On weak compactness in Lebesgue-Bochner spaces

José Rodríguez

Universidad de Murcia

Operators and Banach lattices
Madrid – October 25th, 2012
Strong generation by weakly compact sets

Let $X$ be a Banach space and $B_X = \{x \in X : \|x\| \leq 1\}$.
Let $X$ be a Banach space and $B_X = \{x \in X : \|x\| \leq 1\}$.

**Definition (Schlüchtermann-Wheeler 1988)**

$X$ is called **strongly weakly compactly generated (SWCG)** if there is a weakly compact set $G \subset X$ such that: for every weakly compact set $K \subset X$ and $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that $K \subset nG + \varepsilon B_X$. 

---

Thanks to the Davis-Figiel-Johnson-Pelczyński theorem, $X$ is SWCG $\iff$ $X$ is strongly generated by a reflexive Banach space $Y$, i.e. there is an operator $T : Y \to X$ such that: for every weakly compact set $K \subset X$ and $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that $K \subset nT(B_Y) + \varepsilon B_X$.

**Known facts (Schlüchtermann-Wheeler)**

1. Reflexive $\implies$ SWCG $\implies$ Weakly sequentially complete
2. Weakly sequentially complete $+$ separable $\not\implies$ SWCG
3. Schur $+$ separable $\implies$ SWCG
4. $L^1(\mu)$ is SWCG for any probability measure $\mu$. 

---

Strong generation by weakly compact sets
Let $X$ be a Banach space and $B_X = \{x \in X : \|x\| \leq 1\}$.

**Definition (Schlüchtermann-Wheeler 1988)**

$X$ is called **strongly weakly compactly generated (SWCG)** if there is a weakly compact set $G \subset X$ such that: for every weakly compact set $K \subset X$ and $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that $K \subset nG + \varepsilon B_X$.

**Thanks to the Davis-Figiel-Johnson-Pełczyński theorem...**

$X$ is SWCG $\iff$ $X$ is strongly generated by a reflexive Banach space $Y$, i.e. there is an operator $T : Y \to X$ such that: for every weakly compact set $K \subset X$ and $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that $K \subset nT(B_Y) + \varepsilon B_X$.
Let $X$ be a Banach space and $B_X = \{ x \in X : \| x \| \leq 1 \}$.

**Definition (Schlüchtermann-Wheeler 1988)**

$X$ is called **strongly weakly compactly generated (SWCG)** if there is a weakly compact set $G \subset X$ such that: for every weakly compact set $K \subset X$ and $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that $K \subset nG + \varepsilon B_X$.

**Thanks to the Davis-Figiel-Johnson-Pełczyński theorem...**

$X$ is SWCG $\iff$ $X$ is strongly generated by a reflexive Banach space $Y$, i.e. there is an operator $T : Y \to X$ such that: for every weakly compact set $K \subset X$ and $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that $K \subset nT(B_Y) + \varepsilon B_X$.
Strong generation by weakly compact sets

Let \( X \) be a Banach space and \( B_X = \{ x \in X : \|x\| \leq 1 \} \).

**Definition (Schlüchtermann-Wheeler 1988)**

\( X \) is called **strongly weakly compactly generated (SWCG)** if there is a weakly compact set \( G \subset X \) such that: for every weakly compact set \( K \subset X \) and \( \varepsilon > 0 \) there is \( n \in \mathbb{N} \) such that \( K \subset nG + \varepsilon B_X \).

**Thanks to the Davis-Figiel-Johnson-Pełczyński theorem...**

\( X \) is SWCG \( \iff \) \( X \) is **strongly generated** by a reflexive Banach space \( Y \), i.e. there is an operator \( T : Y \rightarrow X \) such that: for every weakly compact set \( K \subset X \) and \( \varepsilon > 0 \) there is \( n \in \mathbb{N} \) such that \( K \subset nT(B_Y) + \varepsilon B_X \).

**Known facts (Schlüchtermann-Wheeler)**

1. reflexive \( \implies \) SWCG \( \implies \) weakly sequentially complete
Let $X$ be a Banach space and $B_X = \{x \in X : \|x\| \leq 1\}$.

