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ON THE SET OF BOUNDED LINEAR OPERATORS TRANSFORMING A CERTAIN SEQUENCE OF A HILBERT SPACE INTO AN ABSOLUTELY SUMMABLE ONE

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Let $\mathscr{H}$ be a real, separable Hilbert space, $\mathscr{B}$ the set of bounded linear operators on $\mathscr{H}$, and $S = \{a_n \mid n \in N\}$ a fixed sequence in $\mathscr{H}$; we shall denote by

$$C_S = \left\{A \in \mathscr{B} \,\Big|\, \sum_{n=1}^{\infty} \| A a_n \| < \infty\right\}. \text{ Obviously } C_S \neq \phi, \ \forall S \in \mathscr{H}.$$

It is straightforward to check that C_S is a left ideal. We will prove that it never is a proper bilateral ideal, using the following result of Calkin [1]:

Every proper bilateral ideal in the ring of bounded operators of a Hilbert space contains the ideal $\mathfrak{S}_0$ of finite rank operators and is contained in the ideal of the completely continuous ones, $\mathfrak{S}$.

Lemma 1. *It is a necessary and sufficient condition for $\mathfrak{S}_0 \subset C_S$ that S be a summable sequence.*

Proof. Let $S = \{a_n \mid n \in N\}$. S is weakly summable if and only if

$\forall x \in \mathscr{H}, \sum_{n=1}^{\infty} |(a_n, x)| < \infty$, ((,) denotes scalar product). In the Hilbert space $\mathscr{H}$ a sequence is summable iff it is weakly summable. By the assumption $\mathfrak{S}_0 \subset C_S$ we have all the uni-dimensional projectors, $P_{w(x)}$, $x \in \mathscr{H} - \{0\}$, included in C_S and hence

$$\forall x \in \mathscr{H} - \{0\}, \quad P_{w(x)} \in C_S \Leftrightarrow \sum_{n=1}^{\infty} \| P_{w(x)}(a_n) \| < \infty,$$

$$\forall x \in \mathscr{H} - \{0\} \Leftrightarrow \sum_{n=1}^{\infty} |(a_n, x)| < \infty, \quad \forall x \in \mathscr{H}.$$

Thus we also see that whenever S is summable, C_S contains the unidimensional rank operators and from this it is easy to prove that it contains the whole set of finite rank operators.

Recalling that the Hilbert – Schmidt operators on $\mathscr{H}$ coincide with the absolutely p-summing ones, $1 \leqslant p < \infty$, ([4]), one has that there exists no sequence S for which $C_S = \mathfrak{S}_0$. In fact, if $C_S \supset \mathfrak{S}_0$ then $C_S \supset \mathfrak{S}_2$, ideal of the Hilbert – Schmidt operators, since every $A \in \mathfrak{S}_2$ transforms a summable sequence into an absolutely summable one, that is $\sum_{n=1}^{\infty} \| A a_n \| < \infty$, which is to say $A \in C_S$. Nevertheless one can find sequences S such that C_S consists only of finite rank operators, but not of all of them though.

So far we have proved that if a sequence $S \subset \mathscr{H}$ is not summable, C_S cannot be a proper bilateral ideal. Let us now see that for S summable C_S is not proper bilateral either.

Theorem 1. *Let* $S = \{a_n \mid n \in N\}$ *be summable.* C_S *then contains a non completely continuous operator.*

For the proof of this theorem we prove first the following

Lemma 2. *If* S *is summable, for every* $\epsilon > 0$ *there exists a natural number* μ *such that*

$$|(a_{p_1} + \ldots + a_{p_s}, x)| < \epsilon,$$

with $\mu \leqslant p_1 < \ldots < p_s$ *and* $x \in \mathscr{H}$ *only restricted by* $\|x\| = 1$.

