is obvious that \( f \in W \) implies that \( S^k_{\varphi} D^k f \in W \). On the other hand, from Corollary 14 and Theorem 16 it is clear that the hypotheses of Theorem 7 are verified for the test function \( \varphi = \frac{x}{1-|x|} \), the space \( W, \phi(n) = n \) and the multivariate Meyer-König and Zeller operators. Therefore we can apply such a theorem to obtain (25). Then, to prove (26) it suffices to apply Theorem 10 for \( \varphi_1 = t \).

The explicit expressions for \( D^k P_{f,0,r,n}(x) \) follows from (25) by using multivariate Leibnitz’s formula to compute the partial derivative \( D^k [D^0 f \cdot \varphi^{\alpha-\beta}] (x) \) in (21).

In the same way, we compute the explicit form of \( D^k P_{f,0,r,n}(x) \) by means of (26) by differentiating in (22), again with the aid of Leibnitz’s formula to compute

\[ D^k \left[ D^i f \cdot x^{\alpha-\beta+i-\mu} \cdot (1-|t|)^\beta \right] (x) = (-1)^\beta \sum_{\gamma=0}^{k} \binom{k}{\gamma_1} D^{i+\gamma_1} f(x) \times \]
\[ \times \sum_{\gamma_2=0}^{k-\gamma_1+\beta+\alpha} \binom{k-\gamma_1}{\gamma_2} \frac{(i-\mu+\alpha-\beta)! x^{i-\mu+\alpha-\gamma_2}}{(i-\mu+\alpha-\gamma_2)!} \frac{|i|}{|\gamma_1+\gamma_2+\beta-k|!} \]
Corollary 19. Let $\alpha \in \mathbb{N}_0^m$, $x \in H$ and let the function $f \in C^{[\alpha]}(H)$ be such that $D_\psi f \in B(H, \varphi, 2, x)$ for all $i \in \mathbb{N}_0^m$ with $|i| \leq |\alpha|$. Then,

$$\lim_{n \to \infty} 2n \left( D^\alpha M_n f(x) - D^\alpha f(x) \right) = \sum_{i=1}^{m} (1 - |x|)(x_i - x_i^2)D^{\alpha+2\epsilon_i} f(x)$$

$$+ \sum_{i=1}^{m} \alpha_i(1 - |x| - x_i - x_i^2) - 2x_i(1 - |x|)|\alpha| + 2x_i |\alpha x| \ D^\alpha f(x)

+ [ (1 - |x|)(|\alpha| - |\alpha|^2) - 2|\alpha x|(1 - |x|) - |\alpha|^2 ] \ D^\alpha f(x)

+ \sum_{i=1}^{m} \alpha_i(2 - 3|\alpha| + |\alpha|^2)D^{-\epsilon_i} f(x) + \sum_{i,j,s=1}^{m} \alpha_s x_i x_j D^{\alpha + \epsilon_i + \epsilon_j - \epsilon_s} f(x)$$

$$+ \sum_{i,j=1}^{m} \left[ 2\alpha_j^2 x_i - \alpha_i \alpha_j + 2x_i \alpha_j (|\alpha| - \alpha_j) \right] D^{\alpha + \epsilon_i - \epsilon_j} f(x)

- \sum_{i,j=1}^{m} \alpha_j(x_i - x_i^2)D^{\alpha+2\epsilon_i - \epsilon_j} f(x) - \sum_{i,j=1}^{m} (1 - |x|)x_i x_j D^{\alpha+\epsilon_i + \epsilon_j} f(x),$$

where $\alpha^2 = \alpha(\alpha - 1)$.

Proof. Through Theorem 22 it is possible to compute the following expression for $P_{f,0,2,n}(x)$:

$$P_{f,0,2,n}(x) = f(x) + \frac{1 - |x|}{2n} \sum_{i=1}^{m} (x_i - x_i^2)D^{2\epsilon_i} f(x) - \frac{1 - |x|}{n} \sum_{1 \leq i < j}^{m} x_i x_j D^{\epsilon_i + \epsilon_j} f(x).$$

From (26) we know that $P_{f,0,2,n}(x) = D^\alpha P_{f,0,2,n}(x)$. Then, we compute the explicit expression for $D^\alpha P_{f,0,2,n}(x)$ by using Leibnitz’s formula repeatedly. \hfill \Box

5 The Bleimann, Butzer and Hahn operators

Recently, Adell, de la Cal and Miguel [8] have studied a bivariate version of the Bleimann, Butzer and Hahn [9] operators for which Abel [3] has got the complete asymptotic expansion. We will consider a natural multivariate generalization of such operators.

Given $m \in \mathbb{N}$, set $H = \{ x \in (\mathbb{R}_0^+)^m : \frac{x}{1 + x} \leq 1 \}$ and define, for $n \in \mathbb{N}$, $f \in \mathbb{R}^H$ and $x \in H$,

$$L_n f(x) = \sum_{k \in \mathbb{N}_0^m, |k| \leq n} \binom{n}{k} f \left( \frac{k}{n+1} \frac{x}{1+x} \right) \left( 1 - \frac{x}{1+x} \right)^{n-|k|}.$$
To analyze these operators we will follow the same scheme that we have used for the Meyer-König and Zeller ones. Therefore, we first chose a suitable test function for which the operators and their moments present the properties that allow us to apply our results.

Take the test function \( \phi = \frac{t}{1+t} : H \rightarrow \mathbb{R}^m \) whose inverse function is \( \phi^{-1} = \frac{1}{1+t} \) and whose component functions are \( \phi_i = \frac{t_i}{1+t} \), \( i = 1, \ldots, m \). It is immediate that \( \phi \in \text{Dif}(H) \). We begin obtaining a relation between the usual partial derivatives and the ones relative to \( \phi \).

**Lemma 20.** Let \( \alpha \in \mathbb{N}_0^m \), \( x \in H \) and let \( f \in \mathbb{R}^H \) be differentiable with continuity of order \( |\alpha| \) at \( x \). Then,

\[
D^\alpha \phi f(x) = \sum_{i=0}^{\alpha} \binom{\alpha - I}{i - I} \frac{\alpha!}{i!} (1 + x)^{i+\alpha} D^i f(x).
\]

**Proof.** For \( \alpha = 0 \) the proof is clear. For \( \alpha > 0 \) as in Lemma 11, in order to use Lemma 9, we first compute \( D^\alpha ((\phi^{-1} - x)^i) (\phi(x)) \). Such an expression vanishes whenever \( i \not\geq \alpha \) and for \( i \leq \alpha \) we have

\[
D^\alpha ((\phi^{-1} - x)^i) (\phi(x)) = \prod_{j=1}^m D^{\alpha_j e_j} \left( \frac{t_j}{1-t_j} - x_j \right)^{i_j} (\phi(x))
\]

and

\[
D^{\alpha_j e_j} \left( \frac{t_j}{1-t_j} - x_j \right)^{i_j} (\phi(x)) = \alpha_j! \binom{\alpha_j - 1}{i_j - 1} (1 + x_j)^{i_j + \alpha_j},
\]

being this identity true even if \( i_j = 0 \) due to the convention on binomial numbers with negative arguments adopted in page 2. Then,

\[
D^\alpha ((\phi^{-1} - x)^i) (\phi(x)) = \alpha! \binom{\alpha - I}{i - I} (1 + x)^{\alpha + i}
\]

which used with Lemma 9 let us complete the proof. \( \square \)

In the next result we will obtain an expression in terms of forward differences of the partial derivatives with respect to \( \phi \) for the operators \( L_n \). Later, we will use such expressions to settle conservative properties for them.

**Theorem 21.** Given \( n \in \mathbb{N}, \alpha \in \mathbb{N}_0^m \), \( x \in H \) and \( f \in \mathbb{R}^H \),

\[
D^\alpha \phi L_n f(x) = n^{|\alpha|} \sum_{k \in \mathbb{N}_0^m \atop |k| \leq n - |\alpha|} \binom{n - |\alpha|}{k} \Delta^\alpha_{\frac{x}{1+x}} f \left( \frac{k}{(n+1)I - k} \right) \times \left( \frac{x}{I+x} \right)^k \left( 1 - \frac{x}{I+x} \right)^{n - |k| - |\alpha|},
\]

and \( D^\alpha \phi L_n f = 0 \) whenever \( |\alpha| > n \).
Proof. We have

\[ L_n(f) \circ \varphi^{-1} = \sum_{k \in \mathbb{N}_0^n, |k| \leq n} \binom{n}{k} f \left( \frac{k}{(n+1)I - k} \right) t^k (1 - |t|)^{n-|k|}. \]

For \( i = 1, \ldots, m \), let us compute the partial derivative

\[
D^e_i \left( L_n(f) \circ \varphi^{-1} \right) = \sum_{k \in \mathbb{N}_0^n, |k| \leq n, 1 \leq k_i} \binom{n}{k} f \circ \varphi^{-1} \left( \frac{k}{(n+1)I} \right) k_i t^{k-e_i} (1 - |t|)^{n-|k|}
- \sum_{k \in \mathbb{N}_0^n, |k| \leq n-1} \binom{n}{k} f \circ \varphi^{-1} \left( \frac{k}{(n+1)I} \right) t^{k} (n - |k|) (1 - |t|)^{n-|k|-1}
= \sum_{k \in \mathbb{N}_0^n, |k| \leq n-1} \binom{n}{k} (k_i + 1) f \circ \varphi^{-1} \left( \frac{k + e_i}{(n+1)I} \right) t^{k} (1 - |t|)^{n-|k|-1}
- \sum_{k \in \mathbb{N}_0^n, |k| \leq n-1} \binom{n}{k} (n - |k|) f \circ \varphi^{-1} \left( \frac{k}{(n+1)I} \right) t^{k} (1 - |t|)^{n-|k|-1},
\]

where we have obtained the first sum of the last step by making the change \( k = k - e_i \). Since \( \binom{n}{k+e_i}(k_i+1) = \binom{n}{k}(n - |k|) = n\binom{n-1}{k} \) we get

\[
D^e_i L_n(f) \circ \varphi^{-1} = n \sum_{k \in \mathbb{N}_0^n, |k| \leq n-1} \binom{n-1}{k} \Delta^e_i \left( f \circ \varphi^{-1} \right) \left( \frac{k}{(n+1)I} \right) t^{k} (1 - |t|)^{n-|k|-1}
\]

and it is only necessary to consider the definition of \( \Delta^e_i \frac{1}{\varphi^{-1}} \) to obtain the result for \( \alpha = e_i \). The value of \( i \) is arbitrary so we have proved the theorem when \( |\alpha| = 1 \) and reiterating the process we prove the identity for any \( \alpha \) with \( |\alpha| \leq n \). From its definition, it is obvious that \( L_n f \in \mathbb{P}_{\varphi,n} \) so that it is immediate that \( D^e_\varphi L_n f = 0 \) whenever \( |\alpha| > n \).

\[ \square \]

**Corollary 22.** Given \( n \in \mathbb{N} \) and \( f \in \mathbb{R}^H \),

\[
L_n f = \sum_{\alpha \in \mathbb{N}_0^n} \binom{n}{\alpha} \Delta^\alpha_{\varphi,\frac{1}{n+1}} f(0) \varphi^\alpha,
\]

where we assume that \( \binom{n}{\alpha} = 0 \) whenever \( |\alpha| > n \).

Proof. We know that \( L_n f \in \mathbb{P}_{\varphi,n} \) so that from (9) we have

\[
L_n f(x) = \sum_{\alpha \in \mathbb{N}_0^n, |\alpha| \leq n} \frac{D^\alpha L_n f(0)}{\alpha!} \varphi^\alpha(x).
\]

An immediate consequence of Theorem 21 is that \( D^\alpha_\varphi L_n f(0) = n^{|\alpha|} \Delta^\alpha_{\varphi,\frac{1}{n+1}} f(0) \) which included in the above identity proves the result.

\[ \square \]
As we already said, from Theorem 21 and Corollary 22 we conclude that the operators $L_n$ have good preserving properties with respect to the test function $\varphi$. We actually have:

**Corollary 23.** Given $n \in \mathbb{N}$, $f \in \mathbb{R}^H$ and $\alpha \in \mathbb{N}_0^m$, the following holds true:

i) If $f \in \mathcal{P}_{\varphi,\alpha}$ then $L_n f \in \mathcal{P}_{\varphi,\alpha}$.

ii) If $f$ is $\varphi$-convex of order $\alpha$ then $L_n f$ is $\varphi$-convex of order $\alpha$.