**Definition (Schlüchtermann-Wheeler 1988)**

$X$ is called strongly weakly compactly generated (SWCG) if there is a weakly compact set $G \subset X$ such that: for every weakly compact set $K \subset X$ and $\epsilon > 0$ there is $n \in \mathbb{N}$ such that $K \subset nG + \epsilon B_X$.

Thanks to the Davis-Figiel-Johnson-Pełczyński theorem...

$X$ is SWCG $\iff$ $X$ is strongly generated by a reflexive Banach space $Y$, i.e. there is an operator $T : Y \rightarrow X$ such that: for every weakly compact set $K \subset X$ and $\epsilon > 0$ there is $n \in \mathbb{N}$ such that $K \subset nT(B_Y) + \epsilon B_X$.

**Known facts (Schlüchtermann-Wheeler)**

1. reflexive $\Rightarrow$ SWCG $\Rightarrow$ weakly sequentially complete
2. weakly sequentially complete $+$ separable $\not \Rightarrow$ SWCG
Let $X$ be a Banach space and $B_X = \{x \in X : \|x\| \leq 1\}$.

Definition (Schlüchtermann-Wheeler 1988)

$X$ is called **strongly weakly compactly generated (SWCG)** if there is a weakly compact set $G \subset X$ such that: for every weakly compact set $K \subset X$ and $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that $K \subset nG + \varepsilon B_X$.

Thanks to the Davis-Figiel-Johnson-Pełczyński theorem...

$X$ is SWCG $\iff$ $X$ is strongly generated by a reflexive Banach space $Y$, i.e. there is an operator $T : Y \to X$ such that: for every weakly compact set $K \subset X$ and $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that $K \subset nT(B_Y) + \varepsilon B_X$.

Known facts (Schlüchtermann-Wheeler)

1. reflexive $\Rightarrow$ SWCG $\Rightarrow$ weakly sequentially complete
2. weakly sequentially complete + separable $\not\Rightarrow$ SWCG
3. Schur + separable $\Rightarrow$ SWCG
Let $X$ be a Banach space and $B_X = \{x \in X : \|x\| \leq 1\}$.

**Definition (Schlüchtermann-Wheeler 1988)**

$X$ is called **strongly weakly compactly generated (SWCG)** if there is a weakly compact set $G \subset X$ such that: *for every weakly compact set $K \subset X$ and $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that $K \subset nG + \varepsilon B_X$.*

**Thanks to the Davis-Figiel-Johnson-Pełczyński theorem...**

$X$ is SWCG $\iff$ $X$ is strongly generated by a reflexive Banach space $Y$, i.e. there is an operator $T : Y \to X$ such that: *for every weakly compact set $K \subset X$ and $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that $K \subset nT(B_Y) + \varepsilon B_X$.*

**Known facts (Schlüchtermann-Wheeler)**

1. reflexive $\implies$ SWCG $\implies$ weakly sequentially complete
2. weakly sequentially complete $+$ separable $\not\implies$ SWCG
3. Schur $+$ separable $\implies$ SWCG
4. $L^1(\mu)$ is SWCG for any probability measure $\mu$. 

$L^1(\mu)$ is SWCG for any probability measure $\mu$

We shall check that:

$L^1(\mu)$ is strongly generated by $L^2(\mu)$.

Proof: 

1. Let $T: L^2(\mu) \rightarrow L^1(\mu)$ be the identity operator.
2. Fix a weakly compact set $K \subset L^1(\mu)$ and $\varepsilon > 0$.
3. Since $K$ is equi-integrable, there is $\delta > 0$ such that $\mu(A) \leq \delta = \varepsilon$ for every $f \in K$.
4. Pick $n \in \mathbb{N}$ such that $\|f\|_1 \leq \delta$ for every $f \in K$.
5. For every $f \in K$ we can write $f = f_1 \{\{|f| \leq n\} + f_1 \{|f| > n\}$, where $f_1 \{\{|f| \leq n\} \in nT(BL^2(\mu))$ and $f_1 \{|f| > n\} \in \varepsilon BL^1(\mu)$.
6. Hence $K \subset nT(BL^2(\mu)) + \varepsilon BL^1(\mu)$. 
We shall check that:

\[ L^1(\mu) \text{ is strongly generated by } L^2(\mu). \]

**Proof:**

1. Let \( T : L^2(\mu) \to L^1(\mu) \) be the identity operator.
$L^1(\mu)$ is SWCG for any probability measure $\mu$

We shall check that:

$L^1(\mu)$ is strongly generated by $L^2(\mu)$.

**Proof:**

1. Let $T : L^2(\mu) \to L^1(\mu)$ be the identity operator.
2. Fix a weakly compact set $K \subset L^1(\mu)$ and $\varepsilon > 0$. 
We shall check that:

$L^1(\mu)$ is strongly generated by $L^2(\mu)$.

**Proof:**

1. Let $T : L^2(\mu) \to L^1(\mu)$ be the identity operator.
2. Fix a weakly compact set $K \subset L^1(\mu)$ and $\varepsilon > 0$.
3. Since $K$ is equi-integrable, there is $\delta > 0$ such that

\[
\mu(A) \leq \delta \implies \int_A |f| \, d\mu \leq \varepsilon \text{ for every } f \in K.
\]
$L^1(\mu)$ is SWCG for any probability measure $\mu$

We shall check that:

$L^1(\mu)$ is strongly generated by $L^2(\mu)$.

**Proof:**

1. Let $T : L^2(\mu) \to L^1(\mu)$ be the identity operator.
2. Fix a weakly compact set $K \subset L^1(\mu)$ and $\varepsilon > 0$.
3. Since $K$ is equi-integrable, there is $\delta > 0$ such that
   \[
   \mu(A) \leq \delta \implies \int_A |f| \, d\mu \leq \varepsilon \quad \text{for every } f \in K.
   \]
4. Pick $n \in \mathbb{N}$ such that $\frac{\|f\|_1}{n} \leq \delta$ for every $f \in K$. 
$L^1(\mu)$ is SWCG for any probability measure $\mu$

We shall check that:

$L^1(\mu)$ is strongly generated by $L^2(\mu)$.

**Proof:**

1. Let $T : L^2(\mu) \to L^1(\mu)$ be the identity operator.
2. Fix a weakly compact set $K \subset L^1(\mu)$ and $\varepsilon > 0$.
3. Since $K$ is equi-integrable, there is $\delta > 0$ such that
   
   $\mu(A) \leq \delta \implies \int_A |f| \, d\mu \leq \varepsilon$ for every $f \in K$.
4. Pick $n \in \mathbb{N}$ such that $\frac{\|f\|_1}{n} \leq \delta$ for every $f \in K$.
5. For every $f \in K$ we can write $f = f\mathbb{1}_{\{|f| \leq n\}} + f\mathbb{1}_{\{|f| > n\}},$
$L^1(\mu)$ is SWCG for any probability measure $\mu$

We shall check that:

$L^1(\mu)$ is strongly generated by $L^2(\mu)$.

Proof:

1. Let $T : L^2(\mu) \to L^1(\mu)$ be the identity operator.
2. Fix a weakly compact set $K \subset L^1(\mu)$ and $\varepsilon > 0$.
3. Since $K$ is equi-integrable, there is $\delta > 0$ such that
   
   \[ \mu(A) \leq \delta \implies \int_A |f| \, d\mu \leq \varepsilon \text{ for every } f \in K. \]

4. Pick $n \in \mathbb{N}$ such that $\frac{\|f\|_1}{n} \leq \delta$ for every $f \in K$.
5. For every $f \in K$ we can write $f = f \mathbb{1}_{\{|f| \leq n\}} + f \mathbb{1}_{\{|f| > n\}}$, where
   
   $f \mathbb{1}_{\{|f| \leq n\}} \in nT(B_{L^2(\mu)})$ and $f \mathbb{1}_{\{|f| > n\}} \in \varepsilon B_{L^1(\mu)}$. 
$L^1(\mu)$ is SWCG for any probability measure $\mu$

We shall check that:

$L^1(\mu)$ is strongly generated by $L^2(\mu)$.

