Normality on Topological Groups

Elena Martín-peinador

Departamento de Geometría y Topología
Facultad de Ciencias Matemáticas
Universidad Complutense de Madrid
E-28040 Madrid, Spain
peinador@mat.ucm.es

Dedicated to Professor Juan Tarrés on the occasion of his 65th birthday.

Mathematics is, like music, worth doing for its own sake, P. Hilton.
(The Mathematical Component of the Good Education [9])

ABSTRACT

It is a well known fact that every topological group which satisfies a mild separation axiom like being T_0, is automatically Hausdorff and completely regular, thus, a Tychonoff space. Further separation axioms do not hold in general. For instance, the topological product of uncountable many copies of the discrete group of integer numbers, say $\mathbb{Z}^{\mathbb{R}}$ is not normal. Clearly it is a topological Abelian Hausdorff group, with the operation defined pointwise and the product topology τ. With this example in mind, one can ask, are there "many non-normal" groups? Markov asked in 1945 wether every uncountable abstract group admits a non-normal group topology. Van Douwen in 1990 asked if every Abelian group endowed with the weak topology corresponding to the family of all its homomorphisms in the unit circle of the complex plane should be normal. Here we prove that the above group $\mathbb{Z}^{\mathbb{R}}$ endowed with its Bohr topology τ_b is non-normal either, and obtain that all group topologies on $\mathbb{Z}^{\mathbb{R}}$ which lie between τ_b and the original one τ are also non-normal. In fact, every compatible topology for this group lacks normality and we raise the general question about the "normality behaviour" of compatible group topologies.

Key words: Precompact group, normal topological group, Bohr topology, compatible topology, duality.

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1. Introduction, Notation and Auxiliary Results

A topological group is a triple $(G, \cdot, \tau)$ formed by a set G, a binary operation $\cdot$ which provides G with a group structure and a topology τ on G such that the mappings $\varsigma : G \times G \to G$ and $\iota : G \to G$ defined by $\varsigma(x, y) = x \cdot y$ and $\iota(x) = x^{-1}$ are

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continuous with respect to τ on G and the product topology $\tau \times \tau$ on $G \times G$. It is a common practice to write $x + y$ or xy instead of $x \cdot y$, and to write just G to denote a group or a topological group. Any abstract group endowed with the discrete topology is a topological group, and emblematic examples of topological groups are the real numbers $\mathbb{R}$ with ordinary addition and the usual euclidean topology $(\mathbb{R}, +, \tau_u)$, the complex numbers modulus one $\mathbb{T}$, with the ordinary complex multiplication and the topology induced by the euclidean in $\mathbb{C}$. The latter has an important role in the framework of Abelian groups: in fact, $\mathbb{T}$ is the dualizing object for the Pontryagin duality theory. If G is an Abelian group, every homomorphism from G into $\mathbb{T}$ is called a character and the set of all characters $Hom(G, \mathbb{T})$ has a group structure with respect to the pointwise multiplication, namely for $\phi, \varphi \in Hom(G, \mathbb{T})$, $(\phi\varphi)(x) = \phi(x)\varphi(x)$ for all $x \in G$. If G is an Abelian topological group, we will use the symbol $G^\wedge :=$ $CHom(G, \mathbb{T})$ to denote the group of all continuous characters of G.

Topological groups constitute a well behaved subclass of topological spaces. Roughly speaking, the algebraic structure helps the topology giving for free some properties. A well known instance of this effect (besides that mentioned in the abstract) is the Birkhoff-Kakutani Theorem, which states that every first countable Hausdorff topological group is metrizable. This does not hold for topological spaces as the Sorgenfrey line $\mathbb{S}$ shows, and consequently it can be asserted that $\mathbb{S}$ does not admit a topological group structure. A huge, impressive treatise concerning the interaction between Algebra and Topology [1] has been recently published. In this line we shall deal mainly with normality of Abelian topological groups. In the sequel we assume that the groups are Hausdorff. We recall first that a topological space X is normal if every pair of disjoint closed sets can be separated by disjoint open sets. The importance of normal spaces is stressed because of Tietze's Theorem, which places normal spaces as the adequate class where continuous real functions defined on closed subsets can be extended to the whole space.