As we have done before for the Meyer-König and Zeller operators, given $\alpha, \beta \in \mathbb{N}_0^m$ and $x \in H$, we use a recurrence formula to study the expressions $V_{n,\alpha}(x) = L_n ((\varphi - \varphi(x)^n)(x)$.

**Lemma 24.** Given $\alpha \in \mathbb{N}_0^m$ and $i \in \{1, \ldots, m\}$,

$$V_{n,\alpha+e_i} + \frac{\varphi_i}{1 - |\varphi|} \sum_{s=1}^{m} V_{n,\alpha+e_s} = \frac{\varphi_i}{n+1} \left( \alpha_i V_{n,\alpha-e_i} - \frac{1}{1 - |\varphi|} V_{n,\alpha} + D_{\varphi} V_{n,\alpha} \right),$$

where we assume that $V_{n,\beta} = 0$ whenever $\beta \notin 0$.

**Proof.** If we take $x \in H$ such that $t_i(x) = x_i = 0$ then we have that $V_{n,\alpha+e_i}(x) = 0$ and the result holds. Therefore, we only must prove the lemma for the points $x \in H$ such that $t_i(x) \neq 0$, so that in the rest of the proof we will admit that $t_i \neq 0$.

It is easy to check that

$$(n+1)^\alpha V_{n,\alpha} \circ \varphi^{-1} = \sum_{k \in \mathbb{N}_0^m, |k| \leq n} \binom{n}{k} (k - (n+1)t) \alpha^k (1 - |t|)^{n-|k|}.$$

Let us suppose that $\alpha_i \neq 0$. Then, from the last identity, we have

$$(n+1)^\alpha D_{\varphi} (V_{n,\alpha} \circ \varphi^{-1}) = -\alpha_i (n+1) \sum_{k \in \mathbb{N}_0^m, |k| \leq n} \binom{n}{k} (k - (n+1)t) \alpha^k (1 - |t|)^{n-|k|}$$

$$+ \sum_{k \in \mathbb{N}_0^m, |k| \leq n} \binom{n}{k} (k - (n+1)t) \alpha^k t^{-e_i} (1 - |t|)^{n-|k|}$$

$$- \sum_{k \in \mathbb{N}_0^m, |k| \leq n} \binom{n}{k} (k - (n+1)t) \alpha^k (n-|k|) (1 - |t|)^{n-|k|-1}.$$

Write in the second sum $(k_i - (n+1)t_i + (n+1)t_i)$ instead of $k_i$ and in the third one $(n - \sum_{s=1}^{m} (k_s - (n+1)t_s) - (n+1)|t|)$ in place of $(n - |k|)$ to get

$$(n+1)^\alpha D_{\varphi} (V_{n,\alpha} \circ \varphi^{-1}) = -\alpha_i (n+1)^\alpha V_{n,\alpha-e_i} \circ \varphi^{-1} + (n+1)^{\alpha+1} V_{n,\alpha} \circ \varphi^{-1}$$

$$+ \frac{(n+1)^\alpha}{t_i} V_{n,\alpha+e_i} \circ \varphi^{-1} - \frac{n(n+1)^\alpha}{1 - |t|} V_{n,\alpha} \circ \varphi^{-1}$$

$$+ \frac{(n+1)^{\alpha+1}}{1 - |t|} \sum_{s=1}^{m} V_{n,\alpha+e_s} \circ \varphi^{-1} + \frac{(n+1)^{\alpha+1}|t|}{1 - |t|} V_{n,\alpha} \circ \varphi^{-1},$$

which after being simplified lead us to the identity of the lemma. Almost the same proof runs when $\alpha_i = 0$. □
Theorem 25. Given $\alpha \in \mathbb{N}_0^n$ and $x \in H$, $(n + 1)\alpha V_{n,\alpha}(x) \in \mathbb{P}[\frac{|\alpha|}{2}] [n]$.

Proof. We will prove by induction on $|\alpha|$ following the same scheme that we have used in Theorem 16. For $|\alpha| = 0$ the proof is obvious so that let us suppose that the result is true for any $\beta \in \mathbb{N}_0^m$ with $|\beta| \leq |\alpha|$. Take the matrix

$$A(x) = \left( \begin{array}{ccc}
\frac{\varphi_1(x)}{1 - |\varphi(x)|} & \frac{\varphi_1(x)}{1 - |\varphi(x)|} & \cdots \\
\frac{\varphi_2(x)}{1 - |\varphi(x)|} & \frac{\varphi_2(x)}{1 - |\varphi(x)|} & \cdots \\
\vdots & \vdots & \ddots \\
\frac{\varphi_m(x)}{1 - |\varphi(x)|} & \frac{\varphi_m(x)}{1 - |\varphi(x)|} & \cdots \frac{1 + \varphi_m(x)}{1 - |\varphi(x)|}
\end{array} \right).$$

It is a simple matter that $\det(A(x)) = \frac{1}{1 - |\varphi(x)|} \neq 0$ and from Lemma 24 we have

$$(n + 1)^{|\alpha| + 1} A(x) \left( \begin{array}{c} V_{n,\alpha+e_1}(x) \\
V_{n,\alpha+e_2}(x) \\
\vdots \\
V_{n,\alpha+e_m}(x) \end{array} \right) = (n + 1)^\alpha \left( \begin{array}{c} \varphi_1(x) \left( \alpha_1 V_{n,\alpha+e_1}(x) - \frac{V_{n,\alpha+e_1}(x)}{1 - |\varphi(x)|} \right) \\
\varphi_2(x) \left( \alpha_2 V_{n,\alpha+e_2}(x) - \frac{V_{n,\alpha+e_2}(x)}{1 - |\varphi(x)|} \right) \\
\vdots \\
\varphi_m(x) \left( \alpha_m V_{n,\alpha+e_m}(x) - \frac{V_{n,\alpha+e_m}(x)}{1 - |\varphi(x)|} \right) \end{array} \right).$$

Now we can proceed as in Theorem 16 applying the induction hypotheses to end the proof.

Now we are again in a position to apply, first, Theorem 6 to obtain the asymptotic expansion for the sequence $\{L_n\}_{n \in \mathbb{N}}$ and, later, theorems 7 and 10 for its partial derivatives.

Theorem 26. Let $r \in \mathbb{N}_0$ be an even number and let $f \in B(H, \varphi, r, x)$. It is verified that

$$L_n f(x) = P_{f,0,r,n}(x) + o(n^{-\frac{r}{2}}),$$

being possible to write $P_{f,0,r,n}(x)$ in the following two forms:

$$P_{f,0,r,n}(x) = \sum_{j=0}^{\frac{r}{2}} (n + 1)^{-j} \sum_{\substack{i \in \mathbb{N}_0^m \\
|\alpha| \leq j}} \frac{D^\alpha f(x)}{i!} \sum_{\alpha=0}^{i} \varphi(x)^{i-\alpha} H(j, \alpha, i)$$

(27)

and

$$P_{f,0,r,n}(x) = \sum_{j=0}^{\frac{r}{2}} (n + 1)^{-j} \sum_{\beta \in \mathbb{N}_0^m, |\beta| \leq 2j} \frac{D^\beta f(x)}{\beta!} \sum_{\substack{i \in \mathbb{N}_0^m, |\beta| \leq i \\
|\alpha| \leq j}} \left( \begin{array}{c} i \\
\beta - \alpha \end{array} \right) \sum_{\alpha=0}^{i} (\Pi + x)^{\alpha + \beta} x^{i-\alpha} H(j, \alpha, i),$$

(28)

where

$$H(j, \alpha, i) = \sum_{\beta=0}^{i} (-1)^{i-\beta} \left( \begin{array}{c} i \\
\beta \end{array} \right) S_{[\beta - \alpha] + 1}^{[\beta - \alpha] + 1} \sigma_{\beta - \alpha}.$$
Proof. Given $\beta \in \mathbb{N}_0^m$, since $\Delta^{\alpha}_{\frac{\varphi}{\sqrt{n+1}}} \varphi^\beta(0) = 0$ whenever $\beta \leq \alpha$, by Corollary 22 and identity (5) we have

$$L_n(\varphi^\beta) = \sum_{\alpha = 0}^{\beta} \binom{n}{\alpha} \Delta^{\alpha}_{\frac{\varphi}{\sqrt{n+1}}} \varphi^\beta(0) \varphi^\alpha = \sum_{\alpha = 0}^{\beta} n^{\alpha} \sigma_\beta^\alpha (n + 1)^{-|\beta|} \varphi^\alpha. \quad (29)$$

