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Some summability properties of operators on a separable Banach space

By

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Introduction. If H denotes a separable Hilbert space and $S = (a_n) \subset H$ a fundamental fixed sequence, T a bounded linear operator on H, the following are equivalent:

1) $\sum_n Ta_n$ unconditionally convergent $\Rightarrow T$ compact.
2) $\sum_n Ta_n$ unconditionally convergent $\Rightarrow T$ is Hilbert-Schmidt.
3) $\sum \| Ta_n \| < \infty \Rightarrow T$ is nuclear.
4) $\inf_{f \in H^*,\, \|f\| = 1} \sum_n |f(a_n)| > 0.$

$1 \Leftrightarrow 2 \Leftrightarrow 4$ and $4 \Rightarrow 3$ are results appearing in my doctoral dissertation done under Professor Plans' guidance, and $3 \Rightarrow 4$ was proved by H. König, in a personal communication.

It was well-known that operators giving norm-summability in an orthonormal basis of a Hilbert space had to be nuclear (Holub 1972), and those giving unconditional summability in an orthonormal basis had to be Hilbert-Schmidt. With our statement 4), all those sequences of a Hilbert space that in this respect behave as orthonormal bases were intrinsically characterized. The equivalence of 2) and 3) showed that the parallelism among summability and absolute summability in orthonormal bases is a general fact.

In this short note we generalize these results to a separable Banach space X, and then we give a wide class of sequences in X verifying 4), which in particular include the Schauder bases if they exist in X.

Denote by X, Y Banach spaces and $\mathscr{B}(X, Y)$ the set of bounded linear operators from X into Y, by $R(T)$ the range of a linear operator T, by [ ] closed linear span, by (e_n) the standard unit basis of ℓ_1, or the unit vectors in ℓ_∞, and by $\| \; \|_1$ the norm in ℓ_1. Leaning on known properties of operators from a Banach space into ℓ_1 we prove the following:

Theorem 1. *Let X be a separable Banach space, and $S = (a_n) \subset X$ such that $\inf_{\|f\| = 1} \sum_n |f(a_n)| > 0$. For every $T \in \mathscr{B}(X, Y)$, being Y another Banach space, the following holds:*

a) *$\sum_n Ta_n$ unconditionally convergent $\Rightarrow T$ compact.*
b) *$\sum_n \| Ta_n \| < \infty \Rightarrow T^*$ is quasinuclear.*

Proof. Define first $M_s = \{f \in X^*, \text{s.t.} \sum_n |f(a_n)| < \infty\}$ and an operator $\phi: M_s \to \ell_1$ given by $\phi(f) = (f(a_n))_{n=1}^{\infty}$. One easily proves that ϕ is $1-1$ due to the condition $\inf_{\|f\|=1} \sum_n |f(a_n)| > 0$, and that ϕ is a closed linear operator.

Assume now that $\sum_n Ta_n$ is unconditionally convergent for a fixed $T \in \mathscr{B}(X, Y)$. This implies that $\sum_n |g\,T(a_n)| < \infty$, $\forall g \in Y^*$. Equivalently $\sum |T^* g(a_n)| < \infty$, $\forall g \in Y^*$. Thus $R(T^*) \subset M_s$ and $\phi T^* \in \mathscr{B}(Y^*, \ell_1)$ since

$$\|\phi T^*(g)\|_1 = \sum_n |g\,T(a_n)| \leqq \varrho\, \|g\|,$$

being ϱ the weak summability constant of (Ta_n).

Since ϕT^* is a bounded linear operator into ℓ_1 it will be compact if and only if the series $\sum_n (\phi T^*)^* e_n$ is norm subseries convergent in X^* (s. [1]; VII Ex. 3). It is straightforward to check that $(\phi T^*)^* e_n = Ta_n$; by assumption (Ta_n) is summable, thus ϕT^* is compact.

Now T^* is compact; otherwise there would exist $(g_n) \subset Y^*$, $\|g_n\| = 1$ such that $(T^* g_n)$ does not have convergent subsequences. Therefore an $\varepsilon > 0$ and a subsequence (g_{n_i}) could be determined so that $\|T^* g_{n_j} - T^* g_{n_k}\| \geqq \varepsilon$. Then

$$\|\phi T^* g_{n_j} - \phi T^* g_{n_k}\|_1 = \sum_m |(T^* g_{n_j} - T^* g_{n_k})\, a_m| \geqq c\,\varepsilon$$

where $c = \inf_{\|f\|=1} \sum_n |f(a_n)|$. This contradicts the fact that ϕT^* is compact.

By Schauder theorem, T^* compact implies T compact and the statement a) is thus proved.

We remark that when Y is weakly sequentially complete a) in the theorem can be substituted by a′) (Ta_n) weakly Cauchy implies T compact.

In order to prove b), recall that $A \in \mathscr{B}(X, Y)$ is a quasinuclear operator if there exists $(f_n) \subset X^*$ such that $\sum_n \|f_n\| < \infty$ and

$$\|Ax\| \leqq \sum_{n=1} |f_n(x)|, \quad \forall x \in X.$$

Those operators, defined by Pietsch, are a subclass of the absolutely summing one, $\Pi_1(X, Y)$.

Taking into account that

$$\sum_n \left|\left(\frac{T^* g}{\|T^* g\|}, a_n\right)\right| \geqq c := \inf_{\|f\|=1} \sum_n |f(a_n)|,$$

$$\Rightarrow \|T^* g\| \leqq 1/c \sum_n |(g, Ta_n)|.$$

Consider Ta_n as elements in Y^{**}, and by assumption $\sum_n \|Ta_n\| < \infty$. Thus T^* is quasinuclear.

R e m a r k. If ϕ is surjective the condition b) of the theorem can be strengthened in the following sense: every $T \in \mathscr{B}(X, Y)$ such that $\sum_n \| Ta_n \| < \infty$ must have a nuclear adjoint T^*.

Under the assumptions of Theorem 1, if $\sum_n \| Ta_n \| < \infty$ then ϕT^* is nuclear. But then ϕ surjective plus the condition $\inf_{\| f \| = 1} \sum | f(a_n)| > 0$ lead easily to nuclear representations of T^* in terms of those of ϕT^*.

By the other hand ϕ surjective is equivalent to (a_n) has a conjugate sequence $(a_n^*) \subset X^*$, i.e. (a_n, a_n^*) is a biorthogonal system.

A little more can be said if the space X is ℓ_p.

Theorem 2. *If $S = (a_n)$ denotes a fixed total sequence in ℓ_p, $1 < p < \infty$, and $T \in \mathscr{B}(\ell_p)$, the following are equivalent:*

a) $\sum Ta_n$ *unconditionally convergent* $\Rightarrow T$ *compact.*
b) $\sum \| Ta_n \| < \infty \Rightarrow T^*$ *quasinuclear.*
c) $\inf_{\| f \| = 1} \sum_n | f(a_n)| > 0.$

P r o o f. c) $\Rightarrow$ a) and c) $\Rightarrow$ b) have been already proved in the previous theorem. We prove now that a) or b) also imply c).

Suppose that $\inf_{\| f \| = 1} \sum_n | f(a_n)| = 0$ and choose $f_m \in \ell_{p'} (1/p + 1/p' = 1)$ such that $\sum_n | f_m(a_n)| < \frac{1}{m^2}$ and $\| f_m \| = 1$, $m \in \mathbb{N}$. A w^*-convergent subsequence $(f_{m_k}) \subset (f_m)$ can be extracted, say $f_{m_k} \xrightarrow{w^*} f$. But f must be null, otherwise for some $j \in \mathbb{N}, f(a_j) \neq 0$ and $f_{m_k}(a_j) \xrightarrow[k \to \infty]{m} f(a_j) \neq 0$. This contradicts the fact $\sum_n | f_{m_k}(a_n)| < \frac{1}{m_k^2}$ for sufficiently large m_k.

