Chapter 7
Characteristics of the Mackey Topology for Abelian Topological Groups

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The present chapter, written for a joyful event, has suddenly changed its sign: the first author died a few days after its submission. Hiking the mountains of Gredos, near his homeland Ávila, and with a big experience and passion in this sport, José Manuel flew to scale new heights beyond the Mathematics. The chapter contains a part of his Doctoral Thesis, which was to be defended around the coming November.

Abstract This chapter is inspired on the Mackey-Arens Theorem, and consists on a thorough study of its validity in the class of locally quasi-convex abelian topological groups. If G is an abelian group and H is a group of characters which separates points of G, the pair (G, H) is said to be a dual pair. Any group topology on G which has H as its group of continuous characters is said to be compatible with the pair (G, H) or with the group duality (G, H). If the starting group G is already a topological group, a natural duality is obtained taking H as the group of its continuous characters. The family of all locally quasi-convex topologies defined on an abelian group G, with a fixed common character group H, was studied for the first time in [9]. It is a problem not solved yet if the supremum of a family of locally quasi-convex compatible topologies on an abelian topological group G is again a compatible topology. If it is, then it is called the Mackey topology for G. A locally quasi-convex group G is said to be a Mackey group whenever it carries the Mackey topology. Locally quasi-convex topologies can be characterized in terms of the families of equicontinuous subsets that they produce in the corresponding dual group. We have adopted this point of view and we have defined a grading of Mackey-type properties for abelian topological groups. We also study the stability of these properties through quotients.

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7.1 Introduction

The Mackey Theory is a well known topic in Functional Analysis. If (E, τ) denotes a real locally convex topological vector space and E' its dual space, the Mackey topology is defined as the finest locally convex topology on E producing the same continuous linear functionals as (E, τ). The existence of the Mackey topology and an external description of it, are provided by the Mackey-Arens Theorem. Any vector topology ν on E admitting E' as dual space is called a compatible topology for (E, τ) or just for the pair $(E, E^{'})$. Thus the Mackey topology on E is the finest locally convex compatible topology for $(E, E^{'})$, and it can be explicitly described as the topology of uniform convergence on the family of all absolutely convex weakly compact subsets of E'.

J. Kakol was the first to realize that local convexity is an essential requirement in the Theorem of Mackey-Arens. In [14] it is proved that in some classes $\mathscr{R}$ of topological vector spaces, for a fixed $(X, \tau) \in \mathscr{R}$ it is not always possible to obtain a vector space topology μ on X such that $(X, \mu) \in \mathscr{R}$ and it is the finest compatible with τ. Thus, if the class of locally convex spaces is substituted by some other class of topological vector spaces $\mathscr{R}$, the analogous to the Mackey-Arens Theorem does not necessarily hold in $\mathscr{R}$.

A different context to extend Mackey-Arens Theorem is provided by the locally quasi-convex groups. Inspired on the Hahn-Banach Theorem, Vilenkin defined the notion of quasi-convex subset of an abelian topological group [19]. He also introduced the locally quasi-convex groups as those abelian topological groups which admit a basis of quasi-convex zero neighborhoods. On the other hand Banaszczyk proved that a topological vector space is locally convex if and only if it is locally quasi-convex as an abelian topological group [5]. This result permits to conceive the locally quasi-convex groups as a class which extends that of locally convex spaces. The quasi-convexity is by no means as easy to handle as convexity: it requires hard calculations, even if the group involved is the underlying group of a topological vector space. Nevertheless there is some parallellism between duality defined in the category $\mathscr{LCS}$ of locally convex spaces and in the category $\mathscr{LQG}$ of locally quasi-convex groups.

In [9] the Mackey theory for locally convex spaces was generalized—as far as possible—to locally quasi-convex groups. It was also proved that certain classes of groups like the locally compact abelian groups or the complete metrizable locally quasi-convex groups are Mackey groups. Later on, in [3] and in [10] it was proved that completeness cannot be dropped in the previous sentence. The main open problem in this topic is the existence of a Mackey topology in a topological group G, without further constrains on G.

In $\mathscr{LQG}$ the dualizing object is the circle group $\mathbb{T}$, that is, the group of complex numbers of modulus one, endowed with its natural topology induced from $\mathbb{C}$. The homomorphisms from an abelian group G into $\mathbb{T}$, are usually called characters on G. The set of characters on G with respect to the pointwise operation is a group, frequently designated as the algebraic dual of G and represented by the symbol G^*. If (G, τ) is an abelian topological group, the continuous characters on G constitute a subgroup of G^*: it will be denoted by $(G, \tau)^\wedge$ (or simply $G^\wedge$), and called the dual of G. A group topology ν on a topological abelian group (G, τ) is said to be *compatible* for (G, τ)—or for the pair $(G, G^\wedge)$—if $(G, \nu)^\wedge = G^\wedge$, in other words if it gives rise to the same dual group as the original topology.

Varopoulos was the first to consider duality in the framework of abelian groups. In [18], a group duality is defined as a pair (G, H) where G is an abelian group and H is a group of characters on G that separates the points of G. The groups G and H play a symmetric role, since the elements of G can be considered as characters on H, just identifying each point x of G to the evaluation $\tilde{x} : H \to \mathbb{T}$ which carries $\phi \in H$ into $\phi(x)$.

Although the above definition of duality is purely algebraic, it gives rise to two standard topologies: $\sigma(G, H)$ and $\sigma(H, G)$. The first one is the topology on G of pointwise convergence on the elements of H and $\sigma(H, G)$ is the topology on H of pointwise convergence on the elements of G considered as evaluation mappings. It is proved in [18] that $(G, \sigma(G, H))$ has dual group precisely H (Corollary, p. 481) and now it makes sense to consider those topologies on G which admit H as dual group, and call them compatible for the duality (G, H). The symmetric situation for $(H, \sigma(H, G))$ is described in an identical way.

The definition of a duality (G, H) can be done also if H does not separate the points of G. If this is the case, one can replace the group G by the quotient group $G/H^\perp$, where $H^\perp := \bigcap_{\phi \in H} \phi^{-1}(\{0\})$ is the von Neumann kernel. The duality $(G/H^\perp, H)$ is now separating, and loosely speaking, the pair $(G/H^\perp, H)$ carries all the information about the duality (G, H). In our future considerations we do not require the duality (G, H) to be separating and we will write $< G, H >$ instead of (G, H) to stress the fact that we speak of duality in its most general context.

For a topological group (G, τ) there is a standard natural duality obtained by taking H as $G^\wedge$, the dual group of G. As said above, *a topology on G compatible with* τ means a topology compatible with $(G, G^\wedge)$.

Trying to emulate the Mackey theory, Varopoulos considers for a fixed group duality (G, H), the family $\mathscr{P}C$ of all locally precompact compatible topologies on G. The supremum of $\mathscr{P}C$ in the lattice of all topologies, say $\nu(G, H)$, is again a compatible topology, which we will call the Varopoulos topology. However $\nu(G, H)$ cannot be considered a candidate to be the Mackey topology in this new setting of abelian topological groups.

We outline next an example given in [9], which might convince the reader of our point of view. Take in the place of G a topological vector space E and let $E^\wedge$ denote the dual of E as a topological group. The topology $\nu(E, E^\wedge)$ coincides with $\sigma(E, E')$, the weak topology on E as a vector space. Clearly $\sigma(E, E')$ is seldom the Mackey topology on E, as a locally convex topological vector space. For instance, if E is an infinite dimensional Banach space, its Mackey topology is the original one while $\sigma(E, E')$ is not even metrizable.

The authors of [9] had the idea to substitute the above family $\mathscr{P}C$ by the family LQCC of all locally quasi-convex topologies in G which are compatible with (G, H). The framework provided by locally quasi-convex groups, briefly LQC groups, is more suitable for a generalization of the Mackey theory for abelian (topological) groups. However, it is not known for the general case (that is, without any restrictions on the duality) if the supremum of a family of topologies in LQCC is inside that class. It is not hard to prove that the supremum of any family of LQC topologies on a topological group G is again a locally quasi-convex topology, but it is not known so far if the supremum of a family of compatible topologies is compatible.

Following the track of the Mackey-Arens Theorem, two candidates arise to be defined as the Mackey topology of an abelian topological group (G, τ). The first option would be to take the finest among all the topologies which are locally quasi-convex and **compatible** with τ, if such an object exists.

The existence of the Mackey topology in this sense (compatibility of the supremum of locally quasi-convex compatible topologies) is proved in [9] for some particular classes of topological groups and the general case is mentioned as an open problem.

The second candidate to be the Mackey topology on G is the topology $\tau_{\mathscr{Q}}$ of uniform convergence on the family $\mathscr{Q}$ of all the quasi-convex and compact subsets of $G_p^\wedge$, where the latter is the dual group of G endowed with the $\sigma(G^\wedge, G)$-topology. In [9] it is asked if these two approaches are equivalent. A negative answer is provided in [6], by means of examples of topological groups G, for which the finest LQC compatible topology does not coincide with the topology of uniform convergence on the weakly compact and quasi-convex subsets of the dual $G^\wedge$. More precisely, Example (4.2) in [6] describes a topological group which has non-continuous characters, i.e. $G^\wedge \neq G^*$, and nevertheless the topology of uniform convergence on the family $\mathscr{Q}$ is discrete, so $(G, \tau_{\mathscr{Q}})^\wedge = G^*$. Thus, the Mackey-Arens Theorem does not admit a counterpart for abelian locally quasi-convex groups, at least in the most natural way to generalize it.

With the just mentioned Example, it was natural to seek a family $\mathscr{H}$ of subsets of the dual $G^\wedge$ of a topological group G, such that the inverse polars of the members of $\mathscr{H}$ were essentially a basis of neighbourhoods for the greatest compatible topology in G. The authors of [6] pointed out a family $\mathscr{H}$ which could have such a property. This approach has suggested us to define a grading in the property "to be a Mackey group", and this will be done in Sects. 7.6 and 7.7 of the present chapter. The Mackey topology for a topological group (G, τ) (as used subsequently, for instance in [3], [10] and some other papers) is the greatest locally quasi-convex compatible topology, provided it exists. Accordingly, a topological group (G, τ) is said to be a Mackey group if its original topology τ is the Mackey topology. Whenever $\tau_{\mathscr{Q}}$ is compatible, it is precisely the finest with this condition, and therefore the Mackey topology.

We describe now the contents of the present chapter. Sects. 7.2 and 7.3 are introductory, explaining notation and preliminaries. In Sect. 7.4 we introduce some definitions and technical lemmata, relating different topologies in a quotient group G/H with the corresponding of the original group G. These results will be used in Sects. 7.6 and 7.7.

In Sect. 7.5 we prove that the quotients of g-barrelled (pre-Mackey) are essentially g-barrelled (pre-Mackey) and therefore Mackey.

The main results are in Sects. 7.6, 7.7 and 7.8. In Sect. 7.6 we provide a necessary and sufficient condition for the existence of the Mackey topology for an abelian topological group G, and also for the quotient group G/H, where H is any subgroup of G. In order to describe these results, we introduce next some notation.

For an abelian topological group G and for a fixed subset $B \subset G^\wedge$, let $\tau_{\{B\}}$ be the topology on G of uniform convergence on B. It is a group topology on G, which has as a basis of zero-neighborhoods the family

$$\mathscr{B} := \{\cap_{\varphi \in B} \varphi^{-1}(\mathbb{T}_n) : n \in \mathbb{N}\}, \text{ where } \mathbb{T}_n := [-1/4n, 1/4n] + \mathbb{Z} \subset \mathbb{T}.$$

The subset B is said to "determine equicontinuity" for $(G, G^\wedge)$ if $(G, \tau_{\{B\}})^\wedge \subseteq G^\wedge$. If $\mathscr{M}$ is the family of all subsets determining equicontinuity, then $\tau_{\mathscr{M}} := sup_{B \in \mathscr{M}} \tau_{\{B\}}$ is the topology defined by uniform convergence on the sets of the family $\mathscr{M}$. We prove that the above mentioned family $\mathscr{H}$ obtained in [6] is equivalent to the family $\mathscr{M}$. However, the topology $\tau_{\mathscr{M}}$ is not the supremum of all locally quasi convex compatible topologies in (G, τ), as the authors of [6] thought. We characterize the case when $\tau_{\mathscr{M}}$ is a compatible topology (Theorem 7.9), being then $\tau_{\mathscr{M}}$ the Mackey topology corresponding to $(G, G^\wedge)$. We have called $(\mathscr{M})$-Mackey these kind of groups which are Mackey groups in a stronger sense. Thus, compatibility of $\tau_{\mathscr{M}}$ is a sufficient condition for the existence of the Mackey topology, but we do not know if it is also necessary. We end the section with the proof that the quotients of $(\mathscr{M})$-Mackey groups are essentially $(\mathscr{M})$-Mackey groups.

