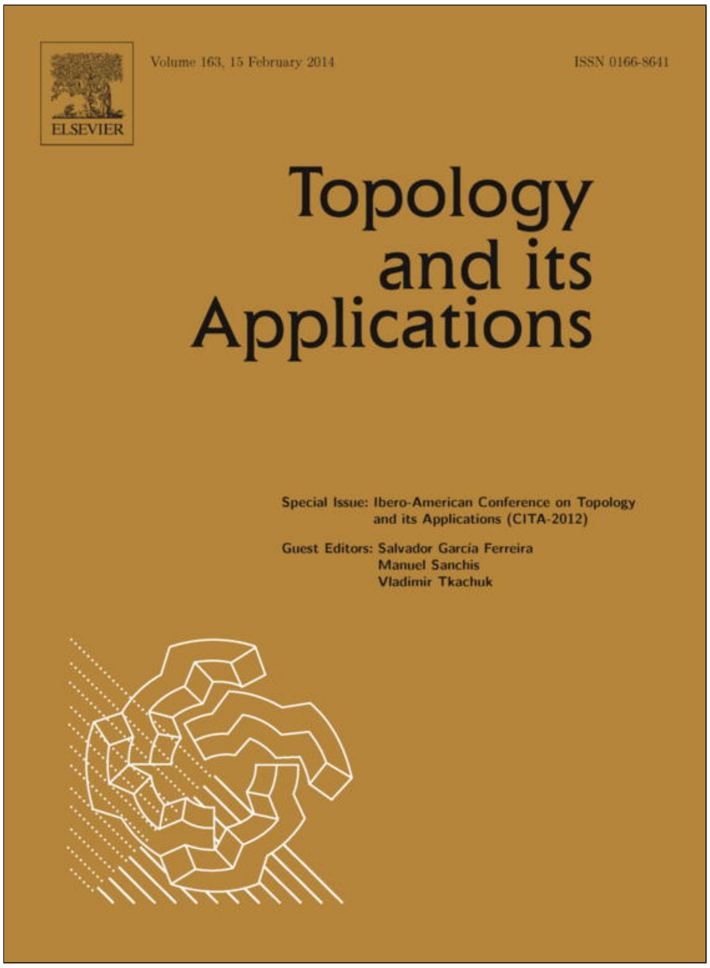
ELSEVIER
Volume 163, 15 February 2014
ISSN 0166-8641
Topology and its Applications
Special Issue: Ibero-American Conference on Topology and its Applications (CITA-2012)
Guest Editors: Salvador García Ferreira
Manuel Sanchis
Vladimir Tkachuk

Topology and its Applications 163 (2014) 11–24

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Equivariant embeddings of metrizable proper G-spaces

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A R T I C L E I N F O

MSC:
22F05
57N20
54C55
54C25
54H15
46B99

Keywords:
Locally compact group
Proper G-space
Invariant metric
Equivariant embedding
Banach G-space

A B S T R A C T

For a locally compact group G we consider the class G-$\mathcal{M}$ of all proper (in the sense of R. Palais) G-spaces that are metrizable by a G-invariant metric. We show that each $X \in G$-$\mathcal{M}$ admits a compatible G-invariant metric whose closed unit balls are small subsets of X. This is a key result to prove that X admits a closed equivariant embedding into an invariant convex subset V of a Banach G-space L such that $L \setminus \{0\} \in G$-$\mathcal{M}$ and V is a G-absolute extensor for the class G-$\mathcal{M}$. On this way we establish two equivariant embedding results for proper G-spaces which may be considered as equivariant versions of the well-known Kuratowski–Wojdyslawski theorem and Arens–Eells theorem, respectively.

1. Introduction

Throughout the paper G will denote a locally compact Hausdorff topological group. All topological spaces under discussion are Tychonoff.

The notion of a proper G-space was introduced in 1961 by R. Palais [28]. It allows to extend a substantial portion of the theory of compact Lie group actions to the case of noncompact ones. Recall that a G-space X is called *proper* (in the sense of Palais [28, Definition 1.2.2]), if X has an open cover consisting of *small* sets. Here a subset $V \subset X$ is called small, if for every point of X there is a neighborhood U with the property that the set $\langle U, V\rangle = \{g \in G \mid gU \cap V \neq \emptyset\}$ has compact closure in G.

Our focus in this paper is on the class G-$\mathcal{M}$ of all proper G-spaces that are metrizable by a G-invariant metric. In his seminal work [28], R. Palais proved that G-$\mathcal{M}$ includes all separable metrizable proper G-spaces

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1 The first author was supported by grant 165195 from CONACYT (Mexico).
2 The second author was supported by grant 165246 from CONACYT (Mexico).

0166-8641/$ – see front matter
http://dx.doi.org/10.1016/j.topol.2013.10.003

provided that G is a Lie group. J. de Vries [31] observed that the same holds true for any locally compact metrizable group G. In [11] it was shown that any locally metrizable proper G-space is in G-$\mathcal{M}$ even for arbitrary locally compact group G. However, it is an open problem of long standing whether the separability condition can entirely be omitted in this result of Palais. In other words, it remains open whether G-$\mathcal{M}$ coincides with the class of all metrizable proper G-spaces (even for $G = \mathbb{R}$ and $G = \mathbb{Z}$). We refer to [11] for a further discussion of this interesting problem.

This paper is devoted to the theory of equivariant embeddings of metrizable proper G-spaces in normed linear G-spaces. Several authors have contributed to this theory. Thus, E. Elfving [17,18] has established equivariant embeddings for proper actions of Lie groups on locally compact metrizable spaces. For more special proper G-spaces similar results were obtained earlier by M. Kankaanrinta [22,23]. A. Feragen [20] obtained equivariant embedding results for an arbitrary G-space X from the class G-$\mathcal{M}$ provided that G is a Lie group. Other related equivariant embedding results the reader can find in the papers [5] and [10].

Here we shall establish equivariant versions of the classical Kuratowski–Wojdyslawski theorem (see [14, Ch. 3, §8]) and Arens–Eells theorem (see [13]) in the class G-$\mathcal{M}$. Our Theorem 5.1 and its Corollary 5.2, roughly, assert that every G-space $X \in G$-$\mathcal{M}$ admits an equivariant embedding $i : X \hookrightarrow \mathcal{Q}(X)$ in an appropriately defined Banach G-space $\mathcal{Q}(X)$ of continuous functions $X \to \mathbb{R}$ such that $\mathcal{Q}(X) \setminus \{0\}$ is a proper G-space and the image $i(X)$ is closed in its convex hull. Theorem 5.4 and its Corollary 5.5 assert that every G-space $X \in G$-$\mathcal{M}$ admits a closed equivariant embedding $i : X \hookrightarrow L$ in a normed linear G-space L such that $L \setminus \{0\}$ is a proper G-space, $\|i(x)\| = 1$ for all $x \in X$ and the image $i(X)$ is a Hamel basis for L. These equivariant embedding results, similar to their nonequivariant counterparts, may play an important role in the equivariant theory of retracts and infinite-dimensional manifolds. At least one example of such an application is shown in Section 6, where in a very short way it is proved that the notions of G-A(N)R and G-A(N)E coincide in G-$\mathcal{M}$. Perhaps, Theorem 3.2 is the key result in our argument. It claims that each $X \in G$-$\mathcal{M}$ admits a compatible G-invariant metric ρ such that every closed unit ball $B_\rho(x, 1)$ is a small set. If in addition, X is locally compact then, as it is proved in [3], an invariant metric on X can be chosen in such a way that every ball of finite radius has compact closure (a so-called, *proper* metric).