Proof. Suppose that there is an $\epsilon > 0$ such that $\forall m$, $\exists x_m$, $\mu_m > m$ verifying $|(a_{\mu_m} + \ldots + a_{\mu_m + s_m}, x_m)| \geqslant \epsilon$. We take $m_1, m_2, \ldots$ such that $m_{i+1} > \mu_{m_i} + s_{m_i}$, and construct the subsequence of S

$$a_{\mu_{m_1}}, \ldots, a_{\mu_{m_1} + s_{m_1}}, a_{\mu_{m_2}}, \ldots, a_{\mu_{m_2} + s_{m_2}}, \ldots$$

for which the corresponding series does not verify the Cauchy condition and therefore is non-summable. This clearly contradicts $\{a_n \mid n \in N\}$ summable. ∎

Proof of the theorem. If $\{a_n \mid n \in N\}$ spans a finite dimensional subspace of $\mathscr{H}$, then $\{a_n \mid n \in N\}$ summable is equivalent to $\{a_n \mid n \in N\}$ absolutely summable, and $C_S = \mathscr{B}$.

Let us now suppose that the closed linear hull $[a_1, \ldots, a_n, \ldots]$ is not finite dimensional, and consider in $\mathscr{H}$ a complete orthonormal system $\{e_n \mid n \in N\}$; let $a_n = \sum_{j=1}^{\infty} a_{nj} e_j$. The series

$$\sum_n |(a_n, e_j)| = \sum_n |a_{nj}|, \qquad (j = 1, 2, \ldots)$$

converge uniformly following the lemma just proved.

If $\eta > 0$ denotes any fixed real number, for $\frac{\eta}{2}$ there exists $\nu\left(\frac{\eta}{2}\right)$ such that $\sum_{n=\nu}^{\infty} |a_{nj}| < \frac{\eta}{2}$ $(\forall j \in N)$. Since for each n, $a_{nj} \to 0$ $(j \to \infty)$, for large enough j we can obtain

$$\sum_{n=1}^{\nu-1} |a_{nj}| < \frac{\eta}{2}$$

and so

$$\sum_{n=1}^{\infty} |a_{nj}| < \eta,$$

thus

$$\sum_{n=1}^{\infty} |a_{nj}| \to 0 \qquad (j \to \infty).$$

Let us denote $\theta_j = \sum_n |a_{nj}|$, $\theta_j \to 0$ $(j \to \infty)$, and choose θ_{l_j} so that $\sum_j \theta_{l_j} < \infty$, that is

$$\sum_{n,l_j} |a_{nl_j}| < \infty.$$

Construct the coordinate subspace $[e_{l_1}, e_{l_2}, \ldots, e_{l_j}, \ldots]$ corresponding to the sequence $\{l_j \mid j \in N\}$, and denote by P the orthogonal projection on it. We have,

$$Pa_n = a_{nl_1}e_{l_1} + \ldots + a_{nl_j}e_{l_j} + \ldots \qquad (j = 1, 2, \ldots)$$

$$\| Pa_n \| \leqslant |a_{nl_1}| + \ldots + |a_{nl_1}| + \ldots \qquad (j = 1, 2, \ldots)$$

thus

$$\sum_n \| Pa_n \| \leqslant \sum_{n,l_j} |a_{nl_j}| < \infty.$$

So we see that P transforms S into $S' = \{Pa_n \mid n \in N\}$ which is absolutely summable, and therefore $P \in C_S$, being P a non completely continuous operator. ∎

The existence of a non completely continuous operator in C_S implies the existence of infinitely many of them.

This theorem shows that when we impose C_S to contain the whole set of finite rank operators, we obtain a very strong condition on S, namely that it must be summable, and because of this fact C_S contains "too many" operators, and it cannot be included in the ideal set of the completely continuous ones $\mathfrak{S}$, proving that C_S is never a proper bilateral ideal.

However C_S can be the whole set $\mathscr{B}$. Indeed, it is so iff $S = \{a_n \mid n \in N\}$ is absolutely summable. The other extreme case, $C_S = \{0\}$, 0 the null operator, is also possible and we consider it now. We quote first the following result [3]:

"Let J designate a left ideal in $\mathscr{B}$. If J does not include projectors, then $J = \{0\}$".

Then it is easy to prove that whenever $C_S \neq \{0\}$, it includes an uni-dimensional projector; hence a *necessary and sufficient condition for* $C_S \neq \{0\}$ *is the existence of a ray* $r \subset \mathscr{H}$ *such that* $\sum_n \| P_r a_n \| < \infty$, *or equivalently:*

$$C_S = \{0\} \Leftrightarrow \sum |(a_n, x)| = \infty, \qquad \forall x \in \mathscr{H} - \{0\}.$$

A geometrical consequence of this is that *for* $C_S = \{0\}$ *the nucleus of* S, $N(S) = \bigcap_1^\infty [a_n, \ldots]$ *must be the whole* $\mathscr{H}$ *hence in particular* $[S] = \mathscr{H}$.