Using now the Bessaga-Pelczynski selection principle we obtain a basic subsequence $(g_r) \subset (f_{m_k})$ which is equivalent to a block basis (b_n) of the unit vector basis of $\ell_{p'}$; furthermore (b_n) can be chosen so that $\| b_n \|$ is close to $\| g_n \| = 1$. As is known for $\ell_{p'}$, $\left(\frac{b_n}{\| b_n \|} \right)$ is equivalent to the unit vector basis (e_n). So there exist isomorphisms from $\ell_{p'}$ into $[(b_n)_{n \in \mathbb{N}}]$ and from $[(b_n)_{n \in \mathbb{N}}]$ into $[(g_n), n \in \mathbb{N}]$. Call J the composition of both; obviously $J(e_n) = g_n$. The operator J^* verifies $\sum_n \| J^* a_n \| < \infty$.

In fact

$$J^*(a_n) = (g_r(a_n))_{r \in \mathbb{N}}.$$

Thus $\| J^* a_n \| \leqq \sum_r |g_r(a_n)| < \infty$ due to the condition $\sum_n |g_r(a_n)| < \frac{1}{r^2}$, and

$$\sum_n \| J^* a_n \| \leqq \sum_n \sum_r |g_r(a_n)| = \sum_r \sum_n |g_r(a_n)| \leqq \frac{1}{r^2} < \infty.$$

By the other hand $(J^*)^* = J$ is not compact, therefore it is not quasinuclear. So we have proved that a) or b) imply c).

Finally we remark that the sequences in X verifying $\inf_{\|f\|=1} \sum_n |f(a_n)| > 0$ are a wide class which includes all the normalized, Markushevich, uniform bounded bases, or the normalized Schauder bases if they exist at all in X.

Proposition. *If* (a_n, a_n^*) *is a biorthogonal, fundamental and total, bounded system and* $\|a_n\| = 1$, *then* $\inf_{\|f\|=1} \sum_n |f(a_n)| > 0$.

Suppose by contradiction that $\inf_{\|f\|=1} \sum_n |f(a_n)| = 0$ and for every integer m take f_m such that $\sum |f_m(a_n)| < \frac{1}{m}$. Every norm-one vector x can be approximated by a sequence of finite linear combinations, say $s_n = \lambda_{n_1} a_{n_1} + \cdots + \lambda_{n_r} a_{n_r} \xrightarrow[n \to \infty]{} x$. The coefficients λ_{n_i} are uniformly bounded since $|\lambda_{n_i}| = |a_{n_i}^*(s_n)| \leqq K \|s_n\|$ with $K = \sup_n \|a_n^*\|$ and $\|s_n\|$ is arbitrarily close to one.

Thus

$$|f_m(s_n)| = |f_m(\lambda_{n_1} a_{n_1} + \cdots + \lambda_{n_r} a_{n_r})| \leqq K' \sum_{i=1}^{r} |f_m(a_{n_i})| \leqq K' \sum_{1}^{\infty} |f_m(a_n)| \leqq K'/m$$

being K' a constant related to K.

The following contradiction: $1 = \|f_m\| = \sup_{\|x\|=1} |f(x)| \leqq K'/m$ proves that $\inf_{\|f\|=1} \sum_n |f(a_n)| > 0$.

References

[1] J. Diestel, Sequences and Series in Banach spaces. Berlin-Heidelberg-New York 1984.
[2] J. R. Holub, Characterization of nuclear operators in Hilbert space. Rev. Roumaine Math. Pures Appl. (5) **16**, 687–690 (1971).
[3] H. König, Personal Communication. 1985.
[4] E. Martín-Peinador, On the set of bounded linear operators transforming a sequence of a Hilbert space into an absolutely summable one. Colloq. Mat. Soc. János Bolyai **23**, 829–837 (1978).
[5] E. Martín-Peinador, The weak summability dominion of a sequence S of a Hilbert space in relation with the set of bounded linear operators. Ann. Inst. Mat. Univ. Nac. Aut. de Mexico. (2) **21**, 149–162 (1981).

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