Math. Z. 215, 195–204 (1994)

Mathematische Zeitschrift

Open subgroups and Pontryagin duality

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Received 21 February 1991; in final form 12 March 1992

0 Introduction

By a *character* of a group G we mean a homomorphism of G into the group $\mathbf{R}/\mathbf{Z}$. If G is an abelian topological group, the set of all its continuous characters, with addition defined pointwise and the compact-open topology, is a Hausdorff abelian group; we call it the *dual group* or the *character group* of G and denote it by $G^\wedge$. We say that G is reflexive if the evaluation map is a topological isomorphism of G onto $G^{\wedge\wedge}$.

Let A be an open subgroup of an abelian topological group G. Venkataraman [5] proved that if G is reflexive, then so is A (see, however, Remark 2.4 below). Under certain additional assumptions, this result had been obtained earlier by Noble [4, Corollary 3.4]. In Sect. 2 of the present paper we show that the reflexivity of G is, in fact, equivalent to the reflexivity of A. We also deal with the relationship between the reflexivity of the groups G and G/K where K is a compact subgroup of G.

An abelian topological group G is called *strongly reflexive* if all closed subgroups and Hausdorff quotient groups of G and $G^\wedge$ are reflexive. This notion was introduced in [2] where countable products of lines and circles were investigated (cf. Remark 3.2 below). The class of strongly reflexive groups comprises, among other things, nuclear Fréchet spaces and countable products of locally compact abelian groups [1, (17.3)]. More information on strong reflexivity can be found in [1, Sect. 17].

Let G be an abelian topological group. Let A be an open and K a compact subgroup of A. In Sect. 3 we prove that if A is strongly reflexive, then so is G. Furthermore, if G/K is strongly reflexive and G admits sufficiently many continuous characters (i.e. if continuous characters separate points of G), then G is strongly reflexive, too. The converse statements are also true: see (3.1.d).

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* Partially supported by D.G.I.C.Y.T. grant BE91-031

1 Preliminaries

The group **R/Z** will be denoted by **T**. It is convenient to identify **T** with the interval $(-\frac{1}{2}, \frac{1}{2}]$ and, consequently, to treat characters as real-valued functions.

Let S be a subset of an abelian topological group G. By gp S we denote the subgroup of G generated by S. Given a character χ of G, we shall write

$$|\chi(S)| = \sup\{|\chi(g)| : g \in S\}.$$

The set

$$\{\chi \in G^{\wedge} : |\chi(S)| \leqq \tfrac{1}{4}\}$$

is called the *polar* of S; we shall denote it by S^0. If S is a subgroup of G, then S^0 is a closed subgroup of G; it consists of all characters of G which vanish on S.

(1.1) Lemma. *The polars of compact subsets of an abelian topological group G form a basis of neighbourhoods of zero in $G^{\wedge}$.*

The easy proof is left for the reader.

(1.2) Lemma. *If U is a neighborhood of zero in an abelian topological group G, then U^0 is a compact subset of $G^{\wedge}$.*

This is a standard fact; see e.g. [4, Lemma 2.2].

Let H be a closed subgroup of an abelian topological group G. We say that H is *dually closed* in G if to each $g \in G \backslash H$ there corresponds a character $\chi \in G^{\wedge}$ with $\chi_{|H} \equiv 0$ and $\chi(g) \neq 0$. Next, H is said to be *dually embedded* in G if each continuous character of H can be extended to a continuous character of G. The canonical homomorphisms $G^{\wedge}/H^0 \to H^{\wedge}$ and $(G/H)^{\wedge} \to H^0$, defined in the obvious way, are denoted by ϕ_H and ϕ^H, respectively. Notice that ϕ_H is a continuous injection; it is surjective if and only if H is dually embedded in G. The mapping ϕ^H is a continuous isomorphism.

