Boyd Indices on Banach Function Spaces

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Rearrangement invariant Banach function space

Let $X$ be a rearrangement invariant Banach function space (r.i.); that is

$$
||f||_X = ||f^*||_{\tilde{X}}
$$

where $f^*$ is the decreasing rearrangement of $f$ respect to the Lebesgue measure:

$$
f^*(t) = \inf\{s > 0; \lambda_f(s) \leq t\}
$$

with

$$
\lambda_f(s) = |\{x; |f(x)| > s\}|.
$$

Examples:

(i) $X = L^p$, $1 \leq p \leq \infty$.

(ii) Orlicz spaces (Ex.: $L \log L$).

(iii) Lorentz spaces $L^{p,q}$. 
**Definition 1** The upper Boyd index $\alpha_X$ is defined as follows:

$$\alpha_X := \lim_{t \to \infty} \frac{\log \|D_t\|_X}{\log t},$$

with $\|D_t\|_X$ the norm of the dilation operator

$$\|D_t\|_X = \sup_{\|f\|_X \leq 1} \|D_t f\|_X, \quad D_t f(s) = f(s/t).$$

And the lower Boyd index $\beta_X$ is defined by

$$\beta_X := \lim_{t \to 0} \frac{\log \|D_t\|_X}{\log t}.$$

We will be interested in the following classical result concerning the following two important operators in Harmonic Analysis: the Hardy-Littlewood maximal operator

$$M f(x) = \sup_{x \in I} \frac{1}{|I|} \int_I |f(y)| dy,$$
where the supremum is taken over all intervals $I$ of the real line, and the Hilbert transform

$$
Hf(x) = \frac{1}{\pi} \lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y} dy,
$$

whenever this limit exists almost everywhere.

**Classical Results**

**Theorem 1** (Lorentz-Shimogaki, 1967)

Given a r.i. Banach Function Space $X$ on $\mathbb{R}$,

$$
M : X \longrightarrow X \quad \text{is bounded} \quad \iff \quad \alpha_X < 1.
$$

**Theorem 2** (Boyd, 1967)

$$
H : X \longrightarrow X \quad \text{is bounded} \quad \iff \quad \alpha_X < 1, \quad \text{and} \quad \beta_X > 0.
$$
Example 1: \( X = L^p \), then \(||D_t||_{L^p} = t^{1/p} \) and therefore

\[
\alpha_X = \beta_X = \frac{1}{p}.
\]

Hence Lorentz-Shimogaki Theorem says that

\[
M : L^p \rightarrow L^p \iff p > 1,
\]

and Boyd’s Theorem

\[
H : L^p \rightarrow L^p \iff 1 < p < \infty.
\]

That is, we recover the Riesz-Kolmogorov theorem (1920’s).
Idea of the Proof

Lorentz-Shimogaki’s

It is known that

\[(Mf)^*(t) \approx \frac{1}{t} \int_0^t f^*(s) ds = \int_0^1 f^*(st) ds,\]

and hence

\[||Mf||_X \leq \int ||f(\cdot s)||_X ds \leq ||f||_X \int_0^1 ||D_{1/s}|| ds.\]

Everything follows now, from the fact that

\[\int_0^1 ||D_{1/s}|| ds < \infty \iff ||D_{1/s}|| \leq \frac{1}{s^\alpha}, \alpha < 1 \iff \alpha_X < 1.\]

and the hidden reason because all these equivalences are true is the fact that the function

\[h(s) = ||D_{1/s}||\]

is submultiplicative; that is \(h(st) \leq h(s)h(t)\).
Boyd’s theorem:

\[(Hf)^*(t) \lesssim \frac{1}{t} \int_0^t f^*(s)ds + \int_t^\infty f^*(s) \frac{ds}{s} = \int_0^1 f^*(st)ds + \int_0^1 f^*(\frac{t}{s}) \frac{ds}{s},\]

and, as before,

\[\|Hf\|_X \lesssim \|f\|_X \left( \int_0^1 \|D_{1/s}\| ds + \int_0^1 \|D_s\| \frac{ds}{s} \right)\]

The first term is controlled by the condition \(\alpha_X < 1\) and the second term by \(\beta_X > 0\) and again, the main property is the submultiplicativity property.

**Example 2:**

**Classical Lorentz spaces**

Let us recall that the Lorentz spaces \(\Lambda^p(w)\) were introduced by Lorentz in 1951 and are defined by the condition

\[\|f\|_{\Lambda^p(w)} = \left( \int_0^\infty (f^*(t))^p w(t) dt \right)^{1/p} < \infty.\]
Examples:

(i) If $w = 1$, $\Lambda^p(w) = L^p$.

(ii) If $w(t) = t^{p/q - 1}$, $\Lambda^p(w) = L^{q,p}$.

(iii) If $w(t) = 1 + \log^+ \frac{1}{t}$, and $p = 1$, $\Lambda^1(w) = L \log L$.

(iv) If $p = 1$, $\Lambda^1(w) = \Lambda_W$ is a minimal Lorentz space with fundamental function $W$.

Question 1:

Since $\Lambda^p(w)$ are r.i., what does the Lorentz-Shimogaki and Boyd’s theorems say?
Proposition 1

For every $0 < p < \infty$,

$$
\alpha_{\Lambda^p(w)} := \lim_{t \to \infty} \frac{\log W^{1/p}(t)}{\log t}, \quad \beta_{\Lambda^p(w)} := \lim_{t \to 0} \frac{\log W^{1/p}(t)}{\log t},
$$

where

$$
\overline{W}(t) := \sup_{s \in [0, +\infty)} \frac{W(st)}{W(s)}.
$$

Then, the Lorentz-Shimogaki’s Theorem applied to $\Lambda^p(w)$ says that

Corollary 3

$$
M : \Lambda^p(w) \longrightarrow \Lambda^p(w) \iff \lim_{t \to \infty} \frac{\log W^{1/p}(t)}{\log t} < 1.
$$

Corollary 4

$$
H : \Lambda^p(w) \longrightarrow \Lambda^p(w) \iff \lim_{t \to \infty} \frac{\log W^{1/p}(t)}{\log t} < 1, \quad \lim_{t \to 0} \frac{\log W^{1/p}(t)}{\log t} > 0.
$$
Theorem 5 (Ariño-Muckenhoupt, 1990)

For every $p > 0$,

$$M : \Lambda^p(w) \rightarrow \Lambda^p(w)$$

if and only if $w \in B_p$, that is

$$r^p \int_r^\infty \frac{w(t)}{t^p} \, dt \lesssim \int_0^r w(t) \, dt.$$  

Theorem 6 (E. Sawyer, 1991)

For every $p > 0$,

$$H : \Lambda^p(w) \rightarrow \Lambda^p(w)$$

if and only if $w \in B_p \cap B_\infty^*$, where $w \in B_\infty^*$ if
\[
\int_0^r \frac{1}{t} \int_0^t w(s)dsdt \lesssim \int_0^r w(s)ds.
\]

As a consequence:

**Corollary 7**

\[
\lim_{t \to \infty} \frac{\log W_{1/p}^{1/p}(t)}{\log t} < 1 \quad \text{if and only if} \quad w \in B_p.
\]

**Corollary 8**

\[
\lim_{t \to 0} \frac{\log W_{1/p}^{1/p}(t)}{\log t} > 0 \quad \text{if and only if} \quad w \in B_\infty^*.
\]

**Remark 1**

\[
w \in B_p \quad \text{if and only if} \quad W_{1/p}^{1/p}(t) \lesssim t^\alpha, \quad \alpha < 1.
\]
Lemma 1

For every submultiplicative increasing function $\varphi$ defined in $[1, \infty)$,

$$\lim_{t \to \infty} \frac{\log \varphi(t)}{\log t} < 1 \quad \text{if and only if} \quad \varphi(x) \lesssim x^\alpha$$

for some $\alpha < 1$ and every $x > 1$.

Question 2: How can we define Boyd’s indices in spaces which are Banach Function Spaces non necessarily rearrangement invariant?
Weighted Lebesgue spaces

In the 70’s the following theorem was proved.

**Theorem 9 (Muckenhoupt, 1972)**

If $p > 1$,

$$M : L^p(u) \rightarrow L^p(u)$$

if and only if $u \in A_p$:

$$\sup_Q \left( \frac{1}{|Q|} \int_Q u(x) dx \right) \left( \frac{1}{|Q|} \int_Q u(x)^{-1/(p-1)} dx \right)^{p-1} < \infty.$$ 

**Theorem 10 (Hunt, Muckenhoupt, Wheeden, 1973)**

$$H : L^p(u) \rightarrow L^p(u) \iff u \in A_p$$
**Question 3:**

Is there an analogue to the Lorentz-Shimogaki and Boyd’s theorem for the spaces $L^p(u)$?

In 2007, Lerner and Pérez defined the upper Boyd’s index for more general spaces than r.i. and proved the analogue to Lorentz-Shimogaki’s theorem.

To pursue this direction we introduce a generalized definition of the upper Boyd index. In this new approach the main role is played by the so-called local maximal operator $m_\lambda f$ defined by

$$m_\lambda f(x) = \sup_{x \in Q} (f\chi_Q)^*(\lambda|Q|), \quad 0 < \lambda < 1.$$  

We give the following generalization of the upper Boyd index.

**Definition 2** For any quasi-Banach function space $X$ over $\mathbb{R}^n$ we define the non-increasing function $\Phi_X$ on $(0, 1)$ as the operator norm
of $m_\lambda$ on $X$, namely,

$$\Phi_X(\lambda) = ||m_\lambda||_X = \sup_{\|f\|_X \leq 1} ||m_\lambda f||_X.$$ 

We define the generalized upper Boyd index as

$$\alpha_X = \lim_{\lambda \to 0} \frac{\log \Phi_X(\lambda)}{\log \frac{1}{\lambda}}.$$ 

**Theorem 11 (Lerner-Pérez, 2007)**

$$M : X \to X$$ 

if and only if $\alpha_X < 1$.

**Examples:**

1) 

$$\alpha_{L^p(u)} = \lim_{t \to \infty} \frac{\log \nu_u^{1/p}(t)}{\log t} < 1,$$

where
\[ \nu_u(t) = \sup \left\{ \frac{u(I)}{u(E)} ; \ E \subset I, \ \frac{|I|}{|E|} = t \right\} \]

**Theorem 12 (Lerner-Pérez, 2007)**

\[ M : L^p(u) \longrightarrow L^p(u) \]

if and only if

\[ \alpha_{L^p(u)} = \lim_{t \to \infty} \frac{\log \nu_u^{1/p}(t)}{\log t} < 1, \]

where

\[ \nu_u(t) = \sup \left\{ \frac{u(I)}{u(E)} ; \ E \subset I, \ \frac{|I|}{|E|} = t \right\} \]

**Question 4:** Where the functions \( m_\lambda \) and \( \Phi_X \) appear?
\[
\frac{1}{|Q|} \int_Q |f(x)| \, dx = \int_0^1 (f \chi_Q)^*(\lambda|Q|) \, d\lambda,
\]
and hence
\[
Mf(x) \leq \int_0^1 m_\lambda f(x) \, d\lambda
\]
\[
||Mf||_X \leq \int_0^1 ||m_\lambda f||_X \, d\lambda \leq ||f||_X \int_0^1 \Phi_X(\lambda) \, dx.
\]

Question 5: how can we define the lower Boyd index for a general BFS?

**Weighted Lorentz Spaces, 1951**

The weighted Lorentz spaces \( \Lambda_p^u(w) \) are defined by the condition
\[
||f||_{\Lambda_p^u(w)} = \left( \int_0^\infty f_u^*(t)^p w(t) \, dt \right)^{1/p} < \infty,
\]
where \( f_u^* \) is the decreasing rearrangement of \( f \) respect to \( u \),
\[ f^*_u(t) = \inf\{ s > 0; \lambda_f^u(s) \leq t \} \]

with

\[ \lambda_f^u(s) = u(\{x; |f(x)| > s\}). \]

**Examples:**

1) If \( w = 1 \), then \( \Lambda^p_u(w) = L^p(u) \)

2) If \( u = 1 \), then \( \Lambda^p_u(w) = \Lambda^p(w) \)

3) If \( u = 1 \) and \( w(t) = t^{p/q-1} \) then \( \Lambda^p_u(w) = L^{q,p} \).

4) If \( w(t) = t^{p/q-1} \) then \( \Lambda^p_u(w) = L^{q,p}(u) \).

**Question 6:** (J. A. Raposo’s thesis)

Which is the characterization of the weights \( u \) and \( w \) such that

\[ M : \Lambda^p_u(w) \longrightarrow \Lambda^p_u(w)? \]
BRIEF ARTICLE
THE AUTHOR

$L^p(u)$ $L^p$ $\Lambda^p(w)$

$70's$ $90's$

$A_p$ $B_p$

$\mathbb{R}$ $\mathbb{R}^+$
Theorem 13 (C-Raposo-Soria, 2007)

If $0 < p < \infty$,

$$M : \Lambda^p_u(w) \longrightarrow \Lambda^p_u(w)$$

is bounded if and only if there exists $\alpha < 1$ such that for every $t > 1$,

$$\overline{W}_{u}^{1/p}(t) \lesssim t^{\alpha}, \quad (1)$$

where, for every $t > 1$,

$$\overline{W}_u(t) := \sup \left\{ \frac{W\left(u\left(\bigcup I_j\right)\right)}{W\left(u\left(\bigcup S_j\right)\right)} : S_j \subseteq I_j, \frac{|I_j|}{|S_j|} = t \right\},$$

with $I_j$ disjoint intervals and all unions are finite.
Remark 2

In the paper of Lerner-Pérez, they compute

\[ \alpha_{\Lambda^p_u(w)} = \lim_{t \to \infty} \frac{\log W^1_{pu}(t)}{\log t}, \]  

(2)

and proved that

\[ M : \Lambda^p_u(w) \longrightarrow \Lambda^p_u(w) \iff \lim_{t \to \infty} \frac{\log W^1_{pu}(t)}{\log t} < 1. \]

Since we also have that

\[ M : \Lambda^p_u(w) \longrightarrow \Lambda^p_u(w) \iff W^{1/p}_u(t) \lesssim t^\alpha, \quad \alpha < 1 \]
Proposition 2

The function $W_u$ is submultiplicative on $[1, \infty)$.

Lemma 2

Let $I$ be an interval and let $S = \bigcup_{k=1}^{N}(a_k, b_k)$ be union of disjoint intervals such that $S \subseteq I$. Then, for every $t \in [1, |I|/|S|]$ there exists a collection of disjoint subintervals $\{I_n\}_{n=1}^{M}$ satisfying that $S \subseteq \bigcup_n I_n$ such that, for every $n$,

$$t|S \cap I_n| = |I_n|.$$  \hfill (3)
Question 6: (E. Agora’s thesis)

Which is the characterization of the weights $u$ and $w$ such that

$$H : \Lambda^p_u(w) \rightarrow \Lambda^p_u(w)?$$
Theorem 14

If $p > 1$ then

$$H : \Lambda^p_u(w) \longrightarrow \Lambda^p_u(w)$$

if and only if the three following condition holds:

(i) $u \in A_\infty = \bigcup_p A_p$.

(ii) $w \in B^*_\infty$.

(iii) $M : \Lambda^p_u(w) \longrightarrow \Lambda^p_u(w)$.

**Question:** Is there some relation between the conditions (i) and (ii) in the previous theorem and some lower Boyd’s index?
The Boyd Theorem for $\Lambda^p_u(w)$

Definition 3

If $\lambda \in (0, 1]$, we define

$$W_u(\lambda) := \sup \left\{ \frac{W(u(\bigcup_j S_j))}{W(u(\bigcup_j I_j))} : S_j \subseteq I_j, \frac{|S_j|}{|I_j|} = \lambda \right\},$$

where $I_j$ are disjoint open intervals and all unions are finite.

Proposition 3

The function $W_u$ is submultiplicative in $[0, 1]$.

By analogy with the case of the upper index, we give the following definition.
Definition 4

We define the generalized lower Boyd index associated to $\Lambda_p^p(u)(w)$ as

$$\beta_{\Lambda_p^p(u)(w)} := \lim_{t \to 0} \frac{\log W_{u}^{1/p}(t)}{\log t}.$$ 

Proposition 4

A couple of weights $u$ and $w$ satisfy that $u \in A_{\infty}$ and $w \in B^\ast_{\infty}$ if and only if

$$\beta_{\Lambda_p^p(u)(w)} > 0.$$

Theorem 15

If $p > 1$ then

$$H : \Lambda_p^p(u)(w) \longrightarrow \Lambda_p^p(u)(w)$$

if and only if

$$\alpha_{\Lambda_p^p(u)(w)} < 1 \quad \text{and} \quad \beta_{\Lambda_p^p(u)(w)} > 0.$$
Theorem 16

Let $0 < p < \infty$. If

$$H : \Lambda^p_u(w) \rightarrow \Lambda^p_u(w)$$

is bounded then

$$\beta_{\Lambda^p_u(w)} > 0.$$

Theorem 17

Let $0 < p < \infty$. If

$$\alpha_{\Lambda^p_u(w)} < 1 \quad \text{and} \quad \beta_{\Lambda^p_u(w)} > 0$$

then

$$H : \Lambda^p_u(w) \rightarrow \Lambda^p_u(w)$$

is bounded.

So it remains to prove that, for every $0 < p \leq 1$,

$$H : \Lambda^p_u(w) \rightarrow \Lambda^p_u(w) \implies \alpha_{\Lambda^p_u(w)} < 1.$$
Indices for Banach Function Spaces

Definition 5

Let \( X \) be a r.i. space with fundamental function \( \varphi_X \) and let

\[
\bar{\varphi}_X(t) = \sup_s \frac{\varphi_X(st)}{\varphi_X(s)}.
\]

Then the lower and upper fundamental indices are defined by

\[
\underline{\beta}_X = \lim_{t \to 0} \frac{\log \varphi_X(t)}{\log t} \quad \text{and} \quad \overline{\beta}_X = \lim_{t \to \infty} \frac{\log \varphi_X(t)}{\log t}.
\]
Remark 3 If we rewrite our function $\overline{W_u^{1/p}}$ we see that, if $X = \Lambda_u^p(w)$,

$$\overline{W_u^{1/p}}(\lambda) = \sup \left\{ \frac{W^{1/p}(u(\bigcup_j I_j))}{W^{1/p}(u(\bigcup_j S_j))} : S_j \subseteq I_j, \frac{|I_j|}{|S_j|} = \lambda \right\}$$

$$= \sup \left\{ \frac{||\chi_{I_j}||_X}{||\chi_{S_j}||_X} : S_j \subseteq I_j, \frac{|I_j|}{|S_j|} = \lambda \right\}$$