2. Notations and terminology

Throughout the paper, unless otherwise is stated, by a *group* we shall mean a locally compact topological group satisfying the Hausdorff separation axiom. All topological spaces are assumed to be Tychonoff (= completely regular and Hausdorff). Basic definitions and facts of the theory of G-spaces or topological transformation groups can be found in the monographs G. Bredon [15], K.H. Hofmann and S.A. Morris [21], and R. Palais [27]. Our basic reference on proper group actions is Palais' article [28]. Other good sources are [24,1,2] (see also [5] and [11]).

However, for the convenience of the reader we recall some more special definitions and facts below.

By a G-space we mean a topological space X together with a fixed continuous action $G \times X \to X$ of a topological group G on X. By gx we shall denote the image of the pair $(g, x) \in G \times X$ under the action.

If Y is another G-space, a map $f : X \to Y$ is called equivariant, if $f(gx) = gf(x)$ for every $x \in X$ and $g \in G$. A continuous equivariant map is called a G-map.

If X is a G-space, then for a subset $S \subset X$ and a subset $H \subset G$, the H-hull (or H-saturation) of S is defined as follows: $H(S) = \{hs \mid h \in H,\ s \in S\}$. If S is the one point set $\{x\}$, then the G-hull $G(\{x\})$ usually is denoted by $G(x)$ and called the orbit of x. The orbit space X/G is always considered in its quotient topology.

A subset $S \subset X$ is called G-invariant or, simply, invariant if it coincides with its G-hull, i.e., $S = G(S)$.

For a closed subgroup $H \subset G$, by G/H we will denote the G-space of cosets $\{gH \mid g \in G\}$ under the action induced by left translations.

A compatible metric ρ on a metrizable G-space X is called invariant or G-invariant, if $\rho(gx, gy) = \rho(x, y)$ for all $g \in G$ and $x, y \in X$. In this case the action is called isometric and the pair (X, ρ) is named a metric G-space.

If ρ is a G-invariant metric on X, then it is easy to verify that the formula

$$\widetilde{\rho}\big(G(x), G(y)\big) = \inf\big\{\rho\big(x', y'\big) \bigm| x' \in G(x),\ y' \in G(y)\big\}$$

defines a pseudometric $\widetilde{\rho}$, compatible with the quotient topology of X/G. If, in addition, X is a proper G-space then $\widetilde{\rho}$ is, in fact, a metric on X/G [28, Theorem 4.3.4].

By a normed linear G-space (respectively, a Banach G-space) we shall mean a G-space L, where L is a normed linear space (respectively, a Banach space) on which G acts by means of *linear isometries*, i.e., $g(\lambda x + \mu y) = \lambda(gx) + \mu(gy)$ and $\|gx\| = \|x\|$ for all $g \in G$, $x, y \in L$ and λ, $\mu \in \mathbb{R}$.

In 1961 Palais [28] introduced the fundamental concept of a *proper action* of an arbitrary locally compact group G and extended a substantial part of the theory of compact Lie transformation groups to noncompact ones.

Let G be a locally compact group and X a G-space. Two subsets U and V in X are called thin relative to each other [28, Definition 1.1.1], if the set

$$\langle U, V\rangle = \{g \in G \mid gU \cap V \neq \emptyset\}$$

called *the transporter* from U to V, has compact closure in G. A subset U of a G-space X is called *G-small*, or just *small*, if every point in X has a neighborhood thin relative to U. A G-space X is called *proper* (in the sense of Palais), if every point in X has a small neighborhood.

Each orbit in a proper G-space is closed, and each stabilizer is compact [28, Proposition 1.1.4]. It is easy to check the following two statements: (1) the product of two G-spaces is proper whenever one of them is so; (2) the inverse image of a proper G-space under a G-map is again a proper G-space.

Important examples of proper G-spaces are the coset spaces G/H with H a compact subgroup of a locally compact group G. Other interesting examples can be found in [1,2,24,28]. The reader is referred to [11] for a discussion of the relationship between Palais proper G-spaces and Bourbaki proper G-spaces.

3. Existence of small metrics

Recall that a compatible metric ρ on a proper G-space X is called *small*, if every closed unit ball $B_\rho(x, 1)$ is a small subset of X.

A subset S of a proper G-space is called *fundamental*, if S is a small set and the saturation $G(S) = \{gs \mid g \in G,\ s \in S\}$ coincides with the whole space.

We begin this section with the following simple

Lemma 3.1. *Let X be a metrizable space and $F \subset X$ a closed subset, and U a neighborhood of F. Then there is a compatible metric d on X such that $d(x, y) > 1$ whenever $x \in F$ and $y \in X \setminus U$.*

Proof. By normality, there exists an open subset $V \subset X$ such that

$$F \subset V \subset \overline{V} \subset U.$$

Let $\varphi : X \to [0, 1]$, be a continuous function such that $\varphi|_{\overline{V}} \equiv 1$ and $\varphi|_{X\setminus U} \equiv 0$. If ρ is a compatible metric on X, then it is easily verified that the formula

$$d(x, y) = \rho(x, y) + \big|\varphi(x) - \varphi(y)\big|, \quad x, y \in X,$$

defines a compatible metric on X.

It follows from the definition of d that $d(x,y)=\rho(x,y)+1>1$ for $x\in F$ and $y\in X\setminus U$, which completes the proof. □

The following theorem is, perhaps, the key result in our argument.

Theorem 3.2. *Let G be a locally compact group. Then every $X\in G\text{-}\mathcal{M}$ admits a compatible G-invariant small metric.*

Proof. Since the orbit space X/G is metrizable, and hence paracompact, one can apply [2, Theorem 1.7], according to which X admits a fundamental set $S\subset X$. Since the closure of each fundamental set is also fundamental, we can assume that S is closed.

Choose identity neighborhoods $V\subset W\subset G$ such that $V=V^{-1}$, $V\cdot V\subset W$ and the closure $\overline{W}$ is compact. In particular, the closure $K=\overline{V}$ is a compact symmetric set and $K\subset W$. Then $K(S)$ is closed (see [2, Proposition 1.4(c)]), $W(S)$ is small (see [2, Proposition 1.2(e)]) and $K(S)\subset W(S)$. Finally, due to paracompactness of the orbit space X/G, one can take an open small subset U of X such that $W(S)\subset U$ (see [2, Proposition 1.8]).