Besides, $n^{|\alpha|} = \frac{(n+1)^{|\alpha|+1}}{n+1} = \sum_{s=1}^{|\alpha|+1} S_{|\alpha|+1}^s (n+1)^{s-1}$ which joined to (29), by making the change $j = |\beta| + 1 - s$ in the sum for $s$ and modifying the order of the sums, leads us to

$$L_n(\varphi^\beta) = \sum_{j=0}^{|\beta|} (n + 1)^{-j} \sum_{\alpha = 0}^{|\beta| - j} S_{|\alpha|+1}^{|\beta|+1-j} \sigma_\beta^\alpha \varphi^\alpha. \quad (30)$$

By means of Newton’s binomial formula (1), for all $i \in \mathbb{N}_0^m$ we know that $L_n \left( ((\varphi - \varphi(x))^i) \right) (x) = \sum_{\beta=0}^i \binom{i}{\beta} (-\varphi(x))^{i-\beta} L_n(\varphi^\beta)(x)$ and then it is sufficient to use (30), to make the change of index $\alpha = \beta - \alpha$ and to rearrange the order of the sums to write

$$L_n \left( ((\varphi - \varphi(x))^i) \right) (x) = \sum_{j=\left[\frac{|i|+1}{2}\right]}^{|i|} (n + 1)^{-j} \sum_{\alpha = 0}^{|i| - j} \varphi(x)^{i-\alpha} H(j, \alpha, i),$$

where the lower limit of the sum for $j$ is a consequence of Theorem 25. From the above identity it is immediate that $L_n \left( ((\varphi - \varphi(x))^i) \right) (x) = O(n^{-\left[\frac{|i|+1}{2}\right]})$ so that we can apply Theorem 6 to get (27) and then Lemma 20 gives us (28). \qed

The results proved in this section about the moments and shape preserving properties and the theorems of Section 3 make possible to study the derivatives of the sequence $\{L_n\}_{n \in \mathbb{N}}$ in terms of the expansion $P_{f,0,r,n}$ that we have just computed in the theorem above.

**Theorem 27.** Let $r \in \mathbb{N}$ be an even number, let $k \in \mathbb{N}_0^m$, $x \in H$ and let $f \in C^{k}(H)$ be such that $D^k_{\varphi} f \in B(H, \varphi, r, x)$ for all $i \in \mathbb{N}_0^m$ with $|i| \leq k$. Then,

$$D^k_{\varphi} L_n f(x) = D^k_{\varphi} P_{f,0,r,n}(x) + o(n^{-\frac{r}{2}}), \quad (31)$$

$$D^k L_n f(x) = D^k P_{f,0,r,n}(x) + o(n^{-\frac{r}{2}}). \quad (32)$$

Furthermore, we have

$$D^k_{\varphi} P_{f,0,r,n}(x) = \sum_{j=0}^\frac{r}{2} (n + 1)^{-j} \sum_{\gamma \in \mathbb{N}_0^m \ k \leq \gamma \ j \leq \gamma - \gamma \ j \leq \gamma \ i \leq \gamma} D^k_{\varphi} f(x) \sum_{\alpha = 0}^{\gamma-k} \varphi(x)^{\gamma-\alpha-k} \times$$

$$\times \sum_{i=\gamma-k}^{\gamma} \frac{H(j, \alpha, i)}{i!} \binom{k}{\gamma-i} \frac{(i-\alpha)!}{(\gamma-\alpha-k)!}.$$
Let us take every $D$ of Theorem 7. Furthermore, for the conditions Corollary 28. which ends the proof after simplifying and putting the sums in the suitable order.

where $H(j, \alpha, i)$ is defined as in Theorem 26.

**Proof.** Let us take $W$ as the set of all bounded functions on $H$. It is immediate that the domain $H, W$ and the test function $\varphi$ chosen for the sequence $\{L_n\}_{n \in \mathbb{N}}$, verify the hypotheses of Theorem 7. Furthermore, for the conditions i), ii) and iii) of such a theorem we have that from Corollary 23 follows i) and Theorem 25 shows that, taking $\phi(n) = n + 1$, ii) also holds. Finally, i) is immediate. Therefore, we can apply Theorem 7 to prove (31). If we take $\varphi_1 = t$, as in Theorem 18, (32) follows from Theorem 10.

Again, we proceed analogously to the proof of Theorem 18 by using Leibnitz’s formula to explicitly compute $D^k f_{j,0,r,n}(x)$ (in this case we have also made a translation for the index $\gamma$).

For $D^k f_{j,0,r,n}(x)$ we differentiate in (28) also with the aid of Leibnitz’s formula to compute

$$D^k f_{j,0,r,n}(x) = k! \sum_{j=0}^{k} (n+1)^{-j} \sum_{\beta \in \mathbb{N}_0^n, \gamma_1=0}^{k} \frac{D^{\beta+\gamma_1} f(x)}{\beta!\gamma_1!} \sum_{i \in \mathbb{N}_0^n, \beta \leq i} \sum_{j \leq i, |\alpha| \leq j} \sum_{\alpha=0}^{i} (i-\frac{\beta}{\alpha-i+\gamma_1+\gamma_2-k}},$$

which ends the proof after simplifying and putting the sums in the suitable order. \hfill \Box

Now we can obtain the Voronovskaja formulae for the partial derivatives of the sequence $\{L_n\}_{n \in \mathbb{N}}$. As in Corollary 19 we give a simplified version of the proof.

**Corollary 28.** Given $\alpha \in \mathbb{N}_0^n$, $x \in H$ and $f \in C^{[\alpha]}(H)$ such that $D^\alpha f \in B(H, \varphi, 2, x)$ for every $i \in \mathbb{N}_0^n$ such that $|i| \leq |\alpha|$, we have

$$\lim_{n \to \infty} 2(n+1) (D^\alpha L_n f(x) - D^\alpha f(x)) =$$

$$= \left(2|\alpha|^2 + 3|x\alpha^2| - (|\alpha| + 2|\alpha x|)^2 + \alpha^2(1 + 2x)^2\right) D^\alpha f(x)$$

$$+ \sum_{i=1}^{m} \left[x_i(1 + x_i)^2 D^{\alpha + 2e_i} f(x) + (\alpha_i(1 + 6x_i + 9x_i^2 + 4x_i^3) - 2(x_i + x_i^2)(|\alpha| - 2|\alpha x|)) D^{\alpha + e_i} f(x) + (2^3 - 2\alpha_i^2(|\alpha||2 + 2x| - \alpha_i(1 + 2x_i)) D^{\alpha - e_i} f(x) \right]$$

29
\[
- \sum_{i,j=1 \atop i \neq j}^{m} x_i x_j (1 + x_i)(1 + x_j) D^{\alpha + \epsilon_i + \epsilon_j} f(x)
+ 2 \alpha_i^2 x_i (1 + x_i) D^{\alpha - \epsilon_i + \epsilon_j} f(x) + \alpha_j^2 \alpha_i^2 D^{\alpha - \epsilon_i - \epsilon_j} f(x)
\]

Proof. By means of Theorem 26, it can be proved that

\[
P_{f,0,2,n}(x) = f(x) + \frac{1}{2(n+1)} \sum_{i=1}^{m} x_i (1+x_i)^2 D^{2\epsilon_i} f(x) - \frac{1}{n+1} \sum_{1 \leq i < j}^{m} x_i x_j (1+x_i)(1+x_j) D^{\epsilon_i + \epsilon_j} f(x)
\]

and (32) guarantees that \( P_{f,k,2,n}(x) = D^k P_{f,0,2,n}(x) \) so it suffices to differentiate by means of Leibnitz formula to obtain the result.

Remark 29. Notice that \( f \in B(H, \varphi, r) \) implies that \( f \) is a bounded function. Abel [3] obtain the result in the case \( m = 2 \) for the non differentiated operators when the functions has polynomial-type growth of certain order. Anyway, the explicit formula that we have obtained for these operators, in the case \( m = 2 \), is more simple than the one given by Abel.

References