A more accurated family must be defined in order to obtain a necessary and sufficient condition for the existence of the Mackey topology. Let $\mathscr{S} \subset \mathscr{M}$ be the subfamily formed by those sets $B \in \mathscr{M}$ such that $\tau_{\{B\}} \vee \sigma(G, G^\wedge)$ is a topology compatible for G. We denote by $\tau_{\mathscr{S}} = sup_{B \in \mathscr{S}} \tau_{\{B\}}$ the topology defined by uniform convergence on the sets of $\mathscr{S}$. Clearly $\tau_{\mathscr{S}} \subset \tau_{\mathscr{M}}$, and $\tau_{\mathscr{S}}$ is the supremum of all compatible topologies (Theorem 7.15). Thus, compatibility of $\tau_{\mathscr{S}}$ is the backbone for the existence of the Mackey topology. In Theorem 7.14, we give a necessary and sufficient condition for it, therefore for the existence of the Mackey topology.

Finally we prove that if (G, τ) is a Mackey group (i.e. $\tau = \tau_{\mathscr{S}}$) and H any subgroup of G, then G/H is also a Mackey group, provided it is a locally quasi-convex group.

In Sect. 7.8 we obtain the following two consequences of our previous results. First, a countable group of bounded exponent does not admit a nondiscrete reflexive group topology. This result has been recently proved in [4] by other means. In the second place a more impressive result: the group of rationals $\mathbb{Q}$ with its usual topology is not a Mackey group.

7.2 Notation and Preliminaries

The chapter deals with abelian groups, therefore we omit the term "abelian" in the sequel and we use additive notation. The symbol 0 denotes the neutral element of a group G. For a group G, a natural number $n \in \mathbb{N}$ and a subset $A \subset G$, we define two new subsets:

$nA := \{nx : x \in A\}$, $\frac{1}{n}A := \{x \in G : nx \in A\}$.

Let A and B be subsets of the group G, we define

$A + B := \{a + b : a \in A, b \in B\} \subset G$.

We denote by $\mathbb{R}$ the additive group of real numbers endowed with the euclidean topology, by $\mathbb{T}$ the quotient topological group $\mathbb{R}/\mathbb{Z}$, and let

$\mathbb{T}_+ := [-1/4, 1/4] + \mathbb{Z} \subset \mathbb{T}$,

$\mathbb{T}_n := [-1/4n, 1/4n] + \mathbb{Z} \subset \mathbb{T}$, where $n \in \mathbb{N}$.

It is easy to prove that $\mathbb{T}_n = \bigcap_{m=1}^{n} \frac{1}{m}\mathbb{T}_+$.

For a group G, the symbol $H \leq G$ means that H is a subgroup of G. If (G, τ) is a topological group, $H \leq (G, \tau)$ means that H is a subgroup of G endowed with the relative topology $\tau|_H$.

A **character** on a group G is a homomorphism from G to $\mathbb{T}$.

The set of all characters of G endowed with the pointwise operation is a group which will be denoted by $G^* := Hom(G, \mathbb{T})$. If G is a topological group, the **dual group** of G, denoted by $G^\wedge$, is the subgroup of G^* formed by the continuous characters. Thus $G^\wedge := CHom(G, \mathbb{T}) \leq G^*$.

A subgroup H of a topological group G is called **dually embedded** in G if every continuous character of H can be extended to a continuous character of G. The subgroup H is said to be **dually closed** if for every $x \in G \setminus H$ there is an element ψ in $G^\wedge$ such that $\psi(x) \neq 0$ and $\psi(H) = \{0\}$.

Definition 7.1 Let G be a topological group and λ another group topology on G. The topology λ is said to be **compatible** with the original topology of G if G and (G, λ) have the same dual group, i.e. if $(G, \lambda)^\wedge = G^\wedge$. Similarly, λ is said to be **subcompatible** if $(G, \lambda)^\wedge \leq G^\wedge$.

The topological group G is said to be **maximally almost periodic (MAP)** if the dual group $G^\wedge \leq G^*$ is a separating subgroup, which means that $G^\wedge$ separates the points of G. In this case we will say that the duality $<G, G^\wedge>$ is a separating duality.

The **Bohr topology** of a topological group G, denoted by $\sigma(G, G^\wedge)$, is the weak topology induced by $G^\wedge$ on G. It is a group topology, and clearly the topological group $(G, \sigma(G, G^\wedge))$ is Hausdorff if and only if G is MAP. In the dual group $G^\wedge$, we denote by $\sigma(G^\wedge, G)$ the topology of pointwise convergence, and we write $G^\wedge_p := (G^\wedge, \sigma(G^\wedge, G))$. Note that $G^\wedge_p$ is always Hausdorff.

For topologies λ and τ defined on a set X, the symbol $\tau \vee \lambda$ will denote the **supremum topology**, i.e. the topology generated by $\tau \cup \lambda$. If $X = G$ is an abelian group, and λ and τ are group topologies, the supremum $\tau \vee \lambda$ is also a group topology with basis of 0-neighbourhoods given by $\{V \cap W : V \in \mathscr{B}_\tau, W \in \mathscr{B}_\lambda\}$, where $\mathscr{B}_\tau$, $\mathscr{B}_\lambda$ are basis of 0-neighborhoods in τ and λ respectively.

Let G be a topological group. For a subset $A \subset G$, the **polar set** of A in G^* is defined as $A^{\blacktriangleright} := \{\chi \in G^* : \chi(C) \subset \mathbb{T}_+\}$, and the polar set of A (in $G^\wedge$) as $A^{\triangleright} := A^{\blacktriangleright} \cap G^\wedge$. If H is a subgroup of G, the **annihilator** of H (in $G^\wedge$) is the subgroup of $G^\wedge$ defined by $H^\perp := \{\chi \in G^\wedge : \chi(H) = \{0\}\}$. Clearly $H^\perp = H^{\triangleright}$ and $H^\perp \leq G^\wedge$. Sometimes we will write $H^{\perp G^*}$, for the annihilator group of H in G^*, hence $H^{\perp G^*} = H^{\blacktriangleright}$ and clearly $H^\perp = H^{\perp G^*} \cap G^\wedge$. It is well-known and easy to prove that if U is a 0-neighborhood, then $U^{\triangleright} = U^{\blacktriangleright}$.

For a topological group G and a subgroup $H \leq G$, G/H denotes the **quotient group** endowed with the quotient topology. It will be explicitly stated if some other topology is considered on the algebraic quotient group G/H. The projection $G \longrightarrow G/H$ is a continuous and open homomorphism. Since we may canonically identify $(G/H)^*$ with the subgroup $H^{\perp G^*} \leq G^*$, it is clear that the dual group of the quotient topological group $(G/H)^\wedge$ may be identified with $H^\perp$, since:

$$(G/H)^\wedge = (G/H)^* \cap G^\wedge = H^{\perp G^*} \cap G^\wedge = H^\perp \leq G^\wedge.$$

Thus the Bohr topology of the quotient group G/H is denoted by $\sigma(G/H, H^\perp)$. It can be also proved that $< G/H, H^\perp >$ is a separating duality whenever H is dually closed in G. In other words: if H is Bohr closed, then G/H is MAP being this fact independent of whether G is MAP or not.

For a subset $B \subset G^\wedge$ we define the **inverse polar** of B (in G) as

$$B^{\triangleleft} := \{x \in G : \chi(x) \in \mathbb{T}_+ \text{ for each } \chi \in B\} = \bigcap_{\chi \in B} \chi^{-1}(\mathbb{T}_+).$$

We also define
$(B, \mathbb{T}_n) := \{x \in G : \chi(x) \in \mathbb{T}_n, \forall \chi \in B\} = \bigcap_{\chi \in B} \chi^{-1}(\mathbb{T}_n)$, $n \in \mathbb{N}$,
$(B, \varepsilon) := \{x \in G : \chi(x) \in [-\varepsilon, \varepsilon] + \mathbb{Z}, \forall \chi \in B\}$, $\varepsilon > 0$.
The following formulae are easy to check:
$(\bigcup_{i \in I} B_i)^{\triangleleft} = \bigcap_{i \in I} (B_i)^{\triangleleft}$, $(mB)^{\triangleleft} = \frac{1}{m} B^{\triangleleft}$, for $m \in \mathbb{N}$
and $(B, \mathbb{T}_n) = \bigcap_{m=1}^{n} (mB)^{\triangleleft} = (\bigcup_{m=1}^{n} mB)^{\triangleleft}$, where $n \in \mathbb{N}$.

A nonempty subset A of a topological group G is said to be **quasi-convex** if for every $x \in G \setminus A$ there exists a continuous character $\chi \in G^\wedge$ so that $\chi \in A^{\triangleright}$ and $\chi(x) \notin \mathbb{T}_+$. It is straightforward to prove that the intersection of any family of quasi-convex subsets is quasi-convex. Hence the quasi-convex hull $q(A)$ of a subset $A \subset G$ can be defined as the smallest quasi-convex subset containing A and explicitly, $q(A)$ is the intersection of all quasi-convex subsets containing A. It is easy to see that $q(A) = A^{\triangleright\triangleleft}$, therefore A is quasi-convex if and only if $A = A^{\triangleright\triangleleft}$. Any quasi-convex subset A is closed in the Bohr topology $\sigma(G, G^\wedge)$, since $A = A^{\triangleright\triangleleft} = \bigcap_{\chi \in A^{\triangleright}} \chi^{-1}(\mathbb{T}_+)$. For $B \subset G^\wedge$, observe that $B^{\triangleleft} = (B^{\triangleleft})^{\triangleright\triangleleft}$ and then the inverse polar $B^{\triangleleft}$ is always quasi-convex.

In a similar way one can define quasi-convex subsets in $G_p^\wedge$. A subset $B \subset G^\wedge$ is quasi-convex if and only if $B = B^{\triangleleft\triangleright}$ and in this case B is closed in $G_p^\wedge$. We also have that $A^{\triangleright}$ is quasi-convex in $G_p^\wedge$, for any $A \subset G$.

Definition 7.2 A topological group G is said to be **locally quasi-convex (LQC)** if there exists a neighborhood basis of 0 consisting of quasi-convex subsets.

If G is a LQC group then $\mathscr{B} := \{U^{\triangleright\triangleleft} : U$ is a 0-neighborhood of $G\}$ is a neighborhood basis of 0 consisting of quasi-convex subsets. As observed in [7], the family $\mathscr{B}$ is always a basis of 0-neighborhoods for a group topology on G, which is strictly coarser than the original one if G is not LQC.

Let (G, τ) be a topological group and $\mathscr{B}$ a neighborhood basis of 0 for (G, τ). From [16] it can be deduced that the family of **τ-equicontinuous subsets** of $G^\wedge$ is:

$$\mathscr{E}(\tau) := \{B \subset G^\wedge : B \text{ is } \tau\text{-equicontinuous } \} = \bigcup_{U \in \mathscr{B}} \mathscr{P}(U^\triangleright) = \bigcup_{U \in \mathscr{B}} \mathscr{P}(U^\blacktriangleright), \tag{7.1}$$

where $\mathscr{P}(A)$ is the power-set of the set A. It is known that a character $\chi \in G^*$ is continuous, i.e. $\chi \in G^\wedge$, if and only if there exists a 0-neighborhood U so that $\chi \in U^\blacktriangleright$. Consequently, $U^\triangleright = U^\blacktriangleright$ for every 0-neighborhood U. Since $U^\blacktriangleright$ is closed in the compact group $G^*_p = (G^*, \sigma(G^*, G))$, we obtain that $U^\triangleright = U^\blacktriangleright$ is compact and quasi-convex in $G^\wedge_p \leq G^*_p$.