The evaluation map of G into $G^{\wedge\wedge}$ is denoted by α_G. The verification of the following simple fact is left to the reader:

(1.3) Lemma. *If H is a dually closed subgroup of G, then $\alpha_G(H) = H^{00} \cap \alpha_G(G)$.*

Let G, H be abelian topological groups and $\psi: G \to H$ a continuous homomorphism. The dual homomorphism $\psi^{\wedge}: H^{\wedge} \to G^{\wedge}$ is defined by $\psi^{\wedge}(\chi) = \chi\psi$, $\chi \in H^{\wedge}$; it is clear that $\psi^{\wedge}$ is continuous. A direct verification shows that the diagram

$$\begin{array}{ccc} G & \xrightarrow{\psi} & H \\ \downarrow{\scriptstyle \alpha_G} & & \downarrow{\scriptstyle \alpha_H} \\ G^{\wedge\wedge} & \xrightarrow{\psi^{\wedge\wedge}} & H^{\wedge\wedge} \end{array}$$

is commutative.

(1.4) Lemma. *Let F, G, H be abelian topological groups. Let $\phi: F \to G$ and $\psi: G \to H$ be continuous homomorphisms such that the sequence*

$$0 \longrightarrow F \xrightarrow{\ \phi\ } G \xrightarrow{\ \psi\ } H \longrightarrow 0$$

is exact. If ψ is open, the sequence

$$F^\wedge \xleftarrow{\ \phi^\wedge\ } G^\wedge \xleftarrow{\ \psi^\wedge\ } H^\wedge \longleftarrow 0$$

is exact. If, in addition, ϕ is open, then $\phi^\wedge$ maps $G^\wedge$ onto $F^\wedge$.

Proof. We have $\ker \psi^\wedge = \{0\}$ because $\psi(G) = H$. We shall prove that $\ker \phi^\wedge = \operatorname{im} \psi^\wedge$. Each character χ belonging to $\ker \phi^\wedge$ vanishes on $\phi(F)$, hence on $\ker \psi$. Since $\psi(G) = H$, it follows that there is a unique homomorphism $\kappa: H \to \mathbf{T}$ such that $\chi = \kappa\psi$. As χ is continuous and ψ open, κ is continuous. We have $\psi^\wedge(\kappa) = \kappa\psi = \chi$. This proves that $\ker \phi^\wedge \subset \operatorname{im} \psi^\wedge$. The opposite inclusion follows from the equalities $\phi^\wedge \psi^\wedge = (\psi\phi)^\wedge = 0^\wedge = 0$.

The last assertion of the lemma follows, for instance, from the fact that open subgroups are dually embedded (cf. (2.2.b)). □

(1.5) Lemma. *Suppose we are given a commutative diagram*

$$\begin{array}{ccccc} F & \xrightarrow{\ \phi\ } & G & \xrightarrow{\ \psi\ } & H \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\ F' & \xrightarrow{\ \phi'\ } & G' & \xrightarrow{\ \psi'\ } & H' \end{array}$$

of abelian groups and their homomorphisms. Suppose further that α and γ are isomorphisms. If $\operatorname{im} \psi = H$ *and* $\ker \psi' = \operatorname{im} \phi'$, *then* $\operatorname{im} \beta = G'$. *If* $\ker \phi' = \{0\}$ *and* $\ker \psi = \operatorname{im} \phi$, *then* $\ker \beta = \{0\}$.

The proof consists in a direct verification.

(1.6) Lemma. *Let G, H be Hausdorff groups (abelian or not) and let $\psi: G \to H$ be a continuous homomorphism. Suppose that ψ is open and its kernel is compact. Then the following statements are true:*

(a) *Let (g_i) be a net in G; if the net $(\psi(g_i))$ has a cluster point in H, then (g_i) has a cluster point in G.*

(b) *ψ is a closed mapping.*

(c) *The inverse images of compact subsets of H are compact subsets of G.*

Proof. Statements (b) and (c) are standard. Besides, they follow easily from (a). We shall prove (a).

Let e_G and e_H denote the neutral elements of G and H, respectively. We may assume that e_H is a cluster point of $(\psi(g_i))$. We shall prove that (g_i) has a cluster point in $K := \ker \psi$. Suppose the contrary. Then to each $p \in K$ there correspond an index i_p and an open neighbourhood U_p of p in G, such that $g_i \notin U_p$ for $i \geqq i_p$. As K is compact, the open covering $\{U_p\}_{p \in K}$ of K has a finite subcovering $\{U_p\}_{p\in S}$. Then $U=\bigcup_{p\in S} U_p$ is an open subset of G containing K, and $g_i\notin U$ for all sufficiently large i, say, for $i\geqq i_0$. As K is compact, there is a neighbourhood V of e_G with $KV\subset U$. Then $\psi(V)$ is a neighbourhood of e_H, and $\psi(g_i)\notin\psi(V)$ for $i\geqq i_0$, contrary to our assumption that e_H is a cluster point of $(\psi(g_i))$. □

2 Open subgroups and duality

(2.1) Lemma. *Let G, H be abelian topological groups and $\psi\colon G\to H$ a continuous homomorphism. Suppose that ψ is open, maps G onto H, and that* ker ψ *is compact. Then the dual homomorphism $\psi^\wedge\colon H^\wedge\to G^\wedge$ is open.*

The assumption that $\psi(G)=H$ may be dropped; cf. Lemma 2.5 below.

Proof. Take an arbitrary compact subset X of H. In view of (1.1), it is enough to show that $\psi^\wedge(X^0)$ is a neighbourhood of zero in $G^\wedge$. Denote $K=\ker\psi$. It follows from (1.6.c) that $K\cup\psi^{-1}(X)$ is compact. Now, it is not hard to see that $(K\cup\psi^{-1}(X))^0\subset\psi^\wedge(X^0)$. □

(2.2) Lemma. *Let A be an open subgroup of an abelian topological group G. Consider the canonical commutative diagram*

$$(*)\qquad\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{\mu} & G & \xrightarrow{\nu} & G/A & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle\alpha_A} & & \downarrow{\scriptstyle\alpha_G} & & \downarrow{\scriptstyle\alpha_{G/A}} & & \\
0 & \longrightarrow & A^{\wedge\wedge} & \xrightarrow{\mu^{\wedge\wedge}} & G^{\wedge\wedge} & \xrightarrow{\nu^{\wedge\wedge}} & (G/A)^{\wedge\wedge} & \longrightarrow & 0.
\end{array}$$

Then the following assertions are true:
(a) *A is dually closed in G;*
(b) *A is dually embedded in G;*
(c) *A^0 is a compact subgroup of $G^\wedge$;*
(d) *$\mu^\wedge\colon G^\wedge\to A^\wedge$ is open and surjective;*
(e) *$\mu^{\wedge\wedge}\colon A^{\wedge\wedge}\to G^{\wedge\wedge}$ is open and injective;*
(f) *$\phi_A\colon G^\wedge/A^0\to A^\wedge$ is a topological isomorphism;*
(g) *$\phi^A\colon (G/A)^\wedge\to A^0$ is a topological isomorphism;*
(h) *both rows in diagram $(*)$ are exact;*
(i) *A^0 is dually embeded in $G^\wedge$;*
(j) *A^0 is dually closed in $G^\wedge$.*

Proof. The discrete group G/A admits sufficiently many continuous characters, which proves (a). For (b), see [4, Lemma 3.3]. Statement (c) follows from (1.2).

Now, (b) says that $\mu^\wedge\colon G^\wedge\to A^\wedge$ is surjective. To prove that $\mu^\wedge$ is open, take an arbitrary compact subset X of G. In view of (1.1), we only have to show that $\mu^\wedge(X^0)$ is a neighbourhood of zero in $A^\wedge$. As $\nu(X)$ is finite, the group $C=\operatorname{gp}\nu(X)$ is a direct sum of some cyclic subgroups $C_1,\dots,C_n$. For each $k=1,\dots,n$, choose a generator c_k of C_k and then some $g_k\in\nu^{-1}(c_k)$; let s_k be the order of C_k.

Let us denote $D = \mathrm{gp}\{g_k\}_{k=1}^{n}$, $I = \{k\colon s_k < \infty\}$ and

$$Q = \{p_1 g_1 + \ldots + p_n g_n \colon p_1, \ldots, p_n \in \mathbf{Z} \text{ and } 0 \leqq p_k < s_k \text{ for } k \in I\}.$$

It is not hard to see that we can find a compact subset Y of A such that

$$(1) \qquad X \subset Y + Q.$$

The set

$$U = \left\{\chi \in A^{\wedge} \colon |\chi(Y \cup \{s_k g_k\}_{k \in I})| \leqq \frac{1}{8n}\right\}$$

is a neighbourhood of zero in $A^{\wedge}$; we shall prove that $U \subset \mu^{\wedge}(X^0)$.

Let us take an arbitrary $\chi \in U$. We shall treat characters as functions with values in the interval $(-\frac{1}{2}, \frac{1}{2}]$. Given a real number x, by $\langle x \rangle$ we shall denote the number $y \in (-\frac{1}{2}, \frac{1}{2}]$ for which $x - y \in \mathbf{Z}$. For each $k \in I$, let us write $r_k = s_k^{-1} \chi(s_k g_k)$. Then the formula

$$(2) \qquad \kappa\left(a + \sum_{k=1}^{n} p_k g_k\right) = \left\langle \chi(a) + \sum_{k \in I} p_k r_k \right\rangle \qquad (a \in A;\, p_1, \ldots, p_n \in \mathbf{Z})$$

defines a character κ of $A + D$; the verification of this simple fact is left to the reader. One has $\kappa_{|A} = \chi$. Since A is open, κ is continuous. Now, by (b), we can extend κ to some $\tilde{\kappa} \in G^{\wedge}$ ($A + D$ is an open subgroup of G). Then $\mu^{\wedge}(\tilde{\kappa}) = \tilde{\kappa}_{|A} = \chi$. Finally, it follows easily from (1), (2) and the definition of U that $|\tilde{\kappa}(X)| = |\kappa(X)| \leqq \frac{1}{4}$, i.e. that $\tilde{\kappa} \in X^0$. This completes the proof of (d).

Statement (e) follows from (c), (d) and (2.1), because $\ker \mu^{\wedge} = A^0$. Next, (f) is a direct consequence of (d), while (g) follows from the compactness of $(G/A)^{\wedge}$.

As the upper row in $(*)$ is exact, (1.4) implies that the sequence

$$0 \longleftarrow A^{\wedge} \xleftarrow{\ \mu^{\wedge}\ } G^{\wedge} \xleftarrow{\ \nu^{\wedge}\ } (G/A)^{\wedge} \longleftarrow 0$$

is exact, too. Applying (1.4) once again and using (d), we see that the lower row in $(*)$ is exact ($\nu^{\wedge\wedge}$ is surjective because such are ν and $\alpha_{G/A}$). This proves (h).

To prove (i), consider the identity embedding $\iota\colon A^0 \to G^{\wedge}$ and the commutative diagram

$$\begin{array}{ccc}
G^{\wedge\wedge} & \xrightarrow{\ \nu^{\wedge\wedge}\ } & (G/A)^{\wedge\wedge} \\
\Big\downarrow \iota^{\wedge} & & \Big\uparrow (\phi^{A})^{\wedge} \\
(A^0)^{\wedge} & \xrightarrow{\ \mathrm{id}\ } & (A^0)^{\wedge}.
\end{array}$$

As $\nu^{\wedge\wedge}$ is surjective, and (g) implies that $(\phi^A)^{\wedge}$ is a topological isomorphism, it follows that $\iota^{\wedge}$ maps $G^{\wedge\wedge}$ onto $(A^0)^{\wedge}$, i.e. that (i) is satisfied. Finally, (j) is a direct consequence of (f). □

(2.3) Theorem. *Let A be an open subgroup of an abelian topological group G. Then A if reflexive if and only if G is reflexive.*

(2.4) Remark. The "if" part was obtained by Venkataraman [5, Corollary 6.3]. However, the proof presented in [5] includes serious inaccuracies and we give another proof below.

Proof of (2.3) Consider the diagram (∗) of (2.2). We may write

$$\mu^{\wedge\wedge}\alpha_A = \alpha_G\,\mu. \tag{1}$$

Suppose that A is reflexive. Then it follows from (1.5) and (2.2.h) that α_G is an algebraic isomorphism ($\alpha_{G/A}$ is an isomorphism because G/A is discrete). Moreover, (1) and (2.2.e) imply that $\alpha_G\,\mu$ is continuous and open. As A is an open subgroup of G, it follows that α_G is a topological isomorphism.