2. Some classes of normal topological groups

(1) $\mathfrak{C}$ (Compact Hausdorff topological groups). We remark that $\mathbb{T}$, $\mathbb{T}^\alpha$ for any cardinal number α are distinguished members of $\mathfrak{C}$.

(2) $\mathfrak{M}$ (Metrizable topological groups). Clearly $\mathbb{R}, \mathbb{Q}, \mathbb{R}^\mathbb{N}$ are in $\mathfrak{M}$.

(3) Countable groups endowed with any Hausdorff group topology. For instance $\mathbb{Z}$ endowed with the p-adic topology is normal. Other groups topologies on $\mathbb{Z}$ have been recently presented in [2]. A consequence is that any monothetic group contains a dense normal subgroup.

(4) Locally compact Hausdorff groups.

Observe that the classes given in (1) and (2) are normal independently of the group structure, since the property holds for topological spaces. However, countable (or locally compact) Hausdorff spaces need not be normal. Thus, the algebraic structure plays some role in the classes mentioned in (3) and (4). Concerning (3), the normality follows from the fact that Hausdorff implies regularity for a group topology, and regular + Lindelöf $\Rightarrow$ normal, even for topological spaces. (4) is proved e.g. in [8, 8.13].

3. The Bohr topology for Abelian groups

For an abstract Abelian group G, there is a topology very related with the algebraic structure which is called called the Bohr topology. It is the weak topology corresponding to the family of all its characters, $Hom(G, \mathbb{T})$. In the seminal paper [6], Van Douwen uses the symbol $G_\natural$ to denote an Abelian group G endowed with the Bohr topology and he studies many important properties of topological groups of this nature. Observe that $Hom(G, \mathbb{T})$ separates the points of G (see e.g. [10]), and therefore, $G_\natural$ can be isomorphically embedded as a subgroup of the product $\mathbb{T}^{Hom(G,\mathbb{T})}$. Thus $G_\natural$ is a precompact group, and the closure of its image in $\mathbb{T}^{Hom(G,\mathbb{T})}$ is called the Bohr compactification of $G_\natural$, and denoted by bG. It is asked in [6] (Question (4.10)) if for an uncountable Abelian group G, $G_\natural$ is normal. The author already knew that such a group $G_\natural$ is not paracompact, and provided the tools so that finally in [14] a negative answer was given. Summarizing all these results, we present the following:

Theorem 3.1. *For an uncountable Abelian group G, $G_\natural$ has the following properties:*

- *(i) $G_\natural$ is 0-dimensional.*

- *(ii) $G_\natural$ is not a Baire space.*

- *(iii) Every infinite subset $A \subset G_\natural$ has a relative discrete subset D with $|D| = |A|$ that is C^*-embedded in the Bohr compactification bG.*

- *(iv) $G_\natural$ is not normal.*

- *(v) No nontrivial sequence in $G_\natural$ converges to a point in bG. In other words, $G_\natural$ is sequentially closed in bG.*

Proof. (i) and (ii) are respectively [6, 4.8 and 4.7 b)]. (iii) and (v) are [6, 1.1.3 (a) and (c)]. (iv) is proved in [14]. A more detailed proof is offered in [1], where important steps taken from [7] are also explained. Item (i) was taken further by Shakmatov in [11] where he proves that it is not strongly 0-dimensional either. For (i) and (v) uncountability of G is not needed. $\qquad\square$

    The notions of the Bohr topology and the Bohr compactification can be also defined for topological groups (and the previous notions $G_\natural$ and bG would correspond for an Abelian group G endowed with the discrete topology). Let (G, τ) be an Abelian topological group. The weak topology on G with respect to the family of its continuous characters $G^\wedge$, is called the *Bohr topology* for the topological group (G, τ). We shall denote it by τ_b.

    If $G^\wedge$ separates the points of G, then the canonical mapping ω from (G, τ_b) into $\mathbb{T}^{G^\wedge}$ defined by $g \mapsto (\phi(g))_{\phi \in G^\wedge}$ is a topological embedding, which is also an isomorphism onto its image. The name Bohr compactification and the symbol bG is used to denote the closure of $\omega(G)$: it is a compactification of (G, τ_b). Although the term "Bohr compactification" and the symbol bG are also referred to the group (G, τ), it is not a compactification of the latter in the usual sense: in fact, ω is only a **continuous** isomorphic embedding from (G, τ) into bG.

    The groups (G, τ) and (G, τ_b) admit the same continuous characters, and clearly this also happens with (G, ν) whenever ν is a group topology lying between τ_b and τ.

A group topology ν on G is called *compatible for* (G, τ) if $(G, \nu)^\wedge = (G, \tau)^\wedge =: G^\wedge$, as abstract groups. The set of all compatible topologies is referred to as the duality $(G, G^\wedge)$. Clearly τ_b is the bottom element for $(G, G^\wedge)$. In [3] compatible topologies are studied in an even more general context.

The constraint of "separating points", mentioned above, is by no means trivial! A topological group is said to be MAP (maximally almost periodic) if the continuous characters of G separate the points of G. The very deep Theorem of Peter-Weyl asserts that a compact Hausdorff Abelian group G is MAP. It is easy to derive from it that a locally compact Hausdorff Abelian group is also MAP, [10]. In this respect we claim (the proof is easy) that the product of MAP-groups is a MAP-group.

Lemma 3.2. *Let $\{G_\iota, \iota \in J\}$ be a family of MAP-groups. Then $G := \prod_{\iota \in J}$ is also a MAP-group.*

4. A topological group which does not admit normal compatible topologies

Before introducing the particular group we are going to deal with, we do some historical considerations. In the paper of Stone [12] where he proves his famous theorem that all metrizable spaces are paracompact, he also proves that the topological space $X := \mathbb{N}^{\mathbb{R}}$ is not normal. To this end he defines two closed disjoint subsets C_1 and C_2 such that every open set U containing C_1 must contain also some element of C_2. Therefore two open sets containing respectively C_1 and C_2 must intersect. This example can be easily parallelized in the space $(\mathbb{Z}^{\mathbb{R}}, \tau)$, and since the latter is a topological group, we have an example of a non-normal topological group. We can consider the Bohr topology for $G := (\mathbb{Z}^{\mathbb{R}}, \tau)$, that is, the weak topology τ_b corresponding to the family of all continuous characters on G, and we shall prove that $\mathbb{Z}^{\mathbb{R}}$ endowed with this weaker topology is neither a normal group.

Theorem 4.1. *Let $G := (\mathbb{Z}^{\mathbb{R}}, \tau)$, where each factor carries the discrete topology and τ is the product topology and let τ_b be the Bohr topology for G. The group $X := (\mathbb{Z}^{\mathbb{R}}, \tau_b)$ is not normal.*

Proof. Let us prove that X is separable and contains a closed discrete subset Y of continuum cardinality $\mathfrak{c}$. Then, applying Jones Lemma (see e.g. [15, 15.2]) we will have the non-normality of X.

Separability of G follows from Hewitt-Marczewski-Pondiczerny Theorem [15, 16.4 c)], which states that the product of $\mathfrak{c}$-many separable spaces is separable. Since $\tau_b < \tau$, separability of G implies the same property for X.

In order to find a "big" discrete subset, decompose the index set $\mathbb{R}$ into a disjoint union of countable sets. For instance, if $\{j_\alpha, \alpha \in A\}$ is a Hamel basis of $\mathbb{R}$ as vector space over $\mathbb{Q}$, $\mathbb{R} = \bigcup_{\alpha \in A} \mathbb{Q} j_\alpha$ and clearly $|A| = \mathfrak{c}$. Write $M_\alpha := (\mathbb{Q} \setminus \{0\}) j_\alpha$. Now $\mathbb{R}$ is a disjoint union of the sets M_α and $\{0\}$. The latter can be inserted in one of the sets M_α, say $M_\beta = \mathbb{Q} j_\beta$ for a fixed $\beta \in A$. Now we claim that $\mathbb{R} = \bigsqcup_{\alpha \in A} M_\alpha$ is written as an uncountable disjoint union of countable sets.

For every $\alpha \in A$ define $y_\alpha \in \mathbb{Z}^{\mathbb{R}}$ as the element whose coordinates, defined through the canonical projections $\pi_j, j \in \mathbb{R}$, are $\pi_j(y_\alpha) = 1$ whenever $j \in M_\alpha$ and 0 otherwise. So the elements of $Y := \{y_\alpha, \alpha \in A\}$ have countable many coordinates which are

equal to 1 and uncountable many which are 0. Let us prove that Y is closed and discrete in X.

In order to see that Y is discrete, it is enough to define a continuous character in G which takes the value -1 in y_α and 1 in y_β for every $\beta \in A \setminus \{\alpha\}$. Consider first $\varphi : \mathbb{Z} \to \mathbb{T}$, the character on $\mathbb{Z}$ such that $\varphi(n) = e^{\pi n i}, \forall n \in \mathbb{Z}$. It gives rise to characters on $\mathbb{Z}^\mathbb{R}$, just composing with the projections, say $\varphi \pi_j$, for all $j \in \mathbb{R}$. Clearly, if we take $j_0 \in M_\alpha$, $\varphi \pi_{j_0}(y_\alpha) = -1$, while $\varphi \pi_{j_0}(y_\beta) = 1$, for all $\beta \neq \alpha$. Thus the subset Y is discrete.

Now we prove that Y is closed. To this end, let $x \in \mathbb{Z}^\mathbb{R} \setminus Y$. Then there exist $j, k \in M_\alpha$ (for some $\alpha \in A$) such that $\pi_j(x) \neq \pi_k(x)$, or else there exists $j \in \mathbb{R}$ such that $\pi_j(x) = x_j \notin \{0, 1\}$. In the first case, a neighborhood of x can be taken just as $V := (\varphi \pi_j)^{-1}(I_1) \cap (\varphi \pi_k)^{-1}(I_2)$, where I_1 and I_2 are disjoint arcs in $\mathbb{T}$, centered respectively in $\varphi \pi_j(x)$ and $\varphi \pi_k(x)$. Now we have $V \cap Y = \emptyset$. For the second case, let $\zeta : \mathbb{Z} \to \mathbb{T}$ be a character such that $1 \neq \zeta(x_j) \neq \zeta(1)$. Clearly $\zeta \pi_j$ is a character on $\mathbb{Z}^\mathbb{R}$ which separates Y and x in the following sense: there exists an arc I in $\mathbb{T}$, centered at $\zeta(x_j)$, such that $(\zeta \pi_j)^{-1}(I) \cap Y = \emptyset$, being $(\zeta \pi_j)^{-1}(I)$ a neighborhood of x in X.

Summarizing, we have obtained a τ_b-closed discrete subset Y, with $|Y| = \mathfrak{c}$, which allows us to claim that X is non-normal. $\qquad \square$

Observe that a closed discrete subset of a topological space (X, τ) is also closed and discrete for any topology τ' on X such that $\tau \leq \tau'$. Next, we use this fact to prove that all group topologies compatible with the duality $(G, G^\wedge)$, for G as in the previous Theorem, are nonnormal. Recall that a topological space (X, τ) is *collectionwise Hausdorff* if for every closed discrete subset $Y \subset X$, there is a family of pairwise disjoint open subsets of X, each containing a point of Y.

Corollary 4.2. *If μ denotes a group topology in $\mathbb{Z}^\mathbb{R}$ such that $\tau_b < \mu \leq \tau$, then $X_\mu := (\mathbb{Z}^\mathbb{R}, \mu)$ is a nonnormal topological group. The group X_μ is neither collectionwise Hausdorff.*

Proof. The subset Y constructed in the proof of the previous Theorem is also μ-closed and discrete, for any topology μ greater than τ_b. On the other hand, the argument proving the separability of X can be used to obtain the same property for X_μ. Thus X_μ is nonnormal. This provides an alternative proof of the fact that $(\mathbb{Z}^\mathbb{R}, \tau)$ is not normal.

For the last part take into account that the product $(\mathbb{Z}^\mathbb{R}, \tau)$ satisfies the countable chain condition, briefly ccc (as a product of σ-compact groups [13, 5.20]). Clearly, $\mu \leq \tau$ implies that also $(\mathbb{Z}^\mathbb{R}, \mu)$ has the ccc. Since Y is μ-closed discrete and $|Y| = \mathfrak{c}$, the claim follows. $\qquad \square$

Next, we obtain the example mentioned in the title of this section.