Proof:

1. Let $T : L^2(\mu) \to L^1(\mu)$ be the identity operator.
2. Fix a weakly compact set $K \subset L^1(\mu)$ and $\varepsilon > 0$.
3. Since $K$ is equi-integrable, there is $\delta > 0$ such that
   $$\mu(A) \leq \delta \implies \int_A |f| \, d\mu \leq \varepsilon$$
   for every $f \in K$.
4. Pick $n \in \mathbb{N}$ such that $\frac{\|f\|_1}{n} \leq \delta$ for every $f \in K$.
5. For every $f \in K$ we can write $f = f 1_{\{|f| \leq n\}} + f 1_{\{|f| > n\}}$, where
   $$f 1_{\{|f| \leq n\}} \in nT(B_{L^2(\mu)}) \quad \text{and} \quad f 1_{\{|f| > n\}} \in \varepsilon B_{L^1(\mu)}.$$
6. Hence $K \subset nT(B_{L^2(\mu)}) + \varepsilon B_{L^1(\mu)}$. \qed
Let $X$ be a Banach space, $(\Omega, \Sigma, \mu)$ a probability space and $L^1(\mu, X)$ the Banach space of all (classes of) Bochner integrable functions $f : \Omega \to X$. 

Theorem (Diestel-Ruess-Schachermayer 1993)

A set $C \subset L^1(\mu, X)$ is relatively weakly compact if and only if

1. $C$ is equi-integrable and bounded;
2. for every sequence $(f_n) \subset C$ there exist $g_n \in \text{co}\{f_k : k \geq n\}$ such that $(g_n(\omega))$ is weakly (resp. norm) convergent in $X$ for $\mu$-a.e. $\omega \in \Omega$. 

Known facts

1. $L^\infty(\mu, X^*) \subset L^1(\mu, X^*)$ in the natural way.
2. $L^\infty(\mu, X^*) = L^1(\mu, X^*)$ if and only if $X^*$ has the Radon-Nikodým property with respect to $\mu$. 

In $L^1(\mu, X)$ we have:

relatively weakly compact $\Rightarrow$ equi-integrable and bounded.

The converse implication holds if and only if $X$ is reflexive.
Weak compactness in $L^1(\mu, X)$

Let $X$ be a Banach space, $(\Omega, \Sigma, \mu)$ a probability space and $L^1(\mu, X)$ the Banach space of all (classes of) Bochner integrable functions $f : \Omega \rightarrow X$.

**Known facts**

1. $L^\infty(\mu, X^*) \subset L^1(\mu, X)^*$ in the natural way.
Weak compactness in $L^1(\mu, X)$

Let $X$ be a Banach space, $(\Omega, \Sigma, \mu)$ a probability space and $L^1(\mu, X)$ the Banach space of all (classes of) Bochner integrable functions $f : \Omega \to X$.

**Known facts**

1. $L^\infty(\mu, X^*) \subset L^1(\mu, X)^*$ in the natural way.
   
   $L^\infty(\mu, X^*) = L^1(\mu, X)^* \iff X^*$ has the Radon-Nikodým property wrt $\mu$. 
Weak compactness in $L^1(\mu, X)$

Let $X$ be a Banach space, $(\Omega, \Sigma, \mu)$ a probability space and $L^1(\mu, X)$ the Banach space of all (classes of) Bochner integrable functions $f : \Omega \to X$.

Known facts

1. $L^\infty(\mu, X^*) \subset L^1(\mu, X)^*$ in the natural way.
   
   $L^\infty(\mu, X^*) = L^1(\mu, X)^* \iff X^*$ has the Radon-Nikodým property wrt $\mu$.

2. In $L^1(\mu, X)$ we have:
   
   relatively weakly compact $\implies$ equi-integrable and bounded.
Weak compactness in $L^1(\mu, X)$

Let $X$ be a Banach space, $(\Omega, \Sigma, \mu)$ a probability space and $L^1(\mu, X)$ the Banach space of all (classes of) Bochner integrable functions $f : \Omega \to X$.

**Known facts**

1. $L^\infty(\mu, X^*) \subset L^1(\mu, X)^*$ in the natural way.
   \[
   L^\infty(\mu, X^*) = L^1(\mu, X)^* \iff X^* \text{ has the Radon-Nikodým property wrt } \mu.
   \]

2. In $L^1(\mu, X)$ we have:
   
   relatively weakly compact $\implies$ equi-integrable and bounded.

   The converse implication holds if and only if $X$ is reflexive.
Weak compactness in $L^1(\mu,X)$

Let $X$ be a Banach space, $(\Omega, \Sigma, \mu)$ a probability space and $L^1(\mu,X)$ the Banach space of all (classes of) Bochner integrable functions $f : \Omega \to X$.

**Known facts**

1. $L^\infty(\mu,X^*) \subset L^1(\mu,X)^*$ in the natural way.
   \[ L^\infty(\mu,X^*) = L^1(\mu,X)^* \iff X^* \text{ has the Radon-Nikodým property wrt } \mu. \]

2. In $L^1(\mu,X)$ we have:
   
   relatively weakly compact $\implies$ equi-integrable and bounded.

   The converse implication holds if and only if $X$ is reflexive.

**Theorem (Diestel-Ruess-Schachermayer 1993)**

A set $C \subset L^1(\mu,X)$ is **relatively weakly compact** if and only if

1. $C$ is equi-integrable and bounded;

2. for every sequence $(f_n) \subset C$ there exist $g_n \in \text{co}\{f_k : k \geq n\}$ such that $(g_n(\omega))$ is weakly (resp. norm) convergent in $X$ for $\mu$-a.e. $\omega \in \Omega$. 
Strong generation by weakly compact sets in $L^1(\mu, X)$

**QUESTION (Schlüchtermann-Wheeler)**

$X$ is SWCG $\Rightarrow L^1(\mu, X)$ is SWCG
Strong generation by weakly compact sets in $L^1(\mu, X)$

**QUESTION (Schlüchtermann-Wheeler)**

$X$ is SWCG $\iff L^1(\mu, X)$ is SWCG

**Affirmative answer in the cases:**

- $X = L^1(\nu)$ for any probability $\nu$. 

Remarks

1. $L^1(\mu, X)$ is strongly generated by $L^2(\mu, X)$.

2. $X$ is WCG $\Rightarrow L^1(\mu, X)$ is WCG. (Diestel 1975)

Proof:

$X$ is WCG $\Rightarrow$ there exist a reflexive Banach space $Y$ and an operator $T: Y \to X$ with dense range (DFJP theorem).

Then $\tilde{T}: L^2(\mu, Y) \to L^1(\mu, X)$ $\tilde{T}(f) = T \circ f$ is an operator with dense range, and $L^2(\mu, Y)$ is reflexive because $Y$ is.

Theorem (Talagrand 1984)

$L^1(\mu, X)$ is weakly sequentially complete if $X$ is.
Strong generation by weakly compact sets in $L^1(\mu, X)$

**QUESTION (Schlüchtermann-Wheeler)**

$X$ is SWCG $\Rightarrow L^1(\mu, X)$ is SWCG

**Affirmative answer in the cases:**

- $X = L^1(\nu)$ for any probability $\nu$.

**Remarks**

1. $L^1(\mu, X)$ is strongly generated by $L^2(\mu, X)$. 

---

**Proof:**

$X$ is WCG $\Rightarrow$ there exist a reflexive Banach space $Y$ and an operator $T: Y \to X$ with dense range (DFJP theorem).

Then $\tilde{T}: L^2(\mu, Y) \to L^1(\mu, X)$ $\tilde{T}(f) = T \circ f$ is an operator with dense range, and $L^2(\mu, Y)$ is reflexive because $Y$ is.

**Theorem (Talagrand 1984)**

$L^1(\mu, X)$ is weakly sequentially complete if $X$ is.
Strong generation by weakly compact sets in $L^1(\mu, X)$

**QUESTION (Schlüchtermann-Wheeler)**

$X$ is SWCG $\iff L^1(\mu, X)$ is SWCG

**Affirmative answer in the cases:**

- $X = L^1(\nu)$ for any probability $\nu$.
- $X$ is reflexive.

**Remarks**

1. $L^1(\mu, X)$ is strongly generated by $L^2(\mu, X)$.  

Proof: $X$ is WCG $\iff$ there exist a reflexive Banach space $Y$ and an operator $T: Y \to X$ with dense range (DFJP theorem). Then $\tilde{T}: L^2(\mu, Y) \to L^1(\mu, X)$ $\tilde{T}(f) = T \circ f$ is an operator with dense range, and $L^2(\mu, Y)$ is reflexive because $Y$ is.