Conversely, suppose that G is reflexive. A direct verification shows that $\alpha_G(A)\subset\mu^{\wedge\wedge}(A^{\wedge\wedge})\subset A^{00}$. From (1.3) and (2.2.a) we get $\alpha_G(\mathrm{A})=\mathrm{A}^{00}$; thus $\alpha_G(A)=\mu^{\wedge\wedge}(A^{\wedge\wedge})=A^{00}$. This allows us to draw the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\ \mathrm{id}\ } & A \\ \Big\downarrow{\scriptstyle \alpha_A} & & \Big\downarrow{\scriptstyle \alpha'_G} \\ A^{\wedge\wedge} & \xrightarrow{\ \mu'\ } & A^{00} \end{array}$$

where $\mu'(\zeta)=\mu^{\wedge\wedge}(\zeta)$ for $\zeta\in A^{\wedge\wedge}$, and $\alpha'_G(a)=\alpha_G(a)$ for $a\in A$. As G is reflexive, α'_G is a topological isomorphism. Next, (2.2.e) implies that μ' is a topological isomorphism, too. Then we may write $\alpha_A=(\mu')^{-1}\alpha'_G$, which means that α_A is a topological isomorphism. □

(2.5) Lemma. *Let G, H be Hausdorff abelian groups and $\psi\colon G\to H$ a continuous homomorphism. Suppose that ψ is open and its kernel is compact. Then the dual homomorphism $\psi^{\wedge}\colon H^{\wedge}\to G^{\wedge}$ is open, its kernel being compact, too.*

Proof. The compactness of $\ker\psi^{\wedge}=\psi(G)^0$ follows from (1.2). To prove that $\psi^{\wedge}$ is open, consider the canonical factorization

$$\begin{array}{ccc} G & \xrightarrow{\ \psi\ } & H \\ \Big\downarrow{\scriptstyle \psi_1} & & \Big\uparrow{\scriptstyle \psi_3} \\ G/K & \xrightarrow{\ \psi_2\ } & \psi(G) \end{array}$$

where $K=\ker\psi$, and ψ_3 is the identity mapping. Naturally, ψ_2 is a topological isomorphism of G/K onto $\psi(G)$, so that $\psi_2^{\wedge}$ is a topological isomorphism of $\psi(G)^{\wedge}$ onto $(G/K)^{\wedge}$. Now, (2.1) and (2.2.d) imply that $\psi_1^{\wedge}$ and $\psi_3^{\wedge}$ are open, and therefore so is $\psi^{\wedge}=\psi_1^{\wedge}\psi_2^{\wedge}\psi_3^{\wedge}$. □

(2.6) Theorem. *Let K be a compact subgroup of a Hausdorff abelian group G. If G admits sufficiently many continuous characters and G/K is reflexive, then*

G is reflexive. Conversely, if G is reflexive and K dually closed in G, then G/K is reflexive.

Proof. Suppose first that G admits sufficiently many characters and G/K is reflexive. Consider the canonical commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{\mu} & G & \xrightarrow{\nu} & G/K & \longrightarrow & 0\\ & & \downarrow{\scriptstyle\alpha_K} & & \downarrow{\scriptstyle\alpha_G} & & \downarrow{\scriptstyle\alpha_{G/K}} & & \\ & & K^{\wedge\wedge} & \xrightarrow{\mu^{\wedge\wedge}} & G^{\wedge\wedge} & \xrightarrow{\nu^{\wedge\wedge}} & (G/K)^{\wedge\wedge}. & & \end{array}$$