$N(S) \neq \mathscr{H}$ would imply the existence of a natural number p such that $E = [a_p, a_{p+1}, \ldots] \neq \mathscr{H}$, consequently $P_{\mathscr{H} \ominus E}$ would belong to C_S, and there would exist uni-dimensional projectors in C_S.

We give an instance of a sequence S for which $C_S = \{0\}$. Let $\{r_n \mid n \in N\}$ be a complete system of rays in $\mathscr{H}$, $[r_1, \ldots, r_n, \ldots] = \mathscr{H}$. If we choose

$$a_1^{(n)}, \ldots, a_m^{(n)}, \ldots$$

in r_n, $r_n = w(a_m^{(n)})$ $(n, m \in N)$ such that

$$\| a_m^{(n)} \| \geqslant k_n > 0 \qquad (n \in N),$$

then $S = \{a_m^{(n)} \mid n, m \in N\}$ verifies $C_S = \{0\}$.

We can even impose conditions on this sequence S so that it becomes a L-system, (that is, the image of an orthonormal basis by a bounded linear operator) with $C_S = \{0\}$. This would be the case if

$$\sum_{n,m=1}^\infty \| a_m^{(n)} \|^2 < \infty \quad \text{and} \quad \sum_{m=1}^\infty \| a_m^{(n)} \| = \infty \qquad (n \in N).$$

If we project S on any ray r of $\mathscr{H}$, we have $\sum_{m=1}^\infty \| P_r a_m^{(p)} \| = \infty$ for a certain $r_p \in \{r_n \mid n \in N\}$ and therefore

$$\sum_{n,m} |(a_m^{(n)}, x)| = \infty, \qquad \forall x \in \mathscr{H} - \{0\},$$

hence $C_S = \{0\}$.

We observe the fact that whenever we have $C_S = \{0\}$ for a given S, we also have $C_{S'} = \{0\}$ for any $S' \supset S$, and this suggests to search the "minimal systems", S, for which $C_S = \{0\}$. We point out that any heterogonal in direction system, S, – even if it is complete – gives $C_S \neq \{0\}$, since it verifies $N(S) = \{0\} \neq \mathscr{H}$.

However the join S of two, or more, heterogonal systems can give $C_S = \{0\}$. To see this, choose an orthonormal complete system of $\mathscr{H}$, $\{e_n \mid n \in N\}$. Let $S' = \{b_n \mid n \in N\}$ be a L-system such that $C_{S'} = \{0\}$ and such that $\{e_n + b_n \mid n \in N\}$ is heterogonal. Then, for

$$S = \{e_n, e_n + b_n \mid n \in N\}$$

we have $C_S = \{0\}$.

Indeed, had we

$$\sum_n |(e_n, x_0)| + \sum_n |(e_n, x_0) + (b_n, x_0)| < \infty$$

for a certain $x_0 \in \mathscr{H} - \{0\}$, then we would have $\sum_n |(b_n, x_0)| < \infty$ contradicting $C_{S'} = \{0\}$.

We summarize the above results in the following:

Proposition. *C_S can never be a bilateral ideal unless either*

(i) *it is the whole $\mathscr{B}$. For this we need a very strong summability condition on S, namely S must be absolutely summable or*

(ii) *C_S is the zero ideal. For this we need a very strong condition of non-summability on S, i.e. for every $x \in \mathscr{H} - \{0\}$, $\{(a_n, x) \mid n \in N\}$ must not be absolutely summable.*

Finally we show that C_S can not contain the ideal $\mathfrak{S}$ of the completely continuous operators, except when C_S equals $\mathscr{B}$, that is, when S is absolutely summable.

Theorem 2. *Let* $S = \{a_n \mid n \in N\}$ *be such that* $\sum_n \| a_n \| = \infty$. *Then there exists a completely continuous operator* C *such that*

$$\sum_{n=1}^{\infty} \| Ca_n \| = \infty.$$

To prove this we give the following

Lemma 2. *Let* $\sum_{n=1}^{\infty} p_n$ *be a numerical divergent series of positive terms. Then there exists a sequence* $\{q_n \mid n \in N\}$ *verifying*

$$q_1 \geqslant q_2 \geqslant \ldots \geqslant q_n \geqslant \ldots > 0, \qquad q_n \to 0 \quad (n \to \infty)$$

such that

$$\sum_{n=1}^{\infty} p_n q_n = \infty.$$

Proof of the theorem. Let us refer $\mathscr{H}$ to an orthonormal basis $\{e_n \mid n \in N\}$. Let $a_n = \sum_{j=1}^{\infty} a_{nj} e_j$. By the assumption made above we have

$$\sum_{n=1}^{\infty} \sqrt{a_{n1}^2 + \ldots + a_{nj}^2 + \ldots} = \infty.$$

We are trying to construct a completely continuous diagonal operator C,

$$C = \begin{pmatrix} \lambda_1 & & & & 0 \\ & \ddots & & & \\ & & \lambda_n & & \\ & & & \ddots & \\ 0 & & & & \end{pmatrix}, \qquad \lambda_j \to 0 \quad (j \to \infty),$$

such that

$$\sum_{n=1}^{\infty} \| Ca_n \| = \sum_{n=1}^{\infty} \sqrt{a_{n1}^2 \lambda_1^2 + \ldots + a_{nj}^2 \lambda_j^2 + \ldots} = \infty.$$

One can find natural numbers ν_n $(n \in N)$ with the condition that

$$\sum_{n=1}^{\infty} \sqrt{a_{n1}^2 + \ldots + a_{n\nu_n}^2} = \infty$$

and we can additionally suppose that $\nu_n < \nu_{n+1}$ $(n \in N)$.

Lemma 2 guarantees the existence of a sequence $\mu_1 \geqslant \mu_2 \geqslant \ldots \ldots \geqslant \mu_n \geqslant \ldots > 0$, $\mu_n \to 0$ $(n \to \infty)$ such that

$$\sum_{n=1}^{\infty} \mu_n \sqrt{a_{n1}^2 + \ldots + a_{n\nu_n}^2} = \infty.$$

Let us take

$$\lambda_1 = \ldots = \lambda_{\nu_1} = \mu_1, \lambda_{\nu_1+1} = \ldots = \lambda_{\nu_2} = \mu_2, \ldots$$
$$\ldots, \lambda_{\nu_{n-1}+1} = \ldots = \lambda_{\nu_n} = \mu_n, \ldots .$$

As

$$\sqrt{a_{n1}^2 \lambda_1^2 + \ldots + a_{n\nu_n}^2 \lambda_{\nu_n}^2} \geqslant \mu_n \sqrt{a_{n1}^2 + \ldots + a_{n\nu_n}^2},$$

we have

$$\sum_{n=1}^{\infty} \sqrt{\lambda_1^2 a_{n1}^2 + \ldots + \lambda_{\nu_n}^2 a_{n\nu_n}^2} = \infty,$$

and consequently

$$\sum_{n=1}^{\infty} \sqrt{\lambda_1^2 a_{n1}^2 + \ldots + \lambda_{\nu_n}^2 a_{n\nu_n}^2 + \ldots} = \infty. \blacksquare$$

Thus we can establish that if a sequence S is such that $\forall C \in \mathfrak{S}$, $\sum_{n=1}^{\infty} \| Ca_n \| < \infty$, then necessarily $\sum_{n=1}^{\infty} \| a_n \| < \infty$, and $C_S = \mathscr{B}$.

This theorem, in a certain sense, can be considered as dual of the one due to Gohberg and Markus [2] which asserts *that for every bounded operator* $A \in \mathscr{B}$, $A \neq 0$, *there exists an orthonormal base* $\{e_n \mid n \in N\}$ *such that*

$$\sum_{n=1}^{\infty} \| Ae_n \| = \infty.$$

We have obtained that *for every non absolutely summable sequence – hence in particular for all the orthonormal bases – there exists not only a bounded operator, but a completely continuous one, C, such that* $\sum_n \| Ca_n \| = \infty$.

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