**Theorem (Talagrand 1984)**

$L^1(\mu, X)$ is weakly sequentially complete if $X$ is.
Strong generation by weakly compact sets in $L^1(\mu, X)$

**QUESTION (Schlüchtermann-Wheeler)**

$X$ is SWCG $\iff$ $L^1(\mu, X)$ is SWCG

**Affirmative answer in the cases:**

- $X = L^1(\nu)$ for any probability $\nu$.
- $X$ is reflexive.

**Remarks**

1. $L^1(\mu, X)$ is strongly generated by $L^2(\mu, X)$.

2. $X$ is WCG $\implies$ $L^1(\mu, X)$ is WCG. (Diestel 1975)

**Proof:**
QUESTION (Schlüchtermann-Wheeler)

\[ X \text{ is SWCG} \iff L^1(\mu, X) \text{ is SWCG} \]

Affirmative answer in the cases:

- \( X = L^1(\nu) \) for any probability \( \nu \).
- \( X \) is reflexive.

Remarks

1. \( L^1(\mu, X) \) is strongly generated by \( L^2(\mu, X) \).
2. \( X \) is WCG \implies L^1(\mu, X) \) is WCG. (Diestel 1975)

**Proof:**

\( X \) is WCG \implies there exist a reflexive Banach space \( Y \) and an operator \( T : Y \to X \) with dense range (DFJP theorem).
Strong generation by weakly compact sets in $L^1(\mu, X)$

**QUESTION (Schlüchtermann-Wheeler)**

$X$ is SWCG $\iff L^1(\mu, X)$ is SWCG

**Affirmative answer in the cases:**

- $X = L^1(\nu)$ for any probability $\nu$.
- $X$ is reflexive.

**Remarks**

1. $L^1(\mu, X)$ is strongly generated by $L^2(\mu, X)$.

2. $X$ is WCG $\implies L^1(\mu, X)$ is WCG. (Diestel 1975)

**Proof:**

$X$ is WCG $\implies$ there exist a reflexive Banach space $Y$ and an operator $T : Y \to X$ with dense range (DFJP theorem). Then

$$\tilde{T} : L^2(\mu, Y) \to L^1(\mu, X)$$

$$\tilde{T}(f) = T \circ f$$

is an operator with dense range,
**Strong generation by weakly compact sets in** \( L^1(\mu, X) \)

**QUESTION (Schlüchtermann-Wheeler)**

\( X \) is SWCG \( \iff \) \( L^1(\mu, X) \) is SWCG

**Affirmative answer in the cases:**

- \( X = L^1(\nu) \) for any probability \( \nu \).
- \( X \) is reflexive.

**Remarks**

1. \( L^1(\mu, X) \) is strongly generated by \( L^2(\mu, X) \).

2. \( X \) is WCG \( \implies \) \( L^1(\mu, X) \) is WCG. (Diestel 1975)

**Proof:**

\( X \) is WCG \( \implies \) there exist a reflexive Banach space \( Y \) and an operator \( T : Y \to X \) with dense range (DFJP theorem). Then

\[
\tilde{T} : L^2(\mu, Y) \to L^1(\mu, X) \quad \tilde{T}(f) = T \circ f
\]

is an operator with dense range, and \( L^2(\mu, Y) \) is reflexive because \( Y \) is.
Strong generation by weakly compact sets in $L^1(\mu, X)$

**QUESTION (Schlüchtermann-Wheeler)**

$X$ is **SWCG** $\iff L^1(\mu, X)$ is **SWCG**

**Affirmative answer in the cases:**

- $X = L^1(\nu)$ for any probability $\nu$.
- $X$ is reflexive.

**Remarks**

1. $L^1(\mu, X)$ is strongly generated by $L^2(\mu, X)$.

2. $X$ is **WCG** $\implies L^1(\mu, X)$ is **WCG**. (Diestel 1975)

**Proof:**

$X$ is **WCG** $\implies$ there exist a reflexive Banach space $Y$ and an operator $T : Y \to X$ with dense range (DFJP theorem). Then

$$\tilde{T} : L^2(\mu, Y) \to L^1(\mu, X), \quad \tilde{T}(f) = T \circ f$$

is an operator with dense range, and $L^2(\mu, Y)$ is reflexive because $Y$ is.

**Theorem (Talagrand 1984)**

$L^1(\mu, X)$ is weakly sequentially complete if $X$ is.
Theorem (Lajara-R. 2012)

If \( X \) is SWCG, then there is a weakly compact set \( G \subset L^1(\mu, X) \) such that:

\[
\text{for every decomposable weakly compact set } K \subset L^1(\mu, X) \text{ and } \varepsilon > 0 \text{ there is } n \in \mathbb{N} \text{ such that } K \subset nG + \varepsilon B_{L^1(\mu, X)}.
\]

Decomposable means: 
\[ f_1 A + g_1 \Omega \setminus A \in K \]
for every \( f, g \in K \) and \( A \in \Sigma \).

\[
\text{A typical decomposable set } L(W) = \{ f \in L^1(\mu, X) : f(\omega) \in W \text{ for } \mu\text{-a.e. } \omega \in \Omega \},
\]
where \( W \subset X \).

\([\text{Diestel 1977}]
\]

1. \( W \) is relatively weakly compact \( \implies L(W) \) is relatively weakly compact.

2. \( W \) weakly compact and convex \( \implies L(W) \) is weakly compact and convex.

\( \Rightarrow \) Decomposable sets arise as collections of selectors of set-valued functions.
Theorem (Lajara-R. 2012)

If $X$ is SWCG, then there is a weakly compact set $G \subset L^1(\mu, X)$ such that:

for every decomposable weakly compact set $K \subset L^1(\mu, X)$ and $\varepsilon > 0$
there is $n \in \mathbb{N}$ such that $K \subset nG + \varepsilon B_{L^1(\mu, X)}$.

Decomposable means: $f \mathbb{1}_A + g \mathbb{1}_{\Omega \setminus A} \in K$ for every $f, g \in K$ and $A \in \Sigma$. 
A partial answer

Theorem (Lajara-R. 2012)

If $X$ is SWCG, then there is a weakly compact set $G \subset L^1(\mu, X)$ such that:

for every decomposable weakly compact set $K \subset L^1(\mu, X)$ and $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that $K \subset nG + \varepsilon B_{L^1(\mu, X)}$.

Decomposable means: $f \mathbb{1}_A + g \mathbb{1}_{\Omega \setminus A} \in K$ for every $f, g \in K$ and $A \in \Sigma$.

A typical decomposable set

$L(W) = \left\{ f \in L^1(\mu, X) : f(\omega) \in W \text{ for } \mu\text{-a.e. } \omega \in \Omega \right\},$ where $W \subset X$. 
Theorem (Lajara-R. 2012)

If $X$ is SWCG, then there is a weakly compact set $G \subset L^1(\mu, X)$ such that:

for every decomposable weakly compact set $K \subset L^1(\mu, X)$ and $\varepsilon > 0$
there is $n \in \mathbb{N}$ such that $K \subset nG + \varepsilon B_{L^1(\mu, X)}$.

Decomposable means: $f \mathbf{1}_A + g \mathbf{1}_{\Omega \setminus A} \in K$ for every $f, g \in K$ and $A \in \Sigma$.

A typical decomposable set

$L(W) = \left\{ f \in L^1(\mu, X) : f(\omega) \in W \text{ for } \mu\text{-a.e. } \omega \in \Omega \right\}$, where $W \subset X$.

(Diestel 1977)

1. $W$ is relatively weakly compact $\implies$ $L(W)$ is relatively weakly compact.
Theorem (Lajara-R. 2012)

If $X$ is SWCG, then there is a weakly compact set $G \subset L^1(\mu, X)$ such that:

for every decomposable weakly compact set $K \subset L^1(\mu, X)$ and $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that $K \subset nG + \varepsilon B_{L^1(\mu, X)}$.

Decomposable means: $f\mathbf{1}_A + g\mathbf{1}_{\Omega \setminus A} \in K$ for every $f, g \in K$ and $A \in \Sigma$.

A typical decomposable set

$L(W) = \left\{ f \in L^1(\mu, X) : f(\omega) \in W \text{ for } \mu\text{-a.e. } \omega \in \Omega \right\}$, where $W \subset X$.

(Diestel 1977)

1. $W$ is relatively weakly compact $\implies L(W)$ is relatively weakly compact.
2. $W$ weakly compact and convex $\implies L(W)$ is weakly compact and convex.
Theorem (Lajara-R. 2012)

If \( X \) is SWCG, then there is a weakly compact set \( G \subset L^1(\mu, X) \) such that:

\[
\text{for every decomposable weakly compact set } K \subset L^1(\mu, X) \text{ and } \varepsilon > 0 \text{ there is } n \in \mathbb{N} \text{ such that } K \subset nG + \varepsilon B_{L^1(\mu, X)}. 
\]

**Decomposable** means: \( f \mathbb{1}_A + g \mathbb{1}_{\Omega \setminus A} \in K \) for every \( f, g \in K \) and \( A \in \Sigma \).

A typical decomposable set

\[
L(W) = \left\{ f \in L^1(\mu, X) : f(\omega) \in W \text{ for } \mu\text{-a.e. } \omega \in \Omega \right\}, \text{ where } W \subset X.
\]

(Diestel 1977)

1. \( W \) is relatively weakly compact \( \implies \) \( L(W) \) is relatively weakly compact.
2. \( W \) weakly compact and convex \( \implies \) \( L(W) \) is weakly compact and convex.

▶ Decomposable sets arise as collections of selectors of set-valued functions.
Decomposability and set-valued functions

Let \( cl(X) = \{ C \subset X : C \text{ is closed and nonempty} \} \).
Let $cl(X) = \{ C \subset X : C \text{ is closed and nonempty} \}$. For any set-valued function $F : \Omega \rightarrow 2^X$ the set

$$S_F^1 = \left\{ f \in L^1(\mu, X) : f(\omega) \in F(\omega) \text{ for } \mu\text{-a.e. } \omega \in \Omega \right\}$$

is decomposable.
Let \( cl(X) = \{ C \subset X : C \text{ is closed and nonempty} \} \).

For any set-valued function \( F : \Omega \to 2^X \) the set

\[
S^1_F = \left\{ f \in L^1(\mu, X) : f(\omega) \in F(\omega) \text{ for } \mu\text{-a.e. } \omega \in \Omega \right\}
\]

is decomposable. Conversely...

**Theorem (Hiai-Umegaki 1977) – assuming that \( X \) is separable**

Let \( D \subset L^1(\mu, X) \) be a decomposable closed nonempty set. Then there is a measurable \( F : \Omega \to cl(X) \) such that

\[
D = S^1_F
\]
Decomposability and set-valued functions

Let \( cl(X) = \{ C \subset X : C \text{ is closed and nonempty} \} \).

For any set-valued function \( F : \Omega \to 2^X \) the set

\[
S_F^1 = \left\{ f \in L^1(\mu, X) : f(\omega) \in F(\omega) \text{ for } \mu\text{-a.e. } \omega \in \Omega \right\}
\]

is decomposable. Conversely...

---

**Theorem (Hiai-Umegaki 1977) – assuming that \( X \) is separable**

Let \( D \subset L^1(\mu, X) \) be a decomposable closed nonempty set. Then there is a measurable \( F : \Omega \to cl(X) \) such that

\[
D = S_F^1
\]

**Measurable** means: \( \{ \omega \in \Omega : F(\omega) \cap U \neq \emptyset \} \in \Sigma \) for every open set \( U \subset X \).
Decomposability and set-valued functions

Let \( cl(X) = \{ C \subset X : C \text{ is closed and nonempty} \} \).
For any set-valued function \( F : \Omega \rightarrow 2^X \) the set

\[
S_F^1 = \left\{ f \in L^1(\mu, X) : f(\omega) \in F(\omega) \text{ for } \mu\text{-a.e. } \omega \in \Omega \right\}
\]

is decomposable. Conversely...

**Theorem (Hiai-Umegaki 1977) – assuming that \( X \) is separable**

Let \( D \subset L^1(\mu, X) \) be a decomposable closed nonempty set. Then there is a measurable \( F : \Omega \rightarrow cl(X) \) such that

\[
D = S_F^1
\]

**Measurable** means: \( \{ \omega \in \Omega : F(\omega) \cap U \neq \emptyset \} \in \Sigma \) for every open set \( U \subset X \).

**Theorem (Klei 1988) – assuming that \( X \) is separable**

Let \( F : \Omega \rightarrow cl(X) \) be measurable. If \( S_F^1 \) is relatively weakly compact, then

\( F(\omega) \) is relatively weakly compact for \( \mu\text{-a.e. } \omega \in \Omega \).