Its upper row is exact. Hence, by (1.4) the sequence

$$K^{\wedge} \xleftarrow{\mu^{\wedge}} G^{\wedge} \xleftarrow{\nu^{\wedge}} (G/K)^{\wedge} \longleftarrow 0$$

is exact, too. Lemma 2.1 says that $\nu^{\wedge}$ is open. Set $F=\mu^{\wedge}(G^{\wedge})$ and let $\sigma\colon G^{\wedge}\to F$ be the homomorphism given by $\sigma(\chi)=\mu^{\wedge}(\chi)$ for $\chi\in G^{\wedge}$. Then the sequence

$$0 \longleftarrow F \xleftarrow{\sigma} G^{\wedge} \xleftarrow{\nu^{\wedge}} (G/K)^{\wedge} \longleftarrow 0$$

is exact, and σ is open because F is discrete. So, by (1.4), the sequence

$$0 \longrightarrow F^{\wedge} \xrightarrow{\sigma^{\wedge}} G^{\wedge\wedge} \xrightarrow{\nu^{\wedge\wedge}} (G/K)^{\wedge\wedge} \longrightarrow 0$$

is exact. As $K^{\wedge}$ is discrete, F is dually embedded in $K^{\wedge}$. Hence $\operatorname{im}\sigma^{\wedge}=\operatorname{im}\mu^{\wedge\wedge}$, so that $\operatorname{im}\mu^{\wedge\wedge}=\ker\nu^{\wedge\wedge}$. Now, (1.5) shows that α_G maps G onto $G^{\wedge\wedge}$. That α_G is injective follows from our assumption that continuous characters separate points of G.

If α_G were not open, there would exist a neighbourhood U of zero in G and a net (g_i) in G, such that $g_i\notin U$ and $\alpha_G(g_i)\to 0$. Hence $\alpha_{G/K}\,\nu(g_i)=\nu^{\wedge\wedge}\alpha_G(g_i)\to 0$ and, consequently, $\nu(g_i)\to 0$ because $\alpha_{G/K}$ is a topological isomorphism. Then, by (1.6.a), we could find a finer net (g'_j) converging to some $g\in G$. As $g'_j\notin U$, we would have $g\neq 0$ and, therefore, $\chi(g)\neq 0$ for some $\chi\in G^{\wedge}$ because $G^{\wedge}$ separates points of G. On the other hand, we would have

$$\chi(g)=\chi(\lim g'_j)=\lim\chi(g'_j)=\lim\alpha_G(g'_j)(\chi)=0,$$

which is a contradiction. Thus α_G is open.

The proof of continuity is similar. Let (g_i) be a net converging to zero in G and V a neighbourhood of zero in $G^{\wedge\wedge}$, such that $\alpha_G(g_i)\notin V$ for every i. Then $\nu^{\wedge\wedge}\alpha_G(g_i)=\alpha_{G/K}\nu(g_i)\to 0$. Applying (2.5) twice, we see that $\nu^{\wedge\wedge}$ is open, and its kernel is compact. So, according to (1.6.a), we can find a finer net (g'_j) such that $\alpha_G(g_j')$ converges to some $\zeta \in G^{\wedge\wedge}$. As $\alpha_G(g_j') \notin V$, we have $\zeta \neq 0$ and hence $\zeta(\chi) \neq 0$ for some $\chi \in G^{\wedge}$. On the other hand,

$$\zeta(\chi) = \lim \alpha_G(g_j')(\chi) = \lim \chi(g_j') = 0,$$

which is impossible. This proves that α_G is continuous.

Conversely, let G be reflexive and K dually closed in G. According to (1.3), we have $\alpha_G(K) = K^{00}$ and we may identify G/K with $G^{\wedge\wedge}/K^{00}$. But $H := K^0$ is an open subgroup of the reflexive group $G^{\wedge}$ and (2.2.f) says that ϕ_H: $G^{\wedge\wedge}/K^{00} \to H^{\wedge}$ is a topological isomorphism. Furthermore, (2.3) implies that H is reflexive. Thus, G/K is topologically isomorphic to the reflexive group $H^{\wedge}$. □

3 Strong reflexivity

(3.1) Proposition. *Let H be a closed subgroup of a strongly reflexive group G. Then*

(a) *H is dually closed in G;*
(b) *H is dually embedded in G;*
(c) *ϕ_H: $G^{\wedge}/H^0 \to H^{\wedge}$ and ϕ^H: $(G/H)^{\wedge} \to H^0$ are topological isomorphisms;*
(d) *H and G/H are strongly reflexive.*

Proof. (a) As G is strongly reflexive, G/H is reflexive. In particular, G/H admits sufficiently many continuous characters, which means that H is dually closed in G.