Then according to Lemma 3.1, one can choose a compatible metric d on X satisfying the following property:

$$d(x,y)>1 \quad \text{whenever} \quad x\in K(S) \quad \text{and} \quad y\in X\setminus U. \tag{3.1}$$

Define

$$r(x)=d(x,X\setminus U), \quad x\in X.$$

Then for any $x,y\in X$, we have $r(x)-r(y)\leqslant d(x,y)$, and hence,

$$r(x)+r(z)\leqslant d(x,y)+r(y)+r(z) \quad \text{for all } x,y,z\in X.$$

Therefore, if we write

$$\mu(x,y)=\min\bigl\{d(x,y),r(x)+r(y)\bigr\}, \quad x,y\in X,$$

then it is obvious that μ is a pseudometric on X. Define

$$\rho(x,y)=\sup_{g\in G}\mu(gx,gy), \quad x,y\in X.$$

It is clear that ρ is a G-invariant pseudometric. Let us check that, in fact, it is a metric. Let x and y be two different points of X. Since $X=G(U)$, we infer that $g_0x\in U$ for some $g_0\in G$, yielding $r(g_0x)>0$. Since the points g_0x and g_0y are also different, we see that $d(g_0x,g_0y)>0$. Consequently, $\mu(g_0x,g_0y)>0$ which yields that $\rho(x,y)>0$. Thus, ρ is a G-invariant metric on X.

We show that ρ is compatible with the topology of X. Let (x_n) be a sequence in X such that $\rho(x_n,x_0)\rightsquigarrow 0$ for some point $x_0\in X$. Take an arbitrary $\varepsilon>0$ and let $O_d(x_0,\varepsilon)$ be the open ε-neighborhood of x_0 in the original metric of X. Since $G(U)=X$, there is an element $g_0\in G$ with $g_0x_0\in U$. Since the map $g_0^{-1}:X\to X$ is continuous and U is open, there is a $\delta>0$ such that $O_d(g_0x_0,\delta)\subset U$ and $g_0^{-1}(O_d(g_0x_0,\delta))\subset O_d(x_0,\varepsilon)$.

The inclusion $O_d(g_0x_0,\delta)\subset U$ implies that $r(g_0x_0)\geqslant\delta>0$. Since $\rho(x_n,x_0)\rightsquigarrow 0$, there exists $n_0\in\mathbb{N}$ such that $\rho(x_n,x_0)<\delta/2$ for all $n\geqslant n_0$. Also, since $\mu(g_0x_n,g_0x_0)\leqslant\rho(x_n,x_0)$, we see that $\mu(g_0x_n,g_0x_0)<\delta/2$. Now, since $r(g_0x_n)+r(g_0x_0)\geqslant r(g_0x_0)\geqslant\delta$, we infer that $\mu(g_0x_n,g_0x_0)=d(g_0x_n,g_0x_0)<\delta/2$; so

$g_0x_n \in O_d(g_0x_0, \delta/2)$. Therefore $x_n = g_0^{-1}(g_0x_n) \in g_0^{-1}(O_d(g_0x_0, \delta/2)) \subset O_d(x_0, \varepsilon)$ for all $n \geqslant n_0$, showing that (x_n) converges to x_0 relative to the original topology of X.

Conversely, assume that $d(x_n, x_0) \rightsquigarrow 0$ for a sequence $(x_n) \subset X$ and a point $x_0 \in X$, while $\rho(x_n, x_0) \not\rightsquigarrow 0$. Then, for some $\varepsilon_0 > 0$, there must be a subsequence $(y_k) \subset (x_n)$ such that $\rho(y_k, x_0) \geqslant \varepsilon_0$ for all indices k. Therefore, $\mu(g_ky_k, g_kx_0) \geqslant \varepsilon_0/2$ for a suitable sequence $(g_k) \subset G$. Consequently,

$$r(g_ky_k) + r(g_kx_0) \geqslant \varepsilon_0/2. \tag{3.2}$$

Next, since U is a small set, one can choose a neighborhood A of the point x_0 such that the transporter $\langle A, U\rangle$ has compact closure in G. Since $d(y_k, x_0) \rightsquigarrow 0$, by passing to a subsequence, we can suppose that $y_k \in A$, $k \geqslant 1$.

Now, since the set $\{x_0\} \cup (y_k)$ is contained in A, the inequality (3.2) implies that $(g_k) \subset \langle A, U\rangle$. But the transporter $\langle A, U\rangle$ has compact closure in G, and hence, the sequence (g_k) has a cluster point, say $g \in G$ (see [19, Theorem 3.1.23]). Then, by continuity of the action of G on X, the point gx_0 is a cluster point for both sequences (g_kx_0) and (g_ky_k) in X. Since X is metrizable (g_kx_0) and (g_ky_k) should contain subsequences which converge to the cluster point gx_0. Without loss of generality, one can assume that the sequences (g_kx_0) and (g_ky_k) themselves converge to gx_0, and hence, there is an index k_0 such that $d(g_ky_k, g_kx_0) < \varepsilon_0/2$ whenever $k \geqslant k_0$. However, this contradicts the condition $d(g_ky_k, g_kx_0) \geqslant \mu(g_ky_k, g_kx_0) \geqslant \varepsilon_0/2$ above.

It remains to show that every closed unit ball $B_\rho(x, 1)$ is a small subset of X. Since S is a fundamental subset of X and ρ is G-invariant, one can assume, without loss of generality, that $x \in S$. We claim the $B_\rho(x, 1)$ is contained in $K(U)$. Indeed, if $y \in X \setminus K(U)$ and $g \in K$, then $gy \in X \setminus U$ because $K = K^{-1}$. Also one has $gx \in K(S)$. Consequently, by virtue of the property (3.1), this yields that $d(gx, gy) > 1$. By the same reason, $r(gx) > 1$. Consequently, $\mu(gx, gy) > 1$ whenever $y \in X \setminus K(U)$ and $g \in K$. This implies that

$$\rho(x, y) \geqslant \sup_{g\in K} \mu(gx, gy) > 1$$

for all $y \in X \setminus K(U)$, i.e., $B_\rho(x, 1) \subset K(U)$. But $K(U)$ is a small set because U is small and K is compact (see e.g., [2, Proposition 1.2(e)]). This yields that $B_\rho(x, 1)$ is small, and the proof is complete. $\square$

4. Important examples of proper G-spaces

Recall that a continuous function $f : X \to \mathbb{R}$ defined on a G-space X is called G-uniform, if for each $\varepsilon > 0$ there is a unity neighborhood U in G such that $|f(gx) - f(x)| < \varepsilon$ for all $x \in X$ and $g \in U$ (see [29] and [12]).

By $\mathcal{A}(X)$ we denote the linear space of all G-uniform bounded functions on X endowed with the sup-norm and the following G-action:

$$(g, f) \mapsto gf, \quad (gf)(x) = f\big(g^{-1}x\big), \quad x \in X.$$

It is known that $\mathcal{A}(X)$ is a Banach G-space (see [7]).

In general, the complement $\mathcal{A}(X) \setminus \{0\}$ may not be a proper G-space even for X a proper G-space. In this connection, for a proper G-space X, we shall define a G-invariant closed linear subspace $\mathcal{Q}(X)$ of $\mathcal{A}(X)$ such that the complement $\mathcal{Q}_0(X) = \mathcal{Q}(X) \setminus \{0\}$ is a proper G-space. Namely, we denote by $\mathcal{Q}(X)$ the subset of $\mathcal{A}(X)$ consisting of all those functions $f \in \mathcal{A}(X)$ which vanish at the infinity in the following sense: for every $\varepsilon > 0$, there exists an open small subset $U \subset X$ such that $|f(x)| \leqslant \varepsilon$ for all $x \in X \setminus U$.