Now, if we take the last expression and we think that $X$ is r.i., we obtain that

$$\sup \left\{ \frac{||\chi_{I_j}||_X}{||\chi_{S_j}||_X} : S_j \subseteq I_j, \frac{|I_j|}{|S_j|} = \lambda \right\} = \sup \left\{ \frac{||\chi_{(0,r)}||_X}{||\chi_{(0,s)}||_X} : \frac{r}{s} = \lambda \right\}$$

$$= \sup \left\{ \frac{\phi_X(r)}{\phi_X(s)} ||_X : r = s\lambda \right\} = \overline{\phi_X}(\lambda)$$
And, if we rewrite our function $W_u^{1/p}$ we see that, if $X = \Lambda_u^p(w)$,

$$W_u^{1/p}(\lambda) = \sup \left\{ \frac{W^{1/p}(u(\bigcup_j S_j))}{W^{1/p}(u(\bigcup_j I_j))} : S_j \subseteq I_j, \frac{|S_j|}{|I_j|} = \lambda \right\}$$

$$= \sup \left\{ \frac{||\chi_{\bigcup_j S_j}||_X}{||\chi_{\bigcup_j I_j}||_X} : S_j \subseteq I_j, \frac{|S_j|}{|I_j|} = \lambda \right\}$$

and, if we take the last expression and we think that $X$ is r.i., we obtain that

$$\sup \left\{ \frac{||\chi_{\bigcup_j S_j}||_X}{||\chi_{\bigcup_j I_j}||_X} : S_j \subseteq I_j, \frac{|S_j|}{|I_j|} = \lambda \right\} = \sup \left\{ \frac{||\chi_{(0,r)}||_X}{||\chi_{(0,s)}||_X} : \frac{r}{s} = \lambda \right\}$$

$$= \sup \left\{ \frac{\varphi_X(r)}{\varphi_X(s)} : r = s\lambda \right\} = \varphi_X(\lambda).$$
Definition 6

Given a Banach function space $X$, we define

$$
\overline{\varphi}_X(\lambda) = \sup \left\{ \frac{\| \chi_{\cup_j I_j} \|_X}{\| \chi_{\cup_j S_j} \|_X} : S_j \subseteq I_j, \frac{|I_j|}{|S_j|} = \lambda \right\}, \quad \lambda \geq 1
$$

and

$$
\underline{\varphi}_X(\lambda) = \sup \left\{ \frac{\| \chi_{\cup_j S_j} \|_X}{\| \chi_{\cup_j I_j} \|_X} : S_j \subseteq I_j, \frac{|S_j|}{|I_j|} = \lambda \right\}, \quad \lambda < 1.
$$

Then, we define the lower and upper fundamental indices as follows:

$$
\beta_X = \lim_{t \to 0} \frac{\log \overline{\varphi}_X(t)}{\log t} \quad \text{and} \quad \underline{\beta}_X = \lim_{t \to \infty} \frac{\log \underline{\varphi}_X(t)}{\log t}.
$$
Questions

(i) Which is the relation between this new upper index and the Boyd index of Pérez-Lerner?

Open question.

(ii) Is the Lorentz-Shimogaki theorem true with this new upper index?

No. In 1970, Shimogaki gave an example of a r.i. space $X$ such that the fundamental function is the same than $L^2$ but the maximal operator is not bounded in $X$.

(iii) Is the Boyd theorem true with these new upper and lower index?

No.

... etc ...
Some progress and recent results

Proposition 5

\( \varphi_X \) is submultiplicative in \([0, \infty)\).

Theorem 18

\[ M : X \to X \implies \bar{\beta}_X < 1. \]

Theorem 19

If \( \bar{\beta}_X < 1 \), then

\[ ||M\chi_E||_X \leq ||\chi_E||_X \]

for every measurable set \( E \).
References (ordered in time)

1) M. Riesz, Sur les fonctions conjuguées, Math. Z. 27 (1928), no. 1, 218–244.


13) E. Agora, M.J. Carro, and J. Soria, Complete characterization of the