(b) Since G is reflexive, it follows from (a) and (1.3) that $\alpha_{G/H}$ is a topological isomorphism of H onto H^{00}; let γ: $H^{00} \to H$ be the inverse isomorphism. Let us write $P = G^{\wedge}$ and $Q = H^0$.

Choose any $\chi \in H^{\wedge}$. Then the sequence

$$(P/Q)^{\wedge} \xrightarrow{\phi^Q} Q^0 = H^{00} \xrightarrow{\gamma} H \xrightarrow{\chi} \mathbf{T}$$

shows that $\chi\gamma\phi^Q \in (P/Q)^{\wedge\wedge}$. By assumption, P/Q is reflexive, so that we may write $\chi\gamma\phi^Q = \alpha_{P/Q}(\xi)$ for some $\xi \in P/Q$. Let ψ: $P \to P/Q$ be the canonical projection. Then $\xi = \psi(\kappa)$ for a certain $\kappa \in G^{\wedge}$ and a direct verification shows that $\kappa_{|H} = \chi$.

Having proved (a) and (b), we can derive (c) and (d) from Propositions 12 and 13 of [2] (cf. Remark 3.2 below). □

(3.2) Remark. Strong reflexivity is closely connected with the notion of strong duality, introduced in [2]. Statements (a) and (b) of (3.1) show that assumption (i) in Proposition 12 from [2] is dispensable. This means that a duality between abelian topological groups G and H is strong if and only if both G and H are strongly reflexive. Thus, an abelian topological group G is strongly reflexive if and only if the natural homomorphism $G^{\wedge} \times G \to \mathbf{T}$ is a strong duality.

(3.3) Theorem. *Let A be an open subgroup of an abelian topological group G. Then A is strongly reflexive if and only if G is strongly reflexive.*

Proof. The "if" part is a consequence of (3.1.d). So, suppose that A is strongly reflexive. Let H be a closed subgroup of G and F a closed subgroup of $E := G\hat{}$. We have to show that the groups H, G/H, F and E/F are reflexive.

The group $H \cap A$ is reflexive, being a closed subgroup of the strongly reflexive group A. On the other hand, $H \cap A$ is an open subgroup of H, and (2.3) implies that H is reflexive.

Next, consider the canonical commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & G \\ \downarrow & & \downarrow \\ A/(H \cap A) & \xrightarrow{\beta} & G/H. \end{array}$$

It is clear that β is open; then it is a topological isomorphism of $A/(H \cap A)$ onto its image in G/H. Now, $A/(H \cap A)$ is reflexive, being a Hausdorff quotient of the strongly reflexive group A. Consequently, im β is a reflexive open subgroup of G/H, and (2.3) implies that G/H is reflexive.

It follows from (2.2.c) that $K := A^0$ is a compact subgroup of E. Next, (2.2.f) implies that

(∗) *E/K is strongly reflexive.*

Consider the canonical commutative diagram

$$\begin{array}{ccccc} F & \xrightarrow{\iota} & F+K & \longrightarrow & E \\ \downarrow & & \downarrow{\scriptstyle \phi} & & \downarrow \\ F/(F \cap K) & \xrightarrow{\rho} & (F+K)/K & \xrightarrow{\sigma} & E/K. \end{array}$$

It is clear that ρ is a continuous isomorphism. Naturally, ι is closed, and ϕ is closed due to (1.6.b). Therefore ρ is closed, which means that ρ is a topological isomorphism of $(F+K)/K$ onto its image in E/K. Consequently, $F/(F \cap K)$ is topologically isomorphic to a closed subgroup of E/K. Now (∗) implies that $F/(F \cap K)$ is reflexive and then F is reflexive owing to (2.6) (being a dual group, E admits sufficiently many continuous characters and, therefore, so does F).