Proposition 4.1. *Let G be locally compact group and X a proper G-space. Then $\mathcal{Q}(X)$ is a closed G-invariant linear subspace of $\mathcal{A}(X)$, and hence, $\mathcal{Q}(X)$ is a Banach G-space.*

Proof. Since the other two properties of $\mathcal{Q}(X)$ are evident, we will show that $\mathcal{Q}(X)$ is closed in $\mathcal{A}(X)$. Indeed, let (f_n) be a sequence in $\mathcal{Q}(X)$ that converges to a limit $f \in \mathcal{A}(X)$. Take $\varepsilon > 0$ arbitrary and choose an index n such that $\|f - f_n\| < \varepsilon/2$. Then $|f(x) - f_n(x)| < \varepsilon/2$ for all $x \in X$. Since $f_n \in \mathcal{Q}(X)$, there exists an open small subset $U \subset X$ such that $|f_n(x)| \leqslant \varepsilon$ for all $x \in X \setminus U$. Then, for every $x \in X \setminus U$ one has $|f(x)| \leqslant |f(x) - f_n(x)| + |f_n(x)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Hence $f \in \mathcal{Q}(X)$, as required. □

Below, in a metric space, we shall denote by $O(x, r)$ the open ball of a radius $r > 0$ centered at the point x.

Proposition 4.2. *Let G be any group and X any proper G-space. Then for every $f \in \mathcal{Q}_0(X) = \mathcal{Q}(X) \setminus \{0\}$, the open ball $O(f, \frac{\|f\|}{2})$ is a small set in $\mathcal{Q}_0(X)$. In particular, $\mathcal{Q}_0(X)$ is a proper G-space.*

Proof. Let $h \in \mathcal{Q}_0(X)$. We are going to show that for the sets $O(f, \frac{\|f\|}{2})$ and $O(h, \frac{\|f\|}{4})$ are relatively thin, i.e., the transporter $\langle O(f, \frac{\|f\|}{2}), O(h, \frac{\|f\|}{4})\rangle$ has compact closure.

Indeed, fix a small set U in X such that $|h(x)| \leqslant \frac{\|f\|}{20}$ whenever $x \in X \setminus U$. Choose $x_0 \in X$ such that $|f(x_0)| > \frac{4\|f\|}{5}$. Since the transporter $\langle \{x_0\}, U\rangle$ has compact closure in G, it suffices to show that

$$\left\langle O\left(f, \frac{\|f\|}{2}\right), O\left(h, \frac{\|f\|}{4}\right)\right\rangle \subset \langle \{x_0\}, U\rangle.$$

Let $g \in \langle O(f, \frac{\|f\|}{2}), O(h, \frac{\|f\|}{4})\rangle$. Then there is $h' \in O(h, \frac{\|f\|}{4})$ such that $g^{-1}h' \in O(f, \frac{\|f\|}{2})$. This implies that

$$\left|h'(gx_0)\right| \geqslant \left|f(x_0)\right| - \frac{\|f\|}{2} \quad \text{and} \quad \left|h(gx_0)\right| \geqslant \left|h'(gx_0)\right| - \frac{\|f\|}{4}.$$

Consequently,

$$\left|h(gx_0)\right| \geqslant \left|f(x_0)\right| - \frac{\|f\|}{2} - \frac{\|f\|}{4} > \frac{4\|f\|}{5} - \frac{\|f\|}{2} - \frac{\|f\|}{4} = \frac{\|f\|}{20}.$$

It then follows that $gx_0 \in U$, i.e., $g \in \langle \{x_0\}, U\rangle$, as required. □

Further, we denote by $\mathcal{P}(X)$ the linear subspace of $\mathcal{Q}(X)$ consisting of all functions $f \in \mathcal{Q}(X)$ whose support

$$\operatorname{supp} f = \{x \in X \mid f(x) \neq 0\}$$

is a small subset of X. It is easy to see that $\mathcal{P}(X)$ is an invariant subset of $\mathcal{Q}(X)$.

Denote the complement $\mathcal{P}(X) \setminus \{0\}$ by $\mathcal{P}_0(X)$.

Since open small sets constitute a base of the (Tychonoff) topology of X, we see that $\mathcal{P}_0(X) \neq \emptyset$.

The G-space $\mathcal{P}(X)$ will play a central role in our further constructions.

In the sequel the following closed convex subsets of $\mathcal{P}_0(X)$ and $\mathcal{Q}_0(X)$, respectively, will play an important role:

$$\mathcal{P}_+(X) = \{f \in \mathcal{P}_0(X) \mid f(x) \geqslant 0,\ \forall x \in X\},$$
$$\mathcal{Q}_+(X) = \{f \in \mathcal{Q}_0(X) \mid f(x) \geqslant 0,\ \forall x \in X\}.$$

Clearly, $\mathcal{Q}_+(X)$ and $\mathcal{P}_+(X)$ are convex G-invariant subsets of $\mathcal{Q}_0(X)$, and hence, they also are proper G-spaces.

The following proposition clarifies the relationship between $\mathcal{P}(X)$ and $\mathcal{Q}(X)$; namely it shows that when the proper G-space X is a normal topological space, then $\mathcal{Q}(X)$ is the completion of $\mathcal{P}(X)$.

Proposition 4.3. *Let G be locally compact group and X a normal proper G-space. Then $\mathcal{P}(X)$ is dense in $\mathcal{Q}(X)$.*

Proof. Indeed, let $f \in \mathcal{Q}(X)$ and $\varepsilon > 0$. We have to find a $\varphi \in \mathcal{P}(X)$ such that $\|f - \varphi\| < \varepsilon$. First we choose an open small subset $U \subset X$ such that $|f(x)| \leqslant \varepsilon/4$ for all $x \in X \setminus U$.

Define $A_\varepsilon = \{x \in X \mid |f(x)| \geqslant \varepsilon/2\}$. Clearly, A_ε is a closed (possibly, empty) subset of X and $A_\varepsilon \subset U$. The fact that f is G-uniform easily implies that the disjoint closed sets A_ε and $X \setminus U$ are, in fact, G-disjoint in the following sense due to M. Megrelishvili [26]: there is a unity neighborhood $O \subset G$ such that $O(A_\varepsilon) \cap O(X \setminus U) = \emptyset$ (see also [4, Corollary 2.2]). By the equivariant Urysohn lemma (see [26,30] and [4, Corollary 2.7]), there exists a G-uniform function $\lambda : X \to [0,1]$ such that $A_\varepsilon \subset \lambda^{-1}(1)$ and $X \setminus U \subset \lambda^{-1}(0)$. Set $\varphi(x) = \lambda(x)f(x)$ for all $x \in X$. Since both λ and f are G-uniform functions, so is their product φ. Further, since U is a small set and $\varphi(x) = \lambda(x)f(x) = 0 \cdot f(x) = 0$ for every $x \in X \setminus U$, we infer that $\varphi \in \mathcal{P}(X)$.

Let us check that $\|f - \varphi\| < \varepsilon$. Indeed, for $x \in A_\varepsilon$ one has $|f(x) - \varphi(x)| = 0$. If $x \in X \setminus U$ then $|f(x)| \leqslant \varepsilon/4$ and $\varphi(x) = \lambda(x)f(x) = 0 \cdot f(x) = 0$. Thus, $|f(x) - \varphi(x)| = |f(x)| \leqslant \varepsilon/4$ for all $x \in (X \setminus U) \cup A_\varepsilon$. Finally, if $x \in U \setminus A_\varepsilon$, then $|f(x)| < \varepsilon/2$, and therefore,

$$\bigl|f(x) - \varphi(x)\bigr| = \bigl|f(x) - \lambda(x)f(x)\bigr| = \bigl|f(x)\bigr| \cdot \bigl|1 - \lambda(x)\bigr| \leqslant \bigl|f(x)\bigr| < \varepsilon/2.$$

Thus, $|f(x) - \varphi(x)| < \varepsilon/2$ for every $x \in X$, yielding that $\|f - \varphi\| = \sup_{x \in X} |f(x) - \varphi(x)| \leqslant \varepsilon/2 < \varepsilon$, as required. □

It is in order to recall here some relevant definitions about equivariant extensors and equivariant retracts.

A G-space Y is called an equivariant absolute neighborhood extensor for the class G-$\mathcal{M}$ (notation: $Y \in G$-ANE), if for every $X \in G$-$\mathcal{M}$, any closed invariant subset $A \subset X$ and any G-map $f : A \to Y$, there exist an invariant neighborhood U of A in X and a G-map $\psi : U \to Y$ such that $\psi|_A = f$. If, in addition, one can always take $U = X$, then we say that Y is an equivariant extensor for X (notation: $Y \in G$-AE). The map ψ is called a G-extension of f.

A G-space $Y \in G$-$\mathcal{M}$ is called a G-equivariant absolute neighborhood retract for the class G-$\mathcal{M}$ (notation: $Y \in G$-ANR), provided that for any closed G-embedding $Y \hookrightarrow X$ in a G-space $X \in G$-$\mathcal{M}$, there exists a G-retraction $r : U \to Y$, where U is an invariant neighborhood of Y in X. If, in addition, one can always take $U = X$, then we say that Y is a G-equivariant absolute retract (notation: $Y \in G$-AR).

We note that, in general, a metrizable G-A(N)E space Y need not be a G-A(N)R, because it may not belong to the class G-$\mathcal{M}$. But if $Y \in G$-$\mathcal{M}$ and $Y \in G$-A(N)E, then clearly $Y \in G$-A(N)R. The converse is also true; below, in Corollary 6.3, we shall give a very transparent prove of this result based on our Theorem 6.1.

Now we establish the following fundamental property of $\mathcal{Q}_+(X)$:

Proposition 4.4. *Let G be locally compact group and X a proper G-space. Then $\mathcal{Q}_+(X)$ is a proper G-AE space.*

Proof. $\mathcal{Q}_+(X)$ is a proper G-space since it is a G-invariant subspace of the proper G-space $\mathcal{Q}_0(X)$ (see Proposition 4.2). In order to prove that $\mathcal{Q}_+(X) \in G$-AE, we aim at applying the following result of Abels [2, Theorem 4.4]: a G-space T is a G-AE if T is a K-AE for each compact subgroup $K \subset G$.

In our case $T = \mathcal{Q}_+(X)$. In order to show that for each compact subgroup $K \subset G$, $\mathcal{Q}_+(X)$ is a K-AE, we will apply the following generalization of the James–Segal theorem [25, Proposition 4.1]:

Theorem. *([9]) Let K be a compact group and T a K-ANR. Then T is a K-AR if and only if for every closed subgroup $H \subset K$, the set of H-fixed points $T^H = \{t \in T \mid ht = t, \ \forall h \in H\}$ is contractible.*

We continue with the proof of Proposition 4.4. Denote

$$E(X) = \{f \in \mathcal{Q}(X) \mid f(x) \geqslant 0, \ \forall x \in X\}.$$

Clearly, $E(X)$ is a closed convex G-invariant subset of the Banach G-space $\mathcal{Q}(X)$, and $\mathcal{Q}_+(X) = E(X) \setminus \{0\}$. Due to completeness of $E(X)$, the equivariant Dugundji extension theorem as proved in [6], is applicable here; according to this result $E(X)$ is a K-AR. Hence, the K-invariant open set $\mathcal{Q}_+(X) = E(X) \setminus \{0\}$ is a K-ANR, and therefore, the above mentioned theorem from [9] is applicable to the K-space $T = \mathcal{Q}_+(X)$.

Let $H \subset K$ be any closed subgroup. Since $\mathcal{Q}_+(X)$ is convex and the action of K is linear, we infer that the H-fixed point set $\mathcal{Q}_+(X)^H$ is a convex subset of $\mathcal{Q}_+(X)$, and hence, it is contractible, as required. Now the above theorem implies that $\mathcal{Q}_+(X)$ is a K-AR. Since the group K is compact, the property K-AR implies the property K-AE (see [7, Theorem 14]), and hence, $\mathcal{Q}_+(X)$ is a K-AE. This completes the proof. □

Remark 4.5. The same proof of Proposition 4.4 implies that $\mathcal{Q}_+(X)$ is a G'-AE space for any closed subgroup G' of G.

5. Equivariant embeddings into convex proper G-spaces

In this section we shall establish two equivariant embedding results for metrizable proper G-spaces which may be considered as equivariant versions of the well-known Kuratowski–Wojdyslawski and Arens–Eells embedding theorems, respectively.

Theorem 5.1. *Let G be an arbitrary group and $X \in G$-$\mathcal{M}$. Then every G-invariant small metric ρ on X defines a G-embedding $i : X \hookrightarrow \mathcal{P}_+(X)$ such that:*

(1) $\|i(x)\| = 1$ *for all* $x \in X$,
(2) $\|i(x) - i(y)\| \leqslant \rho(x, y)$ *for all* $x, y \in X$,
(3) $\rho(x, y) = \|i(x) - i(y)\|$ *whenever* $\rho(x, y) \leqslant 1$,
(4) $\|i(x) - i(y)\| \geqslant 1$ *whenever* $\rho(x, y) > 1$,
(5) *the image $i(X)$ is closed in its convex hull.*

Proof. For every $x \in X$, we define $f^x \in \mathcal{P}_0(X)$ as follows:

$$f^x(y) = \begin{cases} 1 - \rho(x, y), & \text{if } \rho(x, y) \leqslant 1, \\ 0, & \text{if } \rho(x, y) \geqslant 1. \end{cases}$$

Now we define the map $i : X \to \mathcal{P}_0(X)$ by $i(x) = f^x$, $x \in X$.

To see that i is the required map let us check first that $f^x \in \mathcal{P}_0(X)$ for every $x \in X$. Clearly, $f^x : X \to [0, 1]$ is a continuous function. Now, let $\varepsilon > 0$ be arbitrary. Choose a neighborhood W of the unity in G such that

$$\rho(hx, x) < \varepsilon \quad \text{for all } h \in W. \tag{5.1}$$

We claim that (5.1) implies

$$\left|f^x(hy) - f^x(y)\right| < \varepsilon$$

for all $h \in W$ and $y \in X$, which in turn yields that $f^x \in \mathcal{A}(X)$.

Consider all possible cases.

If $\rho(x, hy) \leqslant 1$ and $\rho(x, y) \leqslant 1$, then

$$\begin{aligned}\left|f^x(hy) - f^x(y)\right| &= \left|1 - \rho(x, hy) - 1 + \rho(x, y)\right| = \left|\rho(x, y) - \rho\left(h^{-1}x, y\right)\right| \\ &\leqslant \rho\left(x, h^{-1}x\right) = \rho(hx, x) < \varepsilon.\end{aligned}$$

If $\rho(x, hy) \leqslant 1$ and $\rho(x, y) \geqslant 1$, then

$$\begin{aligned}\left|f^x(hy) - f^x(y)\right| &= 1 - \rho(x, hy) \leqslant \rho(x, y) - \rho(x, hy) = \rho(x, y) - \rho\left(h^{-1}x, y\right) \\ &\leqslant \rho\left(x, h^{-1}x\right) = \rho(hx, x) < \varepsilon.\end{aligned}$$

The case $\rho(x, hy) \geqslant 1$ and $\rho(x, y) \leqslant 1$ reduces to the previous one.

Finally, if $\rho(x, hy) \geqslant 1$ and $\rho(x, y) \geqslant 1$, then $|f^x(hy) - f^x(y)| = 0 < \varepsilon$.

Thus, $f^x \in \mathcal{A}(X)$. Since $\operatorname{supp} f^x \subset B_\rho(x, 1)$ and since ρ is a small metric we infer that $B_\rho(x, 1)$, and hence, its subset $\operatorname{supp} f^x$, are small sets. Thus, f^x belongs to $\mathcal{P}_0(X)$ showing that the map $i : X \to \mathcal{P}_0(X)$ is well-defined.

Further, the following inequality holds:

$$\left\|f^{x_1} - f^{x_2}\right\| \leqslant \rho(x_1, x_2) \quad \text{for all } x_1, x_2 \in X. \tag{5.2}$$

This implies that i is (uniformly) continuous. In order to prove (5.2), again we consider cases. Let $y \in X$ be arbitrary.

If $\rho(x_1, y) \leqslant 1$ and $\rho(x_2, y) \leqslant 1$, then

$$\left|f^{x_1}(y) - f^{x_2}(y)\right| = \left|\rho(x_1, y) - \rho(x_2, y)\right| \leqslant \rho(x_1, x_2).$$

If $\rho(x_1, y) \leqslant 1$ and $\rho(x_2, y) \geqslant 1$, then

$$\left|f^{x_1}(y) - f^{x_2}(y)\right| = \left|1 - \rho(x_1, y)\right| \leqslant \rho(x_2, y) - \rho(x_1, y) \leqslant \rho(x_2, x_1).$$

The case $\rho(x_1, y) \geqslant 1$ and $\rho(x_2, y) \leqslant 1$ reduces to the previous one.

Finally, if $\rho(x_1, y) \geqslant 1$ and $\rho(x_2, y) \geqslant 1$, then

$$\left|f^{x_1}(y) - f^{x_2}(y)\right| = 0 \leqslant \rho(x_1, x_2).$$

Thus (5.2) is proved.

Since $f^x(x) = 1$, it then follows from (5.2) that $\|f^x\| = 1$, so $\|i(x)\| = 1$.

In order to show that i is a topological embedding, we observe the following. If $\rho(x_1, x_2) \leqslant 1$, then $\|f^{x_1} - f^{x_2}\| \geqslant |f^{x_1}(x_1) - f^{x_2}(x_1)| = \rho(x_1, x_2)$, which implies together with (5.2), that

$$\left\|f^{x_1} - f^{x_2}\right\| = \rho(x_1, x_2).$$

Moreover, if $\rho(x_1, x_2) > 1$, then

$$\left\|f^{x_1} - f^{x_2}\right\| \geqslant \left|f^{x_1}(x_1) - f^{x_2}(x_1)\right| = 1.$$

It follows from these observations that i is injective and its inverse is continuous. The equivariance of i is immediate from the G-invariance of the metric ρ.

It remains only to show that $i(X)$ is closed in its convex hull. Let (f^{x_n}) be a sequence in $i(X)$ converging to an $f \in conv(i(X))$. Then f has the form $f = \sum_{i=1}^m t_i f^{z_i}$ with $\sum_{i=1}^m t_i = 1$, $t_i \geqslant 0$ and $z_i \in X$. Without loss of generality, one can assume that $t_1 > 0$. Next, we observe that:

$$\begin{aligned}\|f^{x_n} - f\| &\geqslant |f^{x_n}(x_n) - f(x_n)| = |1 - f(x_n)| \\ &= \left|\sum_{i=1}^m t_i - \sum_{i=1}^m t_i f^{z_i}(x_n)\right| = \sum_{i=1}^m t_i\bigl(1 - f^{z_i}(x_n)\bigr) \geqslant t_1\bigl(1 - f^{z_1}(x_n)\bigr).\end{aligned}$$

Now, $t_1(1 - f^{z_1}(x_n))$ is either $t_1\rho(z_1, x_n)$ or t_1. Since $\|f^{x_n} - f\| \rightsquigarrow 0$, the above inequality implies that $t_1(1-f^{z_1}(x_n)) = t_1\rho(z_1, x_n)$ for all but finitely many indices n. We thus infer that $\rho(z_1, x_n) \rightsquigarrow 0$ as $n \rightsquigarrow \infty$, i.e., $x_n \rightsquigarrow z_1$. Being i a continuous map, we get that $f = \lim i(x_n) = i(z_1)$, as required. $\square$

Since by Theorem 3.2, every proper G-space from G-$\mathcal{M}$ admits a small G-invariant metric and $\mathcal{P}_+(X) \subset \mathcal{Q}_+(X)$, Theorem 5.1 immediately yields the following result which can be viewed as the equivariant analog of the classical Kuratowski–Wojdyslawski embedding theorem for proper G-spaces:

Corollary 5.2. *Let G be a locally compact group and $X \in G$-$\mathcal{M}$. Then X admits a G-embedding $i : X \hookrightarrow \mathcal{Q}_+(X)$ such that $\|i(x)\| = 1$ for all $x \in X$ and the image $i(X)$ is closed in its convex hull.*

For a metric space (X, ρ) we shall denote by $\mathcal{F}(X)$ the metric space of all finite subsets of X endowed with the Hausdorff metric ρ_H. Recall that for $A, B \in \mathcal{F}(X)$, the distance $\rho_H(A, B)$ is defined to be the number $\max\{\max_{a\in A} \rho(a, B), \max_{b\in B} \rho(b, A)\}$.

If, in addition, X is a G-space then a natural action $G \times \mathcal{F}(X) \to \mathcal{F}(X)$ is defined according to the rule

$$(g, A) \mapsto gA = \{ga \mid a \in A\}.$$

The easy verification of the continuity of the action map $G \times \mathcal{F}(X) \to \mathcal{F}(X)$ is left to the reader.

Lemma 5.3. *Let G be an arbitrary group and $X \in G$-$\mathcal{M}$. Then the space $\mathcal{F}(X)$ of all finite subsets of X is a proper G-space. Moreover, if ρ is an invariant small metric for X, then the Hausdorff metric ρ_H is an invariant small metric for $\mathcal{F}(X)$.*

Proof. It is quite easy to check that ρ_H is an invariant metric. Let's check that it is also small. To this end, choose $A = \{a_1, \dots, a_n\} \in \mathcal{F}(X)$ and $Z = \{z_1, \dots, z_p\} \in \mathcal{F}(X)$ arbitrary. Since the closed unit balls $B_\rho(a_1, 1), \dots, B_\rho(a_n, 1)$ are small subsets of X, there must be a number $r > 0$ such that the transporter $\langle \bigcup_{i=1}^n B_\rho(a_i, 1), O_\rho(z_k, r)\rangle$ has compact closure for every $k = 1, \dots, p$.

We claim that the transporter $\langle B_{\rho_H}(A, 1), O_{\rho_H}(Z, r)\rangle$ has compact closure. Indeed, let $g \in \langle B_{\rho_H}(A, 1), O_{\rho_H}(Z, r)\rangle$. Then, there must be an element $C \in B_{\rho_H}(A, 1)$ such that $gC \in O_{\rho_H}(Z, r)$. Take a point $c \in C$. Then there is an $a_j \in A$ such that $c \in B_\rho(a_j, 1)$. Similarly, there must be a $z_k \in Z$ such that $gc \in O_\rho(z_k, r)$. Consequently, $g \in \langle B_\rho(a_j, 1), O_\rho(z_k, r)\rangle$. Since $\langle B_\rho(a_j, 1), O_\rho(z_k, r)\rangle \subset \langle \bigcup_{i=1}^n B_\rho(a_i, 1), O_\rho(z_k, r)\rangle$, we infer that $g \in \langle \bigcup_{i=1}^n B_\rho(a_i, 1), O_\rho(z_k, r)\rangle$. Hence, the transporter $\langle B_{\rho_H}(A, 1), O_{\rho_H}(Z, r)\rangle$ is a subset of the transporter $\langle \bigcup_{i=1}^n B_\rho(a_i, 1), O_\rho(z_k, r)\rangle$ which has compact closure. Therefore $\langle B_{\rho_H}(A, 1), O_{\rho_H}(Z, r)\rangle$ has compact closure, as required. Hence, each closed unit ball $B_{\rho_H}(A, 1)$ is a small subset of $\mathcal{F}(X)$, and the proof is complete. $\square$

Theorem 5.4. *Let G be an arbitrary group, $X \in G$-$\mathcal{M}$ and let $\mathcal{F}(X)$ denote the proper G-space of all finite subsets of X. Then every G-invariant small metric ρ on X defines a G-embedding $i : X \hookrightarrow \mathcal{P}_+(\mathcal{F}(X))$ such that:*

(1) $\|i(x)\| = 1$ *for all* $x \in X$,
(2) $\|i(x) - i(y)\| \leqslant \rho(x, y)$ *for all* $x, y \in X$,
(3) $\rho(x, y) = \|i(x) - i(y)\|$ *whenever* $\rho(x, y) \leqslant 1$,
(4) $\|i(x) - i(y)\| \geqslant 1$ *whenever* $\rho(x, y) > 1$,
(5) $i(X)$ *is linearly independent in* $\mathcal{P}(\mathcal{F}(X))$,
(6) $i(X)$ *is closed in its linear span.*

Proof. By Lemma 5.3, $\mathcal{F}(X) \in G\text{-}\mathcal{M}$. Moreover, if ρ is an invariant small metric for X, then the Hausdorff metric ρ_H is an invariant small metric for $\mathcal{F}(X)$. Hence, we can apply Theorem 5.1 to the G-space $\mathcal{F}(X)$. Let $j : \mathcal{F}(X) \hookrightarrow \mathcal{P}_+(\mathcal{F}(X))$ be the G-embedding defined by the metric ρ_H as in the proof of Theorem 5.1. Then the restriction $i = j|_X : X \hookrightarrow \mathcal{P}_+(\mathcal{F}(X))$ is the desired G-embedding. Indeed, only the properties (5) and (6) require verification. Let us do this.

(5) Let $x_1, \ldots, x_n \in X$ and assume that there exist reals $\lambda_1, \ldots, \lambda_{n-1} \in \mathbb{R}$ such that $f^{x_n} = \sum_{i=1}^{n-1} \lambda_i f^{x_i}$. Denote $A = \{x_1, \ldots, x_n\}$ and $B = \{x_1, \ldots, x_{n-1}\}$. Since $f^{x_i}(A) = 1$ for all $1 \leqslant i \leqslant n$, then from the equality $f^{x_n}(A) = \sum_{i=1}^{n-1} \lambda_i f^{x_i}(A)$ we get that $1 = \sum_{i=1}^{n-1} \lambda_i$. Similarly, since $f^{x_i}(B) = 1$ for all $1 \leqslant i \leqslant n-1$, from the equality $f^{x_n}(B) = \sum_{i=1}^{n-1} \lambda_i f^{x_i}(B)$ we get that $f^{x_n}(B) = \sum_{i=1}^{n-1} \lambda_i$. This yields that $f^{x_n}(B) = 1$ which, however, is impossible because $\rho(x_n, x_i) > 0$ for every $1 \leqslant i \leqslant n-1$.

(6) Now, let L denote the linear span of $i(X)$ in $\mathcal{P}(\mathcal{F}(X))$. We have to show that $i(X)$ is closed in L. Assume that the contrary is true, and let $\varphi \in L \setminus i(X)$ be such that $\varphi \in \overline{i(X)}$. Then $\varphi = \sum_{k=1}^{p} \lambda_k f^{x_k}$ for some $x_k \in X$ and $\lambda_k \in \mathbb{R}$, and there exists a sequence $(a_n) \subset X$ such that $\varphi = \lim f^{a_n}$.

Consider the following finite subsets of X: $A_n = \{a_n, x_1, x_2, \ldots, x_p\}$, $n = 1, 2, \ldots$, and $B = \{x_1, x_2, \ldots, x_p\}$. Since $f^{a_n}(A_n) = 1$ and $f^{x_k}(A_n) = f^{x_k}(B) = 1$ for all $n \geqslant 1$ and $1 \leqslant k \leqslant p$, we get

$$\left\|\varphi - f^{a_n}\right\| \geqslant \left|\varphi(A_n) - f^{a_n}(A_n)\right| = \left|\sum_{k=1}^{p} \lambda_k f^{x_k}(A_n) - f^{a_n}(A_n)\right| = \left|\sum_{k=1}^{p} \lambda_k - 1\right|.$$

Since $\lim \|\varphi - f^{a_n}\| = 0$, we further get that $\sum_{k=1}^{p} \lambda_k = 1$.

Since $\varphi \notin i(X)$, all the distances $\|\varphi - f^{x_k}\|$, $k = 1, \ldots, p$, are positive. Hence, one can choose a real r such that

$$0 < r < \min\left\{\frac{1}{2}, \frac{1}{2} \min_{1 \leqslant k \leqslant n} \left\|\varphi - f^{x_k}\right\|\right\}.$$

Since $\varphi \in \overline{i(X)}$, there must be an $x \in X$ such that $\|\varphi - f^{x}\| < r$.

This yields that $\|f^x - f^{x_k}\| \geqslant r$ for every $1 \leqslant k \leqslant p$. But $i : X \to L$ is a non-expansive map, so

$$\rho(x, x_k) \geqslant \left\|f^x - f^{x_k}\right\| \geqslant r \quad \text{for all } 1 \leqslant k \leqslant p.$$

This implies that

$$\rho(x, B) = \min\left\{\rho(x, x_k) \;\middle|\; 1 \leqslant k \leqslant p\right\} \geqslant r. \tag{5.3}$$

On the other hand, one also has

$$r > \left\|\varphi - f^x\right\| \geqslant \left|\varphi(B) - f^x(B)\right| = \left|\sum_{k=1}^{p} \lambda_k f^{x_k}(B) - f^x(B)\right|.$$

Since $f^{x_k}(B) = 1$ for $k = 1, \ldots, p$, and $\sum_{k=1}^{p} \lambda_k = 1$, we further get

$$r > \left|\sum_{k=1}^{p} \lambda_k f^{x_k}(B) - f^x(B)\right| = \left|\sum_{k=1}^{p} \lambda_k - f^x(B)\right| = \left|1 - f^x(B)\right|. \tag{5.4}$$

Consequently, $r > |1 - f^x(B)|$ yielding $f^x(B) > 1 - r$. Now, remember that $r < 1/2$, so $1 - r > r$, implying that $f^x(B) > r$. Thus $f^x(B) > 0$, and therefore, $f^x(B) = 1 - \rho(x, B)$. Then $|1 - f^x(B)| = \rho(x, B)$, and in combination with (5.4) we get the inequality $\rho(x, B) < r$, which contradicts to (5.3). This contradiction shows that $\overline{i(X)} = i(X)$, and hence, $i(X)$ is closed in L. □

Since by Theorem 3.2, every proper G-space from G-$\mathcal{M}$ admits a small invariant metric, Theorem 5.4 immediately yields the following result which can be viewed as the equivariant analog of the classical Arens–Eells embedding theorem:

Corollary 5.5. *Let G be a locally compact group and $X \in G$-$\mathcal{M}$. Then X admits a closed G-embedding $i : X \hookrightarrow L$ in a normed linear G-space L such that $L \setminus \{0\}$ is a proper G-space, $\|i(x)\| = 1$ for all $x \in X$ and the image $i(X)$ is a Hamel basis for L.*

Since $\mathcal{P}_+(\mathcal{F}(X)) \subset \mathcal{Q}_+(\mathcal{F}(X))$, we further get the following

Corollary 5.6. *Let G be a locally compact group and $X \in G$-$\mathcal{M}$. Then X admits a G-embedding $i : X \hookrightarrow \mathcal{Q}_+(\mathcal{F}(X))$, where $\mathcal{F}(X)$ denotes the proper G-space of all finite subsets of X.*

6. Closed equivariant embeddings into proper G-AE spaces

Recall that in this section the acting group G is always assumed to be locally compact.

For purposes of the equivariant theories of retracts, infinite-dimensional manifolds and equivariant shape it is important to be able to embed every G-space $X \in G$-$\mathcal{M}$ as a *closed* G-invariant subset into some G-AE space. This is achieved in the following

Theorem 6.1. *For each G-space $X \in G$-$\mathcal{M}$, there exist a Banach G-space L, a convex invariant subset $V \subset L$, a normed linear space N and a closed G-embedding $f : X \hookrightarrow V \times N$ such that $V \times N$ is a proper G-space and $V \times N \in G'$-AE for any closed subgroup G' of G.*

For the proof of Theorem 6.1 we shall need the following lemma proved in [5]:

Lemma 6.2. *Let $f : X \to M$ be a G-map between two proper G-spaces and let $p : X \to X/G$ be the orbit map. Then the image of the diagonal map $\varphi : X \to M \times (X/G)$, $\varphi(x) = (f(x), p(x))$, is a closed invariant subset of the product $M \times (X/G)$ endowed with the diagonal G-action, where X/G is equipped with the trivial G-action.*

Proof of Theorem 6.1. Take $L = \mathcal{Q}(X)$ and $V = \mathcal{Q}_+(X)$. Let $j : X \hookrightarrow \mathcal{Q}_+(X)$ be the G-embedding from Corollary 5.2 (or Corollary 5.6) and let $p : X \to X/G$ be the orbit map. Then the diagonal product of j and p is a topological embedding

$$\varphi : X \hookrightarrow \mathcal{Q}_+(X) \times (X/G)$$

(see e.g., [19, Theorem 2.3.20]). Clearly φ is equivariant.

Next, it follows from Lemma 6.2 that φ is a closed embedding. Thus, one can think of X as a closed invariant subset of the product $\mathcal{Q}_+(X) \times (X/G)$. But the orbit space X/G is metrizable (see Section 1), and hence, according to the Arens–Eells embedding theorem (see [13]), one can embed X/G into a normed linear space N as a closed subset.

This generates a closed equivariant embedding of $\mathcal{Q}_+(X) \times (X/G)$ into $\mathcal{Q}_+(X) \times N$. As a result we get an equivariant closed embedding of X into $\mathcal{Q}_+(X) \times N$. Next, by Proposition 4.4, $\mathcal{Q}_+(X)$ is a proper G-space.

Since the product of a proper G-space with any G-space is again a proper G-space we see that $\mathcal{Q}_+(X) \times N$ is a proper G-space.

By Proposition 4.4 and Remark 4.5, $\mathcal{Q}_+(X) \in G'$-AE for any closed subgroup G' of G. By Dugundji extension theorem [16], $N \in$ AE, and hence, N endowed with the trivial G'-action is a G'-AE (see e.g., [5, Lemma 3.12]). Since the product of two G'-AE spaces is again a G'-AE space (this is quite easy to check), we conclude that $\mathcal{Q}_+(X) \times N \in G'$-AE. This completes the proof. □

The following result in different particular cases was proved in [8, Remark 5], [20] and [5]. The general case was handled in [10], however, its proof relies on a complicated [10, Theorem 3.2]. Below we shall give a very transparent prove of this result based on our Theorem 6.1.

Corollary 6.3. *Let $X \in G$-$\mathcal{M}$. Then X is a G-*ANE *(respectively, a G-*AE*) if and only if X is a G-*ANR *(respectively, a G-*AR*).*

Proof. We consider the "G-AR" case only; the "G-ANR" case is quite similar.

As we noticed in Section 4, if $X \in G$-$\mathcal{M}$ and X is a G-AE, then clearly X is a G-AR. Now suppose that X is a G-AR. By Theorem 6.1, one can think of X as a closed invariant subset of a G-space $E \in G$-$\mathcal{M}$ which is a G-AE. Since X is a G-AR, it is an equivariant retract of E, which yields immediately that X is a G-AE. □

Corollary 6.4. *Let X be a G-*ANR *(respectively, a G-*AR*) and $G' \subset G$ a closed subgroup. Then X is a G'-*ANR *(respectively, a G'-*AR*).*

Proof. We consider the "G-AR" case only; the "G-ANR" case is quite similar.

By Theorem 6.1, one can think of X as a closed invariant subset of a proper G-space $V \times N$ with V and N as in Theorem 6.1. Since X is a G-AR, it is a G-equivariant retract (and, in particular, a G'-equivariant retract) of $V \times N$. Since, by Theorem 6.1, $V \times N \in G'$-AR we infer that $X \in G'$-AR, as required. □

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