Theorem 4.3. *Let $G := (\mathbb{Z}^\mathbb{R}, \tau)$, where each factor carries the discrete topology and τ is the product topology. Then, every topology compatible with the duality $(G, G^\wedge)$ is nonnormal.*

Proof. By the previous corollary we have that there are no **compatible normal** topologies for G weaker than τ. On the other hand τ is the greatest topology compatible for the duality $(G, G^\wedge)$. In fact, since the product $G = \mathbb{Z}^\mathbb{R}$ is a separable Baire

space, by [3, 1.6] G is a g-barrelled group. Applying [3, 4.6 (e)], we conclude that there do not exist topologies strictly finer than τ giving rise to the same dual group $G^\wedge$. □

In a context where Jones Lemma cannot be used, is still true the result of the Corollary 4.2? This can be formulated as:

Question 4.4. *(a) Let (G, τ) be a nonnormal topological group such that G endowed with the Bohr topology is neither normal. Are all topologies on G which admit the same dual group $G^\wedge$ also nonnormal?*

  (b) The same question of (a) changing nonnormal, by normal.

  (c) In [5] it is presented an example of a metrizable complete group (G, τ) such that the Bohr topology on τ_b is also metrizable. Therefore (G, τ) and (G, τ_b) are both normal. Are normal (or even metrizable) all the topologies between τ and τ_b?

We are also interested in which different precompact Hausdorff topologies on an abstract Abelian group G are normal. Precompact Hausdorff group topologies on G correspond one to one with dense subgroups of $Hom(G, \mathbb{T})$, being the latter endowed with the pointwise convergence topology. In fact, every precompact group topology ν on G is completely determined as the weak topology on G corresponding to $(G, \nu)^\wedge$ [4]. On the other hand $(G, \nu)^\wedge$ is dense in $Hom(G, \mathbb{T})$ with respect to the pointwise convergence topology if and only if $(G, \nu)^\wedge$ separates the points of G, which in turn is a necessary and sufficient condition for (G, ν) to be a Hausdorff group. The weak topology on G corresponding to the whole group $Hom(G, \mathbb{T})$ is also called in the Literature the maximal precompact topology on G. It is seldom normal: only if G is a countable group, $G_\natural$ is normal, as seen in Theorem 3.1. Leaning on this result, Trigos also proved the following [14]:

Proposition 4.5. *Let (G, τ) be a locally compact Abelian group. Then (G, τ_b) is normal iff (G, τ) is σ-compact.*

Let us clarify the picture with a well known topological group.

Example 4.6. Let $(\mathbb{R}, \tau)$ be the group of the real numbers with its usual topology. Then $\mathbb{R}_b := (\mathbb{R}, \tau_b)$ is normal (since Proposition 4.5 can be applied) but $\mathbb{R}_\natural$ is not normal (Theorem 3.1).

Since $\mathbb{R}_b$ can be embedded as a subgroup of $\mathbb{T}^\mathbb{R}$ (the exponent here stands for the dual of $(\mathbb{R}, \tau)$ which can be identified again with $\mathbb{R}$), while $\mathbb{R}_\natural$ can be embedded as a subgroup of $T^{Hom(\mathbb{R}, \mathbb{T})}$, being the first one normal and the second one nonnormal, an explanation should be in order.

Question 4.7. *Which are the normal nonclosed subgroups of $\mathbb{T}^\alpha$, for any cardinal α?*

Summarizing, the possible cases of normality for the Bohr topology of an Abelian topological group (G, τ) and for the corresponding maximal precompact topology, in the examples which have so far appeared are:

  • (G, τ), (G, τ_b) and $G_\natural$ are nonnormal. (G as in Theorem 4.1).

- (G,τ) and (G,τ_b) are normal but $G_\natural$ is not normal. ((G,τ) a locally compact, σ-compact group).

- (G,τ) normal, but (G,τ_b) and $G_\natural$ nonnormal. ((G,τ) a locally compact, non σ-compact group).

- (G,τ), (G,τ_b) and $G_\natural$ are all normal. (Any countable Hausdorff (G,τ)).

In the second family the assumption "locally compact" can be relaxed: it is enough to take (G,τ) as a σ-compact, uncountable group. In that case it is already Lindelöf (therefore normal), and the same happens to the weaker topology τ_b.

We do not have yet an example of a nonnormal MAP group whose corresponding Bohr topology is normal.

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