Next, consider the canonical commutative diagram

$$\begin{array}{ccccc} E/K & \xleftarrow{\mu} & E & \xrightarrow{\nu} & E/F \\ \downarrow & & \downarrow & & \downarrow \\ (E/K)/\mu(F) & \xleftarrow{\mu'} & E/(K+F) & \xrightarrow{\nu'} & (E/F)/\nu(K). \end{array}$$

It is clear that μ' and ν' are topological isomorphisms (all mappings in the diagram are continuous and open). It follows from (∗) that $(E/K)/\mu(F)$ is reflexive.

Consequently, so is $(E/F)/\nu(K)$ and, in view of (2.6), to prove that E/F is reflexive we only need to verify that F is dually closed in E.

Take any $e \in E \setminus F$. If $e \notin F+K$, then $\mu(e) \notin \mu(F)$. By (*) and (1.6.b), there is some $\chi \in (E/K)^\wedge$ with $\chi_{|\mu(F)} \equiv 0$ and $\chi(\mu(e)) \neq 0$; then $\chi\mu$ is a continuous character of E which takes F, but not e, to zero.

So, suppose that $e \in F+K$; then $e = f+k$ for some $f \in F$ and $k \in K \setminus F$. Consider the canonical sequence

$$K \xrightarrow{\ \gamma\ } F+K \xrightarrow{\ \delta\ } (F+K)/F.$$

It is clear that $\delta\gamma$ maps K onto $(F+K)/F$, and $\delta\gamma(k) \neq 0$. So, $(F+K)/F$ is compact and there is a character χ' of $(F+K)/F$ with $\chi'(\delta\gamma(k)) \neq 0$. Then $\chi := \chi'\delta$ is a continuous character of $F+K$ with $\chi_{|F} \equiv 0$, and $\chi(e) = \chi(f+k) = \chi(f) + \chi(k) = \chi(k) = \chi'\delta(k) = \chi'(\delta\gamma(k)) \neq 0$.

Now, (2.2.i) says that K is dually embedded in E. Therefore we can find some $\kappa \in E^\wedge$ with $\kappa_{|K} = \chi_{|K}$. Let $\kappa' = \kappa_{|F+K}$. Then $\chi - \kappa'$ is a continuous character of $F+K$ vanishing on K, so there is a continuous character ξ of $(F+K)/K$ with $\xi\phi = \chi - \kappa'$. Since σ is a topological embedding, (*) implies that there is some $\eta \in (E/K)^\wedge$ with $\eta\sigma = \xi$. A direct verification shows that the character $\eta\mu + \kappa$ of E is equal to χ on $F+K$; thus $(\eta\mu+\kappa)_{|F} \equiv 0$ and $(\eta\mu+\kappa)(e) \neq 0$, which proves that F is dually closed in E. □

(3.4) Corollary. *Let K be a compact subgroup of a Hausdorff abelian group G. If G admits sufficiently many continuous characters and G/K is strongly reflexive, then G is strongly reflexive.*

Proof. Let $\psi: G \to G/K$ be the canonical projection. Then (2.1) says that $\psi^\wedge$: $(G/K)^\wedge \to G^\wedge$ is open. Hence ϕ^K: $(G/K)^\wedge \to K^0$ is a topological isomorphism. Now, K^0 is an open subgroup of $G^\wedge$, and (3.3) implies that $G^\wedge$ is strongly reflexive. Then $G^{\wedge\wedge}$ is strongly reflexive, too, but, by (2.6), we may identify $G^{\wedge\wedge}$ with G. □

(3.5) Remark. Kaplan [3] proved that the product of an arbitrary family of reflexive groups is reflexive. It follows from (3.3) and (3.4) that the product of a strongly reflexive group and a discrete or compact one is strongly reflexive. It is not known whether the product of a strongly reflexive group and the real line (or an arbitrary locally compact abelian group, which is the same in the given case) has to be strongly reflexive. The product $\mathbf{R}_\omega$ of countably many real lines is strongly reflexive, but $\mathbf{R}_\omega \times \mathbf{R}_\omega^\wedge$ is not [1, (17.7)].

Acknowledgement. The authors wish to thank the referee from the bottom of their hearts for his endurance in the correcting of errors and inaccuracies in the successive versions of the paper.

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