Definition 7.3 For a topological group (G, τ), the **locally quasi-convex modification** (LQC modification) of τ is a new group topology τ_{lqc} defined as the finest among all LQC topologies on G coarser than τ. We denote (G, τ_{lqc}) by $(G, \tau)_{lqc}$.

Remark 7.1 The family $\mathscr{B} := \{U^{\triangleright\triangleleft} : U \text{ is a } 0 - \text{neighborhood for } (G, \tau)\}$ is a neighborhood basis of 0 for $(G, \tau)_{lqc}$. Clearly (G, τ) is LQC iff $(G, \tau)_{lqc} = (G, \tau)$. The LQC modification appears for the first time in [7].

Proposition 7.1 *The topological groups (G, τ) and $(G, \tau)_{lqc}$ admit the same dual group, that is $(G, \tau)^\wedge = (G, \tau_{lqc})^\wedge$. Furthermore, the τ-equicontinuous subsets of $G^\wedge$ coincide with the τ_{lqc}-equicontinuous. Thus we can write: $\mathscr{E}(\tau) = \mathscr{E}(\tau_{lqc})$.*

Proof From $\tau_{lqc} \subset \tau$, we obtain $(G, \tau_{lqc})^\wedge \leq (G, \tau)^\wedge$.

Conversely, if $\chi \in (G, \tau)^\wedge$, there exists a 0-neighborhood U for τ so that $\chi \in U^\triangleright$. Since $(U^\triangleright)^{\triangleleft\triangleright} = U^\triangleright$, $\chi \in (U^{\triangleright\triangleleft})^\triangleright$. As $U^{\triangleright\triangleleft}$ is a 0-neighborhood for τ_{lqc}, then $\chi \in (G, \tau_{lqc})^\wedge$.

The second assertion is derived from the equality $(U^\triangleright)^{\triangleleft\triangleright} = (U^{\triangleright\triangleleft})^\triangleright = U^\triangleright$. □

Thus every group topology gives rise, in a standard way, to a LQC topology which admits the same dual group. This fact allows to replace in some occasions the family of all compatible topologies for a fixed topological group (G, τ), by the family of all compatible LQC topologies for (G, τ).

7.3 Uniform Convergence Topologies on Abelian Groups

Let G be an algebraic group, $G^* = Hom(G, \mathbb{T})$ the group of characters of G, and let $\mathscr{G} \subset \mathscr{P}(G^*)$ be a family of subsets of G^*. We define $\tau_{\mathscr{G}}$ as the **topology of uniform convergence on the family** $\mathscr{G}$. More explicitly, if $\tau_{\{B\}}$ is the topology of uniform convergence on the subset $B \subset G^*$, then $\tau_{\mathscr{G}} = sup_{B\in\mathscr{G}}\tau_{\{B\}}$. The topology $\tau_{\mathscr{G}}$ is a group topology on G, in fact we will see that $(G, \tau_{\mathscr{G}})$ is LQC.

Definition 7.4 The above family $\mathscr{G}$ is **directed** if for every $A \in \mathscr{G}, B \in \mathscr{G}$ there exists $C \in \mathscr{G}$ so that $A \cup B \subset C$. It is **well directed** if it is directed and for every $n \in \mathbb{N}, A \in \mathscr{G}$ there exists $B \in \mathscr{G}$ so that $nA \subset B$.

The following two symbols express respectively the **directed envelope** and the **well directed envelope** of a fixed family $\mathscr{G}$:
$\widetilde{\mathscr{G}} := \{\cup_{i\in F} A_i : A_i \in \mathscr{G}, F \text{ finite}\}$,
$\overline{\mathscr{G}} := \{\cup_{i\in F}\ n_i A_i : A_i \in \mathscr{G}, F \text{ finite}, n_i \in \mathbb{N}\}$.
Clearly $\mathscr{G} \subset \widetilde{\mathscr{G}} \subset \overline{\mathscr{G}}$.

Remark 7.2 For a topological group (G, τ), the family $\mathscr{E}(\tau)$ of τ-equicontinuous subsets of $G^\wedge$ (see (7.1)) is well directed, furthermore $\mathscr{E}(\tau) = \overline{\mathscr{E}(\tau)}$. The proof is straightforward.

Theorem 7.1 *With the above notation, the following statements hold:*

(1) *A neighborhood basis of* 0 *for* $(G, \tau_{\mathscr{G}})$ *is* $\mathscr{B}_1 = \{(B, \mathbb{T}_n) : B \in \widetilde{\mathscr{G}}, n \in \mathbb{N}\}$, *and also* $\mathscr{B}_2 = \{B^{\triangleleft} : B \in \overline{\mathscr{G}}\}$. *It is clear that* $\tau_{\mathscr{G}} = \tau_{\widetilde{\mathscr{G}}} = \tau_{\overline{\mathscr{G}}}$.
(2) $(G, \tau_{\mathscr{G}})$ *is a LQC group, not necessarily Hausdorff.*
(3) *The dual group is* $(G, \tau_{\mathscr{G}})^\wedge = \bigcup_{B\in\widetilde{\mathscr{G}}} \bigcup_{n\in\mathbb{N}} (B, \mathbb{T}_n)^{\blacktriangleright} = \bigcup_{B\in\overline{\mathscr{G}}} B^{\triangleleft\blacktriangleright}$.
(4) *Let* $\overline{\mathscr{G}}^{\triangleleft\blacktriangleright} := \{B^{\triangleleft\blacktriangleright} : B \in \overline{\mathscr{G}}\}$. *Then the family of* $\tau_{\mathscr{G}}$*-equicontinuous is* $\mathscr{E}(\tau_{\mathscr{G}}) = \bigcup_{B\in\overline{\mathscr{G}}} \mathscr{P}(B^{\triangleleft\blacktriangleright}) = \bigcup_{B\in\overline{\mathscr{G}}^{\triangleleft\blacktriangleright}} \mathscr{P}(B)$, *and* $\overline{\mathscr{G}}^{\triangleleft\blacktriangleright}$ *is a well directed family.*

Proof (1) and (2) are in [9]. For the proof of (3), recall that $U^{\triangleright} = U^{\blacktriangleright}$ holds for any 0-neighborhood U, and therefore the dual group can be expressed as:

$$(G, \tau_{\mathscr{G}})^\wedge = \bigcup_{U\in\mathscr{B}_1} U^{\blacktriangleright} = \bigcup_{V\in\mathscr{B}_2} V^{\blacktriangleright}$$

In order to prove 4), we input $\mathscr{B}_2$ to the expression for $\mathscr{E}(\tau)$ obtained in (7.1). □

Corollary 7.1 *Let* (G, τ) *be a topological group. Then the LQC modification of* τ *is the topology of uniform convergence on the family of the* τ*-equicontinuous subsets,* $\tau_{lqc} = \tau_{\mathscr{E}(\tau)}$, *and hence* (G, τ) *is LQC if and only if* $\tau = \tau_{\mathscr{E}(\tau)}$. *On the other hand* $(G, \tau)_{lqc}$ *is Hausdorff if and only if it is MAP.*

The Bohr topology of a topological group G coincides with the topology of the uniform convergence on the well directed family $\mathscr{F} := \{F \subset G^\wedge : F \text{ finite }\}$ of the finite subsets of $G^\wedge$, so $\sigma(G, G^\wedge) = \tau_{\mathscr{F}}$ and $(G, \tau_{\mathscr{F}})$ is LQC.

Definition 7.5 Let (G, τ) be a topological group. We denote by $\tau_g(G, G^\wedge)$ the least upper bound of the family of all LQC topologies compatible with τ and by $\eta_g(G, G^\wedge)$ the least upper bound of the family of all LQC topologies subcompatible with τ. We can write:

$$\tau_g(G, G^\wedge) := \sup\{\lambda : (G, \lambda) \text{ is a LQC group and } (G, \lambda)^\wedge = G^\wedge\}. \tag{7.2}$$

$$\eta_g(G, G^\wedge) := \sup\{\lambda : (G, \lambda) \text{ is a LQC group and } (G, \lambda)^\wedge \leq G^\wedge\}. \tag{7.3}$$

In a similar way, for a subgroup $A \leq G^*$ we define:

$$\tau_g(G, A) := \sup\{\lambda : (G, \lambda) \text{ is a LQC group and } (G, \lambda)^\wedge = A\}. \tag{7.4}$$

$$\eta_g(G, A) := \sup\{\lambda : (G, \lambda) \text{ is a LQC group and } (G, \lambda)^\wedge \leq A\}. \tag{7.5}$$

By their definitions it is clear that $\tau_g \leq \eta_g$. We do not know if they are equal. To see that τ_g and η_g are LQC topologies we need the following theorem.

Theorem 7.2 *Let G be a group and let $\{\tau_i, i \in I\}$ be a family of LQC topologies on G. Then, $\tau := sup\{\tau_i : i \in I\}$ is a LQC topology on G. Moreover τ can be described as the topology of uniform convergence on the family $\mathscr{G} = \bigcup_{i\in I} \mathscr{E}(\tau_i)$.*

Proof The first assertion is in [9, Theorem 3.3]. For the second one, observe that a 0-neighborhood for the supremum topology τ can be written as

$$\bigcap_{i\in F}(B_i, \mathbb{T}_{n_i}) = \bigcap_{i\in F} \bigcap_{m_i=1}^{n_i} (m_i B_i)^{\triangleleft} = (\bigcup_{i\in F} \bigcup_{m_i=1}^{n_i} m_i B_i)^{\triangleleft}$$

where $B_i \in \mathscr{E}(\tau_i)$, $n_i \in \mathbb{N}$ for every $i \in F \subset I$, with F finite.

Since $\bigcup_{i\in F} \bigcup_{m_i=1}^{n_i} m_i B_i \in \overline{\mathscr{G}}$ and any element of $\overline{\mathscr{G}}$ can be expressed in this way, the second assertion follows from Theorem 7.1. □

Remark 7.3 In the last theorem if $\tau_i = \tau_{\mathscr{G}_i}$, where $\mathscr{G}_i$ is a family of subsets of G^* for every $i \in I$, then the supremum topology is
$\tau = \sup\{\tau_i : i \in I\} = \tau_{\mathscr{G}}$, being now $\mathscr{G} = \bigcup_{i\in I} \mathscr{G}_i$.

The proof of this remark is totally analogous to the above theorem.

Corollary 7.2 *Let (G, τ) be a topological group. Let us define the families*
$\mathscr{S} := \bigcup_{\lambda,(G,\lambda)^\wedge = G^\wedge} \mathscr{E}(\lambda)$ *and* $\mathscr{M} := \bigcup_{\lambda,(G,\lambda)^\wedge \leq G^\wedge} \mathscr{E}(\lambda)$.
Then: $(G, \tau_g(G, G^\wedge)) = (G, \tau_{\mathscr{S}})$ *and* $(G, \eta_g(G, G^\wedge)) = (G, \tau_{\mathscr{M}})$.

It is straightforward to check that $A \subset B$ and $B \in \mathscr{M}$ imply $A \in \mathscr{M}$. The same is true for $\mathscr{S}$. It is also clear that $\mathscr{S} \subset \mathscr{M}$.

Observe that $(G, \tau_g(G, G^\wedge))$ and $(G, \eta_g(G, G^\wedge))$ are LQC groups (Theorem 7.2), but the topologies τ_g and η_g might not be compatible. This justifies the following definitions:

Definition 7.6 (1) We will say that the **Mackey topology** exists for the duality $< G, G^\wedge >$ if $\tau_g(G, G^\wedge)$ is compatible with the duality $< G, G^\wedge >$. That is, if $(G, \tau_g(G, G^\wedge))^\wedge = G^\wedge$.

(2) A LQC group (G, τ) is a **Mackey group** if $(G, \tau) = (G, \tau_g(G, G^\wedge))$. In other words, a LQC group (G, τ) is said to be a Mackey group if for any locally quasi-convex topology ν on G such that $(G, \nu)^\wedge = (G, \tau)^\wedge$, it holds $\nu \leq \tau$.

It is an open problem to know if the equality $(G, \tau_g(G, G^\wedge))^\wedge = G^\wedge$ holds for every topological group G. For some classes of topological groups an affirmative answer was obtained in [9].

It is clear that if the topological group (G, τ) is Mackey then $\mathscr{S} = \mathscr{E}(\tau)$ and $\tau = \tau_g(G, G^\wedge) = \tau_{\mathscr{S}}$. For a topological group (G, τ), it may happen that $\mathscr{M} = \mathscr{E}(\tau)$ and then $\tau = \eta_g(G, G^\wedge) = \tau_{\mathscr{M}}$. In this case (G, τ) is also a Mackey group, which justifies the following definition.

Definition 7.7 Let us say that a topological group (G, τ) is $(\mathscr{M})$-Mackey if it is LQC and the family of its equicontinuous subsets $\mathscr{E}(\tau)$ coincides with the family $\mathscr{M}$, therefore $\tau = \tau_{\mathscr{M}} = \eta_g(G, G^\wedge)$.

Thus a $(\mathscr{M})$-Mackey group is Mackey in a stronger sense. In the following theorem we relate the topologies τ_g and η_g.

Theorem 7.3 *Let G be a topological group. Then:*

$$\tau_{\mathscr{M}} = sup\{\lambda : (G, \lambda) \text{ is LQC with } (G, \lambda)^\wedge \leq G^\wedge\} = sup\{\tau_g(G, A) : A \leq G^\wedge\}.$$

It may happen that $\tau_g(G, A)$ is not Hausdorff.

Proof With the definition of the family $\mathscr{M}$ and Theorem 7.2 the first equality is clear. Now, for each $A \leq G^\wedge$, write $\mathscr{S}(A) := \bigcup_{\lambda, (G,\lambda)^\wedge = A} \mathscr{E}(\lambda) \subset \mathscr{M}$. By the definition of $\mathscr{M}$ we can set

$\mathscr{M} = \bigcup_{A \leq G^\wedge} \bigcup_{\lambda, (G,\lambda)^\wedge = A} \mathscr{E}(\lambda) = \bigcup_{A \leq G^\wedge} \mathscr{S}(A)$

Recall that $\tau_{\mathscr{S}(A)} = \tau_g(G, A)$ and therefore the second equality again follows from Remark 7.3. □

Let (G, τ) be a topological group. Besides the families $\mathscr{M}$ and $\mathscr{S}$ of Corollary 7.2, the following families will be also used in the sequel:

$\mathscr{F} := \{F \subset G^\wedge : F \text{ finite}\}$.

$\mathscr{Q} := \{B \subset G_p^\wedge : \exists K \text{ compact and quasi-convex in } G_p^\wedge \text{ with } B \subset K\} = \bigcup_K \mathscr{P}(K)$ with K compact and quasi-convex subset of $G_p^\wedge$.

$\mathscr{K} := \{B \subset G_p^\wedge : \exists K \text{ compact in } G_p^\wedge \text{ with } B \subset K\} = \bigcup_K \mathscr{P}(K)$ with K a compact subset of $G_p^\wedge$.

The families $\mathscr{F}$, $\mathscr{E}(\tau)$ and $\mathscr{K}$ are well directed, and from their definitions it follows that $\mathscr{F} \subset \mathscr{E}(\tau) \subset \mathscr{S} \subset \mathscr{M}$ and $\mathscr{Q} \subset \mathscr{K}$. Further, by Theorem 7.1, if $B \in \mathscr{M}$ then $B \subset B^{\triangleleft\blacktriangleright} \subset G^\wedge$ and clearly $B^{\triangleleft\blacktriangleright}$ is compact and quasi-convex in $G_p^\wedge$. Thus $\mathscr{M} \subset \mathscr{Q}$.

We define an order on the families of subsets of G^*, by establishing $\mathscr{G} < \mathscr{H}$ if $\tau_{\mathscr{G}} \subset \tau_{\mathscr{H}}$. If $\mathscr{G} \subset \mathscr{H}$ then $\mathscr{G} < \mathscr{H}$, and we will say $\mathscr{G} \approx \mathscr{H}$ when $\tau_{\mathscr{G}} = \tau_{\mathscr{H}}$. If (G, τ) is a LQC group then:

$$\mathscr{F} \subset \mathscr{E}(\tau) \subset \mathscr{S} \subset \mathscr{M} \subset \mathscr{Q} \subset \mathscr{K}.$$

Consequently,

$$\sigma(G, G^\wedge) = \tau_{\mathscr{F}} \subset \tau = \tau_{\mathscr{E}(\tau)} \subset \tau_{\mathscr{S}} = \tau_g(G, G^\wedge) \subset \tau_{\mathscr{M}} = \eta_g(G, G^\wedge) \subset \tau_{\mathscr{Q}} \subset \tau_{\mathscr{K}},$$

and $G^\wedge \leq (G, \tau_{\mathscr{S}})^\wedge \leq (G, \tau_{\mathscr{M}})^\wedge \leq (G, \tau_{\mathscr{Q}})^\wedge \leq (G, \tau_{\mathscr{K}})^\wedge$.

We now recall some definitions which are related with the families $\mathscr{Q}$ and $\mathscr{K}$ above defined.

Definition 7.8 [9] A LQC group (G, τ) is said to be:

1. **g-barrelled**, if every compact subset of $G_p^\wedge$ is equicontinuous. Equivalently, if $\mathscr{E}(\tau) = \mathscr{K}$, and then $(G, \tau) = (G, \tau_{\mathscr{K}})$.
2. **pre-Mackey**, if every compact and quasi-convex subset of $G_p^\wedge$ is equicontinuous, or equivalently, if $\mathscr{E}(\tau) = \mathscr{Q}$, and then $(G, \tau) = (G, \tau_{\mathscr{Q}})$.

Theorem 7.4 *[9] Let G be a LQC group. The following implications hold true: G is g-barrelled $\Longrightarrow$ G is pre-Mackey $\Longrightarrow$ G is Mackey.*

With Definition 7.7, and since $\tau_{\mathscr{M}} \subset \tau_{\mathscr{Q}}$, we may introduce a new implication: G is pre-Mackey $\Longrightarrow$ G is $(\mathscr{M})$-Mackey $\Longrightarrow$ G is Mackey.

Remark 7.4 The g-barrelled groups were introduced in [9] by the following statement: An abelian topological group (G, τ) is g-barrelled if every $\sigma(G^\wedge, G)$-compact subset is equicontinuous. With this definition it can be easily checked (through Proposition 7.1) that (G, τ) is g-barrelled iff (G, τ_{lqc}) is g-barrelled. Since we only deal with LQC g-barrelled groups, we find convenient to define g-barrelled groups only in the class of LQC groups.

As proved in [9], the class of g-barrelled groups includes several well known types of groups like the Čech-complete, the metrizable hereditarily Baire, the separable Baire and the pseudocompact groups. Taking into account our modified definition, we should require further local quasi-convexity for the validity of this assertion, or else do the assertion for the LQC modification. For example, if G is a metrizable and complete group, then G_{lqc} is g-barrelled, however it may not be complete (Example 7.1).

In [6] there are examples of Mackey groups that are not g-barrelled neither pre-Mackey. These examples are $(\mathscr{M})$-Mackey groups, therefore $(G, \tau) = (G, \tau_{\mathscr{M}})$, that is $\mathscr{E}(\tau) = \mathscr{M} \neq \mathscr{Q}$.

In Definition 7.7 a new class of Mackey groups has been obtained by means of the condition $\mathscr{E}(\tau) = \mathscr{M}$ and clearly $\mathscr{M} \subset \mathscr{Q} \subset \mathscr{K}$. In the root paper [9] the groups (G, τ) for which $\mathscr{E}(\tau) = \mathscr{K}$ or $\mathscr{E}(\tau) = \mathscr{Q}$ were respectively called g-barrelled or pre-Mackey. In order to ensamble this new class in the existing schema it is convenient to introduce new names.

Definition 7.9 Let us say that a topological group (G, τ) is $(\mathscr{K})$-Mackey or $(\mathscr{Q})$-Mackey if it is locally quasi-convex and the family of its equicontinuous subsets $\mathscr{E}(\tau)$ coincides respectively with $\mathscr{K}$ or $\mathscr{Q}$, therefore $\tau = \tau_{\mathscr{K}}$ or $\tau = \tau_{\mathscr{Q}}$.

From now on the g-barrelled (pre-Mackey) groups will be named $(\mathscr{K})$-Mackey ($(\mathscr{Q})$-Mackey) groups.

Observe that the conditions $\mathscr{E}(\tau) = \mathscr{K}$ and $\mathscr{E}(\tau) = \mathscr{Q}$—which define respectively when a group (G, τ) is g-barrelled or pre-Mackey—can be weakened to simply require $\mathscr{E}(\tau) = \mathscr{M}$ or $\mathscr{E}(\tau) = \mathscr{S}$. A topological group G for which these equalities occur is also a Mackey group, and so we could establish a grading in "being a Mackey group". We shall use the term **Mackey-type properties** to indicate these sort of properties which in particular imply that (G, τ) is a Mackey group.

Example 7.1 Consider the topological group $G = (l_p, \tau_p)$, $p \in (0, 1)$, where (l_p, τ_p) is the topological vector space formed by the sequences $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ such that $\sum_{n\in\mathbb{N}} |x_n|^p < \infty$. The topology τ_p is given by the metric

$\rho(x, y) = \sum_{n\in\mathbb{N}} |x_n - y_n|^p$, where $x = (x_n)_{n\in\mathbb{N}}$ and $y = (y_n)_{n\in\mathbb{N}}$.

It is known that G is metrizable and complete. In [14] it is proved that (l_p, τ_p) is a topological vector space which is not locally convex. Therefore G is not a LQC group [5].

The LQC modification G_{lqc} is metrizable non complete. In fact the locally quasi convex modification of the topology τ_p coincides with the restriction to l_p of τ_1, the standard topology of l_1. Summarizing, $G_{lqc} = (l_p, \tau_1) \leq (l_1, \tau_1)$, and G_{lqc} is dense in $l_1 \neq G_{lqc}$.

We conclude that $G_{lqc} = (l_p, \tau_1)$ is metrizable, non complete, $(\mathscr{K})$-Mackey (or g-barrelled), and then it is a Mackey group.

This example provides a metrizable complete group G, such that its LQC modification G_{lqc} is a $(\mathscr{K})$-Mackey metrizable non complete group.

7.4 Quotient Groups of LQC Groups

In this section we first focus our attention on the Hausdorff property. If a topological group G is non-Hausdorff, it is well known that $G/\overline{\{0\}}$ is a Hausdorff group very related to G. In a natural sense, it can be said that $G/\overline{\{0\}}$ is the Hausdorff-ication of G. We next study this operation in the context of LQC groups.

If a topological group G is not MAP, then the closure of $\{0\}$ in the Bohr Topology of G is $\overline{\{0\}} = (G^\wedge)^\perp$, the von Neumann kernel. It is straightforward to prove that a Hausdorff LQC group is MAP. It is frequent in the literature, to define LQC groups already in the class of Hausdorff abelian groups, to make sure that they are MAP groups, a convenient property. Clearly, a non-Hausdorff LQC group G has nontrivial von Neumann kernel, $(G^\wedge)^\perp$. We next formulate explicitly some items essential for our future considerations.

Definition 7.10 Let (G, λ) be a LQC group and let $A := (G, \lambda)^\wedge$. We define a LQC and Haudorff group topology $h(\lambda)$ on $G/A^\perp$, by stating $(G/A^\perp, h(\lambda)) := (G, \lambda)/A^\perp$.

Lemma 7.1 *The topological group $(G/A^\perp, h(\lambda))$ is LQC and Hausdorff. If the dual group of $G/A^\perp$ is identified with A, from the equality $(G/A^\perp, h(\lambda))^\wedge = A$, we obtain:*

$$\mathscr{E}(\lambda) = \mathscr{E}(h(\lambda)) \text{ and } h(\lambda) = \tau_{\mathscr{E}(\lambda)}.$$

Proof Since $< G/A^\perp, A >$ is a separating duality it is clear that $h(\lambda)$ is Hausdorff.

Let us prove that $(G, \lambda)/A^\perp$ is a LQC group. Fix U, a quasi-convex neighborhood of 0 for (G, λ). Then, $A^\perp \subset U$ since the non-null elements of $A^\perp$ cannot be separated from 0 by continuous characters. It is easy to deduce that $(U + A^\perp)^\triangleright = U^\triangleright \cap A = U^\triangleright$. In order to see that the 0-neighborhood $U + A^\perp$ of $(G/A^\perp, h(\lambda))$ is quasi-convex, take into account that:

$(U + A^\perp)^{\triangleright\triangleleft G/A^\perp} = (U^\triangleright \cap A)^{\triangleleft G/A^\perp} = (U^\triangleright \cap A)^\triangleleft + A^\perp = U^{\triangleright\triangleleft} + A^\perp = U + A^\perp \subset G/A^\perp$.

As $(U + A^\perp)^\triangleright = U^\triangleright$, with Eq. (7.1) we obtain that $\mathscr{E}(\lambda) = \mathscr{E}(h(\lambda))$.

Since $h(\lambda)$ is LQC and $\mathscr{E}(\lambda) = \mathscr{E}(h(\lambda))$, it follows from Corollary 7.1 that $h(\lambda) = \tau_{\mathscr{E}(\lambda)}$. □

Summarizing, by means of h we have constructed a locally quasi-convex topology on the quotient of a LQC group G by its von Neumann kernel, which loosely speaking has the same dual group as the original group G and the same equicontinuous family. Now we try to do the inverse operation, that is, starting with a locally quasi-convex topology on a quotient group G/H, we construct a locally quasi-convex topology in G so that both dual groups together with the corresponding equicontinuous families might be identified.

Notation 7.1 If $(G/H, \nu)$ is a topological group and $h : G \longrightarrow G/H$ the canonical mapping, we denote by $h^{-1}(\nu)$ the inverse image (or initial) topology defined on G, whose neighborhood basis of 0 is $\mathscr{B}_{h^{-1}(\nu)} = \{h^{-1}(U) : U \text{ is a 0-neighborhood for } \nu\}$.

Lemma 7.2 *Let $(G/H, \nu)$ be a LQC group (non necessarily Hausdorff), let $A := (G/H, \nu)^\wedge$ and $h : G \longrightarrow G/H$ the canonical mapping. Then $(G, h^{-1}(\nu))$ is a LQC group, where $(G, h^{-1}(\nu))^\wedge$ may be identified with A and $\mathscr{E}(h^{-1}(\nu)) = \mathscr{E}(\nu)$.*

Proof It is straightforward to prove that $(G, h^{-1}(\nu))$ is LQC. Now let U be a 0-neighborhood for ν, then

$(h^{-1}(U))^\blacktriangleright = (h^{-1}(U) + H)^\blacktriangleright = (h^{-1}(U))^\blacktriangleright \cap H^{\perp G^*}$.

After identifying $(G/H)^*$ with $H^{\perp G^*}$,

$(h^{-1}(U))^\blacktriangleright \cap H^{\perp G^*} = U^{\blacktriangleright (G/H)^*} = U^\triangleright$.

Therefore $(G, h^{-1}(\nu))^\wedge = A$ and $\mathscr{E}(h^{-1}(\nu)) = \mathscr{E}(\nu)$.

The groups $(G/H, \nu)$ and $(G, h^{-1}(\nu))$ in Notation 7.1 and Lemma 7.2 are not necessarily Hausdorff. The operations h and h^{-1} performed respectively in Definition 7.10 and in Notation 7.1 are inverse to each other as we specify next.

Lemma 7.3 *With the above definitions, the following assertions hold:*

(1) *If* (G, λ) *is a LQC group and* $(G, \lambda)^\wedge = A$*, then* $(G, \lambda) = (G, h^{-1}(h(\lambda)))$.
(2) *If* $(G/H, \nu)$ *is a Hausdorff LQC group and* $(G/H, \nu)^\wedge = A$*, then:*

$$(G/H, \nu) = (G/H, h(h^{-1}(\nu)) = (G, h^{-1}(\nu))/H.$$

Proof The same argument proves (1) and (2). By Lemmas 7.1 and 7.2, it is only necessary to observe that all the topological groups involved in the equalities of (1) and (2) are LQC, with identical dual groups and identical equicontinuous families. In fact, (1) follows from

$$(G, \lambda)^\wedge = (G, h^{-1}(h(\lambda)))^\wedge \text{ and } \mathscr{E}(\lambda) = \mathscr{E}(h(\lambda)) = \mathscr{E}(h^{-1}(h(\lambda))),$$

and (2) follows from $(G/H, \nu)^\wedge = (G/H, h(h^{-1}(\nu)))^\wedge = ((G, h^{-1}(\nu))/H)^\wedge$
and from $\mathscr{E}(\nu) = \mathscr{E}(h^{-1}(\nu)) = \mathscr{E}(h(h^{-1}(\nu)))$. □

The next result will be useful in order to prove some of the main theorems. It also has interest on its own.

Theorem 7.5 *Let* H *be a subgroup of a topological group* G*. Let* $(G/H, \nu)$ *be a LQC group so that* $(G/H, \nu)^\wedge = A \leq H^\perp \leq G^\wedge$*. Then:*

(1) $(G/H, \nu \vee \sigma(G/H, H^\perp)) = (G, h^{-1}(\nu) \vee \sigma(G, G^\wedge))/H$.
(2) $(G/H, \nu \vee \sigma(G/H, H^\perp))^\wedge = H^\perp \Leftrightarrow (G, h^{-1}(\nu) \vee \sigma(G, G^\wedge))^\wedge = G^\wedge$.

Proof (1) Denote by U a 0-neighborhood for ν and by V a 0-neighborhood for $\sigma(G, G^\wedge)$. A 0-neighborhood for $(G, h^{-1}(\nu) \vee \sigma(G, G^\wedge))/H$ can be written as $h^{-1}(U) \cap V + H$. Clearly $h^{-1}(U) \cap V + H = U \cap h(V)$, and since $h(V)$ is a 0-neighborhood of $(G, \sigma(G, G^\wedge))/H = (G/H, \sigma(G/H, H^\perp))$ we can deduce that $(G, h^{-1}(\nu) \vee \sigma(G, G^\wedge))/H = (G/H, \nu \vee \sigma(G/H, H^\perp))$.

(2) The implication $\Leftarrow$) follows from (1). In order to prove the reverse implication, assume

$$(G, h^{-1}(\nu) \vee \sigma(G, G^\wedge))^\wedge = D. \qquad (\dagger)$$

It is easy to observe that
$(H, (h^{-1}(\nu) \vee \sigma(G, G^\wedge))|_H) = (H, \sigma(G, G^\wedge)|_H) = (H, \sigma(H, G^\wedge/H^\perp))$.
From (†) we have that $D \supset G^\wedge$ and $h^{-1}(\nu) \vee \sigma(G, G^\wedge) = h^{-1}(\nu) \vee \sigma(G, D)$.
As above:
$(h^{-1}(\nu) \vee \sigma(G, D))|_H = \sigma(G, D)|_H = \sigma(H, D/H^{\perp D})$, where $H^{\perp D} = H^{\perp G^*} \cap D$.
Then $\sigma(H, G^\wedge/H^\perp) = \sigma(H, D/H^{\perp D})$, Which implies that $D/H^{\perp D} = G^\wedge/H^\perp$. From this and from $H^\perp = H^{\perp D} \cap G^\wedge$, by the Second Isomorphism Theorem (Noether) we deduce that $D = G^\wedge + H^{\perp D}$. By (†) applied to 1), we obtain that

$$(G/H, \nu \vee \sigma(G/H, H^\perp)) = (G, h^{-1}(\nu) \vee \sigma(G, D))/H = (G/H, \nu \vee \sigma(G/H, H^{\perp D})).$$

Since $(G/H, \nu \vee \sigma(G/H, H^{\perp}))^{\wedge} = H^{\perp}$ and $H^{\perp} \leq H^{\perp D}$, it yields that $H^{\perp} = H^{\perp D}$, and finally

$D = G^{\wedge} + H^{\perp D} = G^{\wedge} + H^{\perp} = G^{\wedge}$. □

7.5 Stability of Mackey-Type Properties with Respect to Quotients

As indicated in the title, in this section we deal with quotients of Mackey groups. We need to realize first that a LQC group might have quotients groups that are not LQC groups, [5, (5.3)]. This fact is not disturbing for our considerations on compatible topologies, since we can replace such a quotient by its locally quasi-convex modification, which has the same dual group.

Based upon Theorem 7.4 and the comments after it, together with the stability properties listed in the following Lemma, we had the conjecture that all the Mackey-type properties behave well with respect to the operation of taking quotients. We shall prove it in this and in subsequent sections.

Lemma 7.4 *Let G be a topological group and H a closed subgroup of G. Then the following assertions hold:*

1. *If G is completely metrizable, then G/H is also completely metrizable.*
2. *If G is Čech-complete, then G/H is also Čech-complete.*
3. *If G is Baire separable, then G/H is also Baire separable.*

Proof The proofs of 1 and 2 are respectively [1, 4.3.26] and [20, (6.10, p. 47)] . The proof of 3 follows from the facts that separability is preserved through continuous mappings, and the Baire property is preserved for quotients as can be easily derived from Exercise B in [15, Sect. 9]. □

For all the classes of groups considered in Lemma 7.4, the locally quasi-convex modification of the corresponding groups and quotient groups are $(\mathscr{K})$-Mackey (g-barrelled) groups. Thus, one suspects that the locally quasi-convex modification of the quotient of a $(\mathscr{K})$-Mackey (g-barrelled) group is again $(\mathscr{K})$-Mackey (g-barrelled). In order to have the tools to analyze these questions, we describe next the family $\mathscr{G}$ of equicontinuous subsets for the quotient group G/H of a topological group G. Clearly, the LQC modification is given through: $(G/H)_{lqc} = (G/H, \tau_{\mathscr{G}})$.

Lemma 7.5 *Let (G, τ) be a topological group and $H \leq G$ a fixed subgroup. If we denote the quotient group by $(G/H, \lambda) := (G, \tau)/H$, then the family of λ-equicontinuous subsets is $\mathscr{E}(\lambda) = \{B \cap H^{\perp} : B \in \mathscr{E}(\tau)\}$. Here we have identified $((G, \tau)/H)^{\wedge}$ with the subgroup $H^{\perp} \leq G^{\wedge}$.*

Proof If $\mathscr{B}$ is a neighborhood basis of 0 for (G, τ), then $\mathscr{B}' = \{U + H \subset G/H : U \in \mathscr{B}\}$ is a neighborhood basis of 0 for $(G/H, \lambda) = (G, \tau)/H$. In terms of the basis $\mathscr{B}'$, the family of λ-equicontinuous subsets is $\mathscr{E}(\lambda) = \bigcup_{V \in \mathscr{B}'} \mathscr{P}(V^{\triangleright H^{\perp}})$. Let us describe $V^{\triangleright H^{\perp}}$ for $V \in \mathscr{B}'$. To that end, fix $V = U + H \in \mathscr{B}'$ with $U \in \mathscr{B}$. We have:

$$\begin{aligned} V^{\triangleright H^{\perp}} &= (U+H)^{\triangleright H^{\perp}} = (\bigcup_{x \in U} \{x+H\})^{\triangleright H^{\perp}} \\ &= \bigcap_{x \in U} \{x+H\}^{\triangleright H^{\perp}} = \bigcap_{x \in U} (\{x\}^{\triangleright} \cap H^{\perp}) \\ &= U^{\triangleright} \cap H^{\perp}. \end{aligned}$$

Since $U^{\triangleright} \in \mathscr{E}(\tau)$, and every τ-equicontinuous subset is contained in the polar of a zero neighborhood, we obtain the proposed expression for $\mathscr{E}(\lambda)$. □

With this Lemma we are ready to prove the stability through quotients of the Mackey-type properties. In this section we study quotients of $(\mathscr{K})$-Mackey and $(\mathscr{Q})$-Mackey groups.

Theorem 7.6 *Let G be a $(\mathscr{K})$-Mackey group and H a subgroup of G. Then the topological group $(G/H)_{lqc}$ is a $(\mathscr{K})$-Mackey group.*

Proof As (G, τ) is $(\mathscr{K})$-Mackey, $(G, \tau) = (G, \tau_{\mathscr{K}})$ where $\mathscr{E}(\tau) = \mathscr{K}$. Let us define now:

$(G/H, \lambda) := (G/H)_{lqc} = ((G, \tau_{\mathscr{K}})/H)_{lqc}$.

By Lemma 7.5 and Proposition 7.1, $\mathscr{E}(\lambda) = \{B \cap H^{\perp} : B \in \mathscr{E}(\tau) = \mathscr{K}\}$. Let $\mathscr{K}' := \{B \subset H^{\perp} : \exists K \text{ compact in } H^{\perp} \leq G_p^{\wedge} \text{ with } B \subset K\} = \bigcup_K \mathscr{P}(K)$ where K runs over the compact subsets of $H^{\perp} = (G/H)_p^{\wedge}$.

The family $\mathscr{K}'$ considered in $(G/H)_p^{\wedge}$ plays the same role as the family $\mathscr{K}$ in $G_p^{\wedge}$. It is clear that $\mathscr{K}' \subset \mathscr{K}$ and since for $B \in \mathscr{K}'$ we have $B \subset H^{\perp}$, it yields $\mathscr{K}' \subset \mathscr{E}(\lambda)$.

It also holds $\mathscr{E}(\lambda) \subset \mathscr{K}'$, and hence $\mathscr{K}' = \mathscr{E}(\lambda)$ and $(G/H)_{lqc} = (G/H, \tau_{\mathscr{K}'})$. The dual group is $(G/H, \tau_{\mathscr{K}'})^{\wedge} = (G/H)^{\wedge} = H^{\perp}$ concluding that $(G/H, \tau_{\mathscr{K}'})$ is $(\mathscr{K})$-Mackey. □

Theorem 7.7 *Let G be a $(\mathscr{Q})$-Mackey group and H a subgroup of G. Then the topological group $(G/H)_{lqc}$ is a $(\mathscr{Q})$-Mackey group.*

Proof Since (G, τ) is a $(\mathscr{Q})$-Mackey group, $(G, \tau) = (G, \tau_{\mathscr{Q}})$ where $\mathscr{E}(\tau) = \mathscr{Q}$. Let now

$(G/H, \lambda) := (G/H)_{lqc} = ((G, \tau_{\mathscr{Q}})/H)_{lqc}$.

By Lemma 7.5, $\mathscr{E}(\lambda) = \{B \cap H^{\perp} : B \in \mathscr{E}(\tau) = \mathscr{Q}\}$ is the family of the λ-equicontinuous subsets. Define:

$\mathscr{Q}' := \{B \subset H^{\perp} : \exists K \text{ compact and quasi-convex in } H^{\perp} \leq G_p^{\wedge} \text{ with } B \subset K\} = \bigcup_K \mathscr{P}(K)$ where K is a compact and quasi-convex subset of $H^{\perp} = (G/H)_p^{\wedge}$.

The family $\mathscr{Q}'$ in $(G/H)_p^{\wedge}$ plays the same role as the family $\mathscr{Q}$ in $G_p^{\wedge}$.

From the fact that $H^\perp$ is a dually closed and dually embedded subgroup of $G_p^\wedge$, by [11, (2.2)] we obtain that any quasi-convex subset $K \subset H^\perp$ is also a quasi-convex subset of $G_p^\wedge$. Thus, $\mathscr{Q}' \subset \mathscr{Q}$. Since for $B \in \mathscr{Q}'$ we have $B \subset H^\perp$, $\mathscr{Q}' \subset \mathscr{E}(\lambda)$ derives from $\mathscr{Q} \subset \mathscr{E}(\tau)$. Clearly $\mathscr{E}(\lambda) \subset \mathscr{Q}'$ also holds, hence $\mathscr{Q}' = \mathscr{E}(\lambda)$ and $(G/H)_{lqc} = (G/H, \tau_{\mathscr{Q}'})$. The dual group is $(G/H, \tau_{\mathscr{Q}'})^\wedge = (G/H)^\wedge = H^\perp$, concluding that $(G/H, \tau_{\mathscr{Q}'})$ is a $(\mathscr{Q})$-Mackey group. □

We shall prove in Sects. 7.6 and 7.7 respectively, that being $\mathscr{M}$-Mackey, or in general Mackey, are also properties stable through quotients.

7.6 The Family $\mathscr{M}$ of Subsets Determining Equicontinuity

For a topological group (G, τ) the family $\mathscr{M}$ was explicitly defined in Corollary 7.2. We start this section giving new representations of it. By means of these representations, we characterize when $\tau_{\mathscr{M}}$ is compatible and consequently when $(G, \tau_{\mathscr{M}})$ is a Mackey group. In that case it will be moreover $(\mathscr{M})$-Mackey.

Theorem 7.8 *Let G be a topological group. In the duality $< G, G^\wedge >$, the family $\mathscr{M} := \bigcup_{\lambda, (G,\lambda)^\wedge \le G^\wedge} \mathscr{E}(\lambda)$ can also be represented as:*

(1) $\mathscr{M}_1 = \bigcup_{H,\lambda} \mathscr{E}(\lambda)$ *where H and λ run respectively over all dually closed subgroups of G and over all the group topologies on G/H such that $(G/H, \lambda)$ is MAP with $(G/H, \lambda)^\wedge = A \le H^\perp$.*

(1′) $\mathscr{M}_1' = \bigcup_{H,\rho} \mathscr{E}(\rho)$ *where H and ρ run respectively over all subgroups of G and group topologies on the quotient G/H such that $(G/H, \rho)^\wedge \le H^\perp$ (here $(G/H, \rho)$ is not necessarily MAP).*

(2) $\mathscr{M}_2 = \{B \subset G^\wedge : (G, \tau_{\{B\}})^\wedge \le G^\wedge\} = \{B \subset G^\wedge : (G/B^\perp, \tau_{\{B\}})^\wedge \le B^{\perp\perp}\}$. *Here $B^{\perp\perp} = \overline{< B >}^{G_p^\wedge}$ and $(G/B^\perp, \tau_{\{B\}})$ is Hausdorff and LQC, therefore MAP.*

(3) $\mathscr{M}_3 = \{B \subset G^\wedge : (B, \mathbb{T}_n)^\triangleright = (B, \mathbb{T}_n)^\blacktriangleright, \forall n \in \mathbb{N}\}$.

(3′) $\mathscr{M}'_3 = \{B \subset G^\wedge : (\bigcup_{m=1}^n mB)^{\triangleleft\triangleright} = (\bigcup_{m=1}^n mB)^{\triangleleft\blacktriangleright}, \forall n \in \mathbb{N}\}$.

(4) $\mathscr{M}_4 = \{B \subset G^\wedge : (B, \varepsilon)^\triangleright = (B, \varepsilon)^\blacktriangleright, \forall \varepsilon \in (0, 1/4]\}$.

Proof For (1) and (1′), it is clear that $\mathscr{M}_1 \subset \mathscr{M}_1'$. By Lemma 7.2 it is straightforward to see that $\mathscr{M}_1' \subset \mathscr{M}$. From Lemma 7.1 it follows that $\mathscr{M} \subset \mathscr{M}_1$, concluding that $\mathscr{M}_1 = \mathscr{M}_1' = \mathscr{M}$.

(2) Recall that $(G, \tau_{\{B\}})$ is the group G with the topology of the uniform convergence on $B \subset G^\wedge$. Clearly $\mathscr{M}_2 \subset \mathscr{M}$. For the converse, take $B \in \mathscr{M}$. By the definition of $\mathscr{M}$ there exists a LQC topology on G, say λ, such that $B \in \mathscr{E}(\lambda)$ and $(G, \lambda)^\wedge \le G^\wedge$. Therefore $\tau_{\{B\}} \le \lambda$ and then $(G, \tau_{\{B\}})^\wedge \le (G, \lambda)^\wedge \le G^\wedge$, so $B \in \mathscr{M}_2$. With Lemmas 7.1 and 7.2 we obtain the second equality of (2).

(3) We check that $\mathscr{M}_3 = \mathscr{M}_2$. Take $B \in \mathscr{M}_2$. By Theorem 7.1 (3) we deduce that $(G, \tau_{\{B\}})^\wedge = \bigcup_{n\in\mathbb{N}} (B, \mathbb{T}_n)^\blacktriangleright$, and from here it follows that:

$$B \in \mathscr{M}_2 \Leftrightarrow \bigcup_{n\in\mathbb{N}} (B, \mathbb{T}_n)^{\blacktriangleright} \leq G^{\wedge} \Leftrightarrow (B, \mathbb{T}_n)^{\blacktriangleright} = (B, \mathbb{T}_n)^{\triangleright}, \forall n \in \mathbb{N} \Leftrightarrow B \in \mathscr{M}_3$$

(3′) follows from the fact that $(B, \mathbb{T}_n) = \bigcap_{m=1}^{n} (mB)^{\triangleleft} = (\bigcup_{m=1}^{n} mB)^{\triangleleft}$.

(4) Let us prove that $\mathscr{M}_4 = \mathscr{M}_3$. Since $(B, \mathbb{T}_n) = (B, 1/4n)$ where $n \in \mathbb{N}$, it is clear that $\mathscr{M}_4 \subset \mathscr{M}_3$. Let now $B \in \mathscr{M}_3$. For $\varepsilon \in (0, 1/4]$ there exists $n \in \mathbb{N}$ so that $\frac{1}{4n} \leq \varepsilon$, and hence $(B, \mathbb{T}_n) \subset (B, \varepsilon)$. This implies that $(B, \mathbb{T}_n)^{\blacktriangleright} \supset (B, \varepsilon)^{\blacktriangleright}$. From $(B, \mathbb{T}_n)^{\triangleright} = (B, \mathbb{T}_n)^{\blacktriangleright}$, we obtain $(B, \varepsilon)^{\triangleright} = (B, \varepsilon)^{\blacktriangleright}$. □

Remark 7.5 The family $\mathscr{M}_4$ was already considered in [6]. The authors proved that if $\mathscr{M}_4$ is a well directed family, then there exists a Mackey topology in the duality $< G, G^{\wedge} >$. In Theorem 7.9 we prove that "well directed" can be weakened to "directed" in their statement. However we do not know if the existence of the Mackey topology implies that $\mathscr{M}$ must be a directed family, or equivalently if G is Mackey implies that G is $(\mathscr{M})$-Mackey.

Theorem 7.9 *Let G be a topological group. The following assertions are equivalent:*

(1) *The group topology $\tau_{\mathscr{M}}$ is compatible (i.e. $(G, \tau_{\mathscr{M}})^{\wedge} = G^{\wedge}$ and $(G, \tau_{\mathscr{M}})$ is a $(\mathscr{M})$-Mackey group), and hence it is the Mackey topology in $< G, G^{\wedge} >$.*
(2) *The family $\mathscr{M}$ is directed (i.e. $\mathscr{M} = \widetilde{\mathscr{M}}$).*

Proof (1) ⇒ (2). Since $(G, \tau_{\mathscr{M}})^{\wedge} = G^{\wedge}$, we have $\mathscr{E}(\tau_{\mathscr{M}}) \subset \mathscr{M}$. It is obvious that $\mathscr{M} \subset \mathscr{E}(\tau_{\mathscr{M}})$ and, by Remark 7.2, the family $\mathscr{M} = \mathscr{E}(\tau_{\mathscr{M}})$ is directed.

(2) ⇒ (1). By Theorem 7.8. (2), if $B \in \mathscr{M}$ then:
$\mathscr{E}(\tau_{\{B\}}) = \bigcup_{n\in\mathbb{N}} \mathscr{P}((B, \mathbb{T}_n)^{\triangleright}) = \bigcup_{n\in\mathbb{N}} \mathscr{P}((\bigcup_{m=1}^{n} mB)^{\triangleleft\triangleright}) \subset \mathscr{M}$,
so $B^{\triangleleft\triangleright} \in \mathscr{M}$ and $nB \in \mathscr{M}$, $\forall n \in \mathbb{N}$.

Since $\mathscr{M} = \widetilde{\mathscr{M}}$ and $nB \in \mathscr{M}$ for every $B \in \mathscr{M}$, $n \in \mathbb{N}$, it follows that $\mathscr{M} = \widetilde{\mathscr{M}} = \overline{\mathscr{M}}$ is well directed and with Theorem 7.1 a neighborhood basis of 0 for $\tau_{\mathscr{M}}$ is $\mathscr{B}(\tau_{\mathscr{M}}) = \{B^{\triangleleft} : B \in \mathscr{M}\}$. For this basis we can set $(G, \tau_{\mathscr{M}})^{\wedge} = \bigcup_{B\in\mathscr{M}} B^{\triangleleft\blacktriangleright}$. By Theorem 7.8 (3′) $B^{\triangleleft\blacktriangleright} = B^{\triangleleft\triangleright}$, and $(G, \tau_{\mathscr{M}})^{\wedge} = \bigcup_{B\in\mathscr{M}} B^{\triangleleft\triangleright} = G^{\wedge}$. Thus $\tau_{\mathscr{M}}$ is compatible and $(G, \tau_{\mathscr{M}})$ is a Mackey group, in this case it is a $(\mathscr{M})$-Mackey group. □

As a corollary of the above theorem, taking into account Remark 7.2, we obtain the following important result:

Theorem 7.10 *Let (G, τ) be a LQC topological group. If every $B \in \mathscr{M}$ is equicontinuous (i.e. $\mathscr{E}(\tau) = \mathscr{M}$), then (G, τ) is a $(\mathscr{M})$-Mackey group.*

In Theorems 7.6 and 7.7 we proved that the properties of being $(\mathscr{K})$-Mackey (g-barrelled) or $(\mathscr{Q})$-Mackey (pre-Mackey) are essentially preserved by quotients. We will prove next (Corollary 7.4) that the same pattern is followed by the property of being $(\mathscr{M})$-Mackey. More precisely, if G is a topological group such that $G = (G, \tau_{\mathscr{M}})$, i.e. G is a $(\mathscr{M})$-Mackey group, then for any subgroup $H \leq G$, $(G/H)_{lqc} = (G/H, \tau_{\mathscr{M}'})$, where the family $\mathscr{M}'$ defined for the duality $< G/H, H^{\perp} >$ plays the same role as the family $\mathscr{M}$ in the duality $< G, G^{\wedge} >$. In other words, $(G/H)_{lqc}$ is also a $(\mathscr{M})$-Mackey group.

Theorem 7.11 *Let G be a topological group, H a subgroup of G and* $A \leq H^{\perp}$*. Let* $\mathscr{M}' := \bigcup_{\rho,(G/H,\rho)^{\wedge} \leq A} \mathscr{E}(\rho)$*. Then, the natural mappings*

$$(G, \tau_{\mathscr{M}}) \longrightarrow ((G, \tau_{\mathscr{M}})/H)_{lqc} \longrightarrow (G/H, \tau_{\mathscr{M}'}) \longrightarrow (G/H, \tau_g(G/H, A))$$

are continuous homomorphisms.

Proof Set $(G/H, \lambda) := ((G, \tau_{\mathscr{M}})/H)_{lqc}$. By Lemma 7.5 and Proposition 7.1 $\mathscr{E}(\lambda) = \{B \cap H^{\perp} : B \in \mathscr{E}(\tau_{\mathscr{M}})\}$, where $H^{\perp}$ is the annihilator of H in $(G, \tau_{\mathscr{M}})^{\wedge}$. Now by Theorem 7.8 1′) $\mathscr{M}' \subset \mathscr{M}$ and since $B \in \mathscr{M}'$ implies $B \subset A \subset H^{\perp}$ and $\mathscr{M} \subset \mathscr{E}(\tau_{\mathscr{M}})$, we deduce $\mathscr{M}' \subset \mathscr{E}(\lambda)$, concluding that $((G, \tau_{\mathscr{M}})/H)_{lqc} = (G/H, \tau_{\mathscr{E}(\lambda)}) \to (G/H, \tau_{\mathscr{M}'})$ is a continuous homomorphism.

By the definition of $\mathscr{M}'$ it is clear that $\mathscr{M}' \supset \bigcup_{\nu} \mathscr{E}(\nu)$ where $(G/H, \nu)^{\wedge} = A$ and then $(G/H, \tau_{\mathscr{M}'}) \longrightarrow (G/H, \tau_g(G/H, A))$ is a continuous homomorphism. □

Corollary 7.3 *Let G be a* $(\mathscr{M})$*-Mackey group, so* $G = (G, \tau_{\mathscr{M}})$*. Let* $H \leq G$*,* $A \leq H^{\perp}$*, and let* $\mathscr{M}' = \bigcup_{\rho,(G/H,\rho)^{\wedge} \leq A} \mathscr{E}(\rho)$*. Then* $(G/H, \tau_{\mathscr{M}'})^{\wedge} \leq H^{\perp}$*.*

Proof As $(G, \tau_{\mathscr{M}})^{\wedge} = G^{\wedge}$, $((G, \tau_{\mathscr{M}})/H)^{\wedge} = H^{\perp}$. In Theorem 7.11 we proved that the natural mapping

$$(G, \tau_{\mathscr{M}})/H \longrightarrow (G/H, \tau_{\mathscr{M}'})$$

is a continuous homomorphism. Dualizing this expression, we obtain:

$$(G/H, \tau_{\mathscr{M}'})^{\wedge} \leq ((G, \tau_{\mathscr{M}})/H)^{\wedge} = H^{\perp}. \qquad \square$$

Corollary 7.4 *Let G be a* $(\mathscr{M})$*-Mackey group, so* $G = (G, \tau_{\mathscr{M}})$*. Let H be a subgroup of G. Then the LQC modification of G/H is a* $(\mathscr{M})$*-Mackey group such that* $(G/H)_{lqc} = (G/H, \tau_{\mathscr{M}'})$*, with* $\mathscr{M}' = \bigcup_{\rho,(G/H,\rho)^{\wedge} \leq H^{\perp}} \mathscr{E}(\rho)$*, i.e.* $(G/H, \tau_{\mathscr{M}'})^{\wedge} = H^{\perp}$*.*

Proof If $(G/H, \lambda) := G/H$, it holds: $(G/H)_{lqc} = (G/H, \tau_{\mathscr{E}(\lambda)})$ and $(G/H)^{\wedge} = H^{\perp}$. Now we apply Corollary 7.3 with $A = H^{\perp}$, and we obtain that $(G/H, \tau_{\mathscr{M}'})^{\wedge} = (G/H)^{\wedge} = H^{\perp}$. By Theorem 7.11 $\mathscr{M}' \subset \mathscr{E}(\lambda)$ and by the definition of $\mathscr{M}'$ also $\mathscr{E}(\lambda) \subset \mathscr{M}'$, thus $\mathscr{M}' = \mathscr{E}(\lambda)$ and we have that $(G/H)_{lqc} = (G/H, \tau_{\mathscr{M}'})$. □

Theorem 7.12 *Let* (G, τ) *be a* $(\mathscr{M})$*-Mackey group, so that* $\tau = \tau_{\mathscr{M}}$ *and* $\mathscr{E}(\tau) = \mathscr{M}$*. The following assertions hold:*

(1) *If* ν *denotes a new LQC topology on G, then* $(G, \nu)^{\wedge} \leq G^{\wedge} \Leftrightarrow \nu \subset \tau$*.*
(2) *If H is a subgroup of* (G, τ) *and* λ *a LQC topology on G/H, then the canonical projection* $(G, \tau) \longrightarrow (G/H, \lambda)$ *is continuous if and only if* $(G/H, \lambda)^{\wedge} \leq H^{\perp} \leq G^{\wedge}$*.*

Proof In order to prove (1), assume that $\mathscr{E}(\tau) = \mathscr{M}$. From the definition of the family $\mathscr{M}$, it is clear that $\nu \subset \tau$ if and only if $\mathscr{E}(\nu) \subset \mathscr{E}(\tau) = \mathscr{M}$ if and only if $(G, \nu)^{\wedge} \leq G^{\wedge}$.

The proof of (2) follows from the fact that $(G, \tau) \longrightarrow (G/H, \lambda)$ is a continuous homomorphism if and only if $\mathscr{E}(\lambda) \subset \mathscr{E}(\tau) = \mathscr{M}$ and by Theorem 7.8. (1′) this last condition is equivalent to $(G/H, \lambda)^\wedge \leq G^\wedge$. □

7.7 The Family $\mathscr{S}$

The family $\mathscr{S}$ defined in Sect. 7.3 turns out to be optimal in the following sense: the topology $\tau_{\mathscr{S}}$ of uniform convergence on $\mathscr{S}$ coincides with τ_g, the least upper bound of all LQC topologies compatible with τ. Therefore a ($\mathscr{S}$)-Mackey group defined in the above spirit, is simply a Mackey group.

We begin this section with new representations for the family $\mathscr{S}$. From these representations we characterize when $\tau_g(G, G^\wedge)$ is a compatible topology and thus the duality $< G, G^\wedge >$ admits a Mackey topology.

Theorem 7.13 *Let G be a topological group. For the duality* $<G, G^\wedge>$*, the family* $\mathscr{S} := \bigcup_{\lambda, (G,\lambda)^\wedge = G^\wedge} \mathscr{E}(\lambda)$ *above defined admits the following representations:*

(1) $\mathscr{S}_1 = \bigcup_{H,\nu} \mathscr{E}(\nu)$ *where* $H \leq G$ *and* ν *is a topology on* G/H *such that* $(G/H, \nu)^\wedge = H^\perp$.
(2) $\mathscr{S}_2 = \{B \subset G^\wedge : (G, \tau_{\{B\}} \vee \sigma(G, G^\wedge))^\wedge = G^\wedge\}$.
It also holds: $\mathscr{S}_2 = \{B \subset G^\wedge : (G/B^\perp, \tau_{\{B\}} \vee \sigma(G/B^\perp, B^{\perp\perp}))^\wedge = B^{\perp\perp}\}$.
(3) $\mathscr{S}_3 = \{B \in \mathscr{M} : \forall F \subset G^\wedge, F$ *finite* $, B \cup F \in \mathscr{M}\}$.

Proof (1) In order to prove $\mathscr{S} \subset \mathscr{S}_1$, just take $H = G^{\wedge\perp}$ and apply Lemma 7.1.

Now, by Theorem 7.5 applied to the particular case $A = H^\perp$, if $(G/H, \nu)^\wedge = H^\perp$ then $(G, h^{-1}(\nu) \vee \sigma(G, G^\wedge))^\wedge = G^\wedge$. So $B \in \mathscr{E}(\nu)$ implies
$B \in \mathscr{E}(h^{-1}(\nu)) \subset \mathscr{E}(h^{-1}(\nu) \vee \sigma(G, G^\wedge)) \subset \mathscr{S}$, concluding $\mathscr{S}_1 \subset \mathscr{S}$.

(2) $\mathscr{S}_2 \subset \mathscr{S}$ is trivial. Conversely, if $B \in \mathscr{S}$ there exists then a LQC topology λ so that $B \in \mathscr{E}(\lambda)$ and $(G, \lambda)^\wedge = G^\wedge$. Therefore $\tau_{\{B\}} \vee \sigma(G, G^\wedge) \leq \lambda$ and we have:

$$G^\wedge = (G, \sigma(G, G^\wedge))^\wedge \leq (G, \tau_{\{B\}} \vee \sigma(G, G^\wedge))^\wedge \leq (G, \lambda)^\wedge = G^\wedge.$$

Hence $\mathscr{S} \subset \mathscr{S}_2$.

For the second part of (2) apply Theorem 7.5.

(3) We check that $\mathscr{S}_2 = \mathscr{S}_3$. Let $B \in \mathscr{S}_2$, and therefore $(G, \tau_{\{B\}} \vee \sigma(G, G^\wedge))^\wedge = G^\wedge$. It is easy to deduce that

$$\tau_{\{B\}} \vee \sigma(G, G^\wedge) = \tau_{\{B\} \cup \mathscr{F}}, \text{ where } \mathscr{F} = \{F \subset G^\wedge : F \text{ finite }\}.$$

Now, applying Theorem 7.1 (3) we obtain:

$$(G, \tau_{\{B\}} \vee \sigma(G, G^\wedge))^\wedge = (G, \tau_{\{B\} \cup \mathscr{F}})^\wedge = \bigcup_{F \in \mathscr{F}} \bigcup_{n \in \mathbb{N}} (B \cup F, \mathbb{T}_n)^\blacktriangleright.$$

Since $B \in \mathscr{S}_2$, B is already in $\mathscr{M}$ and $(B \cup F, \mathbb{T}_n)^{\blacktriangleright} = (B \cup F, \mathbb{T}_n)^{\triangleright}$ for every $F \in \mathscr{F}$ and $n \in \mathbb{N}$. By Theorem 7.8. (3) this assertion is equivalent to $B \cup F \in \mathscr{M}$ for each $F \in \mathscr{F}$. Thus it follows $\mathscr{S}_2 = \mathscr{S}_3$. □

Theorem 7.14 *Let G be a topological group. Then, the Mackey topology exists for the duality* $< G, G^\wedge >$ *(i.e.* $(G, \tau_g(G, G^\wedge))^\wedge = (G, \tau_{\mathscr{S}})^\wedge = G^\wedge$*) if and only if the family* $\mathscr{S}$ *is directed (i.e.* $\mathscr{S} = \widetilde{\mathscr{S}}$*).*

Proof $\Rightarrow$): From $(G, \tau_{\mathscr{S}})^\wedge = G^\wedge$ and the definition of the family $\mathscr{S}$, we obtain $\mathscr{E}(\tau_{\mathscr{S}}) \subset \mathscr{S}$. On the other hand $\mathscr{S} \subset \mathscr{E}(\tau_{\mathscr{S}})$ and again by Remark 7.2 the family $\mathscr{S} = \mathscr{E}(\tau_{\mathscr{S}})$ must be directed.

$\Leftarrow$): By Theorem 7.13 (2), if $B \in \mathscr{S}$ then:

$$\mathscr{E}(\tau_{\{B\}}) = \bigcup_{n\in\mathbb{N}} \mathscr{P}((B, \mathbb{T}_n)^{\triangleright}) = \bigcup_{n\in\mathbb{N}} \mathscr{P}((\bigcup_{m=1}^{n} mB)^{\triangleleft\triangleright}) \subset \mathscr{S},$$

so $B^{\triangleleft\triangleright} \in \mathscr{S}$ and $nB \in \mathscr{S}$ for every $n \in \mathbb{N}$.

Since $\mathscr{S} = \widetilde{\mathscr{S}}$ and $nB \in \mathscr{S}$ for every $B \in \mathscr{S}$ and $n \in \mathbb{N}$, we have that $\mathscr{S} = \widetilde{\mathscr{S}} = \overline{\mathscr{S}}$ is a well directed family. Applying then Theorem 7.1, we obtain the following neighborhood basis of 0 for $\tau_{\mathscr{S}}$, $\mathscr{B}(\tau_{\mathscr{S}}) = \{B^{\triangleleft} : B \in \mathscr{S}\}$. With respect to this basis, we can set $(G, \tau_{\mathscr{S}})^\wedge = \bigcup_{B\in\mathscr{S}} B^{\triangleleft\blacktriangleright}$. Since $B \in \mathscr{S} \subset \mathscr{M}$, $B^{\triangleleft\blacktriangleright} = B^{\triangleleft\triangleright}$ and hence $(G, \tau_{\mathscr{S}})^\wedge = \bigcup_{B\in\mathscr{S}} B^{\triangleleft\triangleright} = G^\wedge$.

We now definitely prove (Theorem 7.16) the stability of the Mackey property through quotients. Explicitly, if G is a Mackey group, for any $H \leq G$, the LQC modification of the quotient group $(G/H)_{lqc}$ is also a Mackey group.

Theorem 7.15 *Let G be a topological group. Then* $\tau_g(G, G^\wedge) = \tau_{\mathscr{S}}$ *and for every subgroup* $H \leq G$*, the following natural mappings*

$$(G, \tau_g(G, G^\wedge)) \longrightarrow ((G, \tau_g(G, G^\wedge))/H)_{lqc} \longrightarrow (G/H, \tau_g(G/H, H^\perp))$$

are continuous homomorphisms.

Proof The first homomorphism is continuous just by the definition of the LQC-modification. In order to obtain that the second one is also continuous, set $(G/H, \lambda) := ((G, \tau_{\mathscr{S}})/H)_{lqc}$. By Lemma 7.5, $\mathscr{E}(\lambda) = \{B \cap H^\perp : B \in \mathscr{E}(\tau_{\mathscr{S}})\}$, where $H^\perp$ is now the annihilator of H in $(G, \tau_{\mathscr{S}})^\wedge$. Let $\mathscr{S}' = \bigcup_\nu \mathscr{E}(\nu)$ where ν runs over the LQC topologies with $(G/H, \nu)^\wedge = H^\perp$. The family $\mathscr{S}'$ in $(G/H)^\wedge$ has the same role as the family $\mathscr{S}$ in $G^\wedge$.

Clearly $(G/H, \tau_g(G/H, H^\perp)) = (G/H, \tau_{\mathscr{S}'})$. By Theorem 7.13 (1), $\mathscr{S}' \subset \mathscr{S}$ and if $B \in \mathscr{S}'$ then $B \subset H^\perp$ and we deduce that $\mathscr{S}' \subset \mathscr{E}(\lambda)$. Therefore the mapping

$$((G, \tau_{\mathscr{S}})/H)_{lqc} = (G/H, \tau_{\mathscr{E}(\lambda)}) \longrightarrow (G/H, \tau_{\mathscr{S}'})$$

is a continuous homomorphism. □

Theorem 7.16 *Let G be a Mackey group and let $H \leq G$. Then the LQC modification of G/H is a Mackey group, so $(G/H)_{lqc} = (G/H, \tau_g(G/H, H^{\perp}))$.*

Proof As G is a Mackey group, $G = (G, \tau_{\mathscr{S}})$. From the equality $(G, \tau_{\mathscr{S}})^\wedge = G^\wedge$ we obtain $((G, \tau_{\mathscr{S}})/H)^\wedge = H^{\perp}$, and now dualizing Theorem 7.15 we have

$$(G/H, \tau_g(G/H, H^{\perp}))^\wedge = ((G, \tau_{\mathscr{S}})/H)^\wedge = H^{\perp}.$$

By the definition of the topology $\tau_g(G/H, H^{\perp})$, we conclude that
$((G, \tau_{\mathscr{S}})/H)_{lqc} = (G/H, \tau_g(G/H, H^{\perp}))$ is a Mackey group.

Corollary 7.5 *Let (G, τ) be a LQC topological group. The following assertions are equivalent:*

(1) *(G, τ) is Mackey.*
(2) *Every $B \in \mathscr{S}$ is equicontinuous, i.e. $\mathscr{E}(\tau) = \mathscr{S}$.*
(3) *For every subgroup $H \leq G$, $(G/H)_{lqc}$ is Mackey.*

7.8 Two Consequences

The following theorem was proved in [4]. Here we use an observation of [6] (see the claim) to give a more direct proof.

Theorem 7.17 *Let G be a countable, Pontryagin reflexive group of bounded torsion. Then G is discrete.*

Proof **Claim** [6]. Let H be a MAP group of bounded torsion such that $|H^\wedge| < \mathfrak{c}$. Then H carries the Bohr topology and it is a Mackey group. Furthermore, $\mathscr{M} = \mathscr{F}$.

Since G is reflexive, then the dual group with the compact-open topology $H := G_c^\wedge$ is also reflexive and of bounded torsion. It is clear that $H^\wedge = G^{\wedge\wedge} = G$ is countable. Now we apply the claim to H, deducing that H must be precompact. As $H^\wedge$ is countable we obtain that H is precompact and metrizable. By [2] or [8], $H_c^\wedge = G$ must be discrete. □

Theorem 7.18 *The topological group $\mathbb{Q} \leq \mathbb{R}$ is not a Mackey group.*

Proof Assume $\mathbb{Q}$ is Mackey and consider the subgroup $\mathbb{Z} \leq \mathbb{Q}$. By Theorem 7.16 the group $\mathbb{Q}/\mathbb{Z} \leq \mathbb{T}$ is a Mackey group. On the other hand $\mathbb{Q}/\mathbb{Z}$ is the torsion group of $\mathbb{T}$, i.e. $\mathbb{Q}/\mathbb{Z} = tor\mathbb{T} = \bigoplus \mathbb{Z}(p^\infty)$, where $\mathbb{Z}(p^\infty)$ is the Prüfer group, and p runs over the prime numbers.

As $\mathbb{T}$ is a compact group, $\mathbb{Q}/\mathbb{Z}$ must be a precompact group (with the induced quotient topology).

In order to obtain a contradiction we have to prove that the precompact group $\mathbb{Q}/\mathbb{Z} = (tor\mathbb{T}, \sigma(tor\mathbb{T}, \mathbb{Z}))$ is not Mackey. This will follow from the following facts.

Fact 1 Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{N}$ such that $\sum_{n\in\mathbb{N}} \frac{1}{a_n} < \infty$. Define the sequence $(\gamma_n)_{n\in\mathbb{N}}$ by $\gamma_1 = 1$ and $\gamma_n = \prod_{i=1}^{i=n-1} a_i$, for $n > 1$. Following [21], we denote by $\mathbb{Z}\{\gamma_n\}$ the group $\mathbb{Z}$ endowed with the finest group topology in which the sequence γ_n converges to zero. Then, the topological group $\mathbb{Z}\{\gamma_n\}$ is Pontryagin reflexive (See [12, Theorems 2, 3. p. 2797]) and $\mathbb{Z}\{\gamma_n\}_c^\wedge$ is a Polish group.

Fact 2 [12] Consider the group $\mathbb{Z}\{\gamma_n\}_c^\wedge$ of the previous fact, it is clear that algebraically $\mathbb{Z}\{\gamma_n\}^\wedge \leq \mathbb{T}$. Define the group $G = \{z \in \mathbb{T} : \gamma_n z = 0 \text{ for some } n \in \mathbb{N}\}$. Then the group G is dense in $\mathbb{Z}\{\gamma_n\}_c^\wedge$.

Let τ be the topology of $G \leq \mathbb{Z}\{\gamma_n\}_c^\wedge$. As G is dense in $\mathbb{Z}\{\gamma_n\}_c^\wedge$, $G^\wedge = (\mathbb{Z}\{\gamma_n\}_c^\wedge)^\wedge = \mathbb{Z}$ and the family of equicontinuous $\mathscr{E}(\tau)$ coincides with that of $(\mathbb{Z}\{\gamma_n\}_c^\wedge)^\wedge = \mathbb{Z}$.

It is clear that $\mathscr{E}(\tau) \neq \mathscr{F}$, therefore G cannot be precompact, i.e. $\sigma(G, G^\wedge) = \sigma(G, \mathbb{Z}) < \tau$. In other words, the Bohr topology is strictly coarser than the original.

Observe that for the particular case of $a_n = n!$, the group G coincides with the torsion of $\mathbb{T}$, $tor\mathbb{T}$. So we obtain that there is a finer compatible topology on the quotient $\mathbb{Q}/\mathbb{Z}$, the searched contradiction. □

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