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COUNTABLE POWERS OF COMPACT ABELIAN GROUPS IN THE UNIFORM TOPOLOGY AND CARDINALITY OF THEIR DUAL GROUPS

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ABSTRACT. For a topological Abelian group X we consider in the group $X^{\mathbb{N}}$ the uniform topology and study some properties of the obtained topological group. We show, in particular, that if $X = \mathbb{S}$ is the circle group, then the group $\mathbb{S}^{\mathbb{N}}$ endowed with the uniform topology has dual group with cardinality $2^{\mathfrak{c}}$.

1. Introduction

For a set A we denote by $\mathrm{Card}(A)$ or by $|A|$ the cardinality of A.

For a topological space A we denote by:

- $w(A)$ the *weight* of A, i.e. the smallest cardinality of a base of A,
- $d(A)$ the *density character* of A, i.e., the smallest cardinality of a dense subset of A,
- $c(A)$ the *cellularity* of A, i.e., the smallest cardinal κ such that every family of non-empty pairwise disjoint open sets has cardinality $\leq \kappa$.

All groups considered in this note will be Abelian.

Let G, Y be groups. We denote by $\mathrm{Hom}(G, Y)$ the group of all group homomorphisms from G to Y. If G, Y are topological groups, $\mathrm{CHom}(G, Y)$ stands for the continuous elements of $\mathrm{Hom}(G, Y)$.

A set $\Gamma \subset \mathrm{Hom}(G, Y)$ will be called *separating*, if

$$(x_1, x_2) \in G \times G,\ x_1 \neq x_2 \implies \exists \gamma \in \Gamma,\ \gamma(x_1) \neq \gamma(x_2)\,.$$

In what follows the letter $\mathbb{S}$ will stand for the multiplicative group of complex numbers of modulus one endowed with the usual compact topology. We write:

$$\mathbb{S}_+ = \{s \in \mathbb{S} : \mathrm{Re(s)} \geq 0\}\,.$$

For a topological group G we denote by $\mathcal{N}(G)$ the collection of all neighborhoods of the neutral element of G.

A subset K of a topological group G is called *precompact* if for every $V \in \mathcal{N}(G)$ there exists a finite non-empty $A \subset G$ such that $K \subset A + V (:= \{a + v : a \in A, v \in V\})$.

A topological group G is called *locally precompact* if $\mathcal{N}(G)$ admits a basis consisting of precompact subsets of G.

For a group G an element of $\mathrm{Hom}(G, \mathbb{S})$ is called a (*multiplicative*) *character*.

For a topological group G we write

$$G^\wedge := \mathrm{CHom}(G, \mathbb{S})\,.$$

The elements of $G^\wedge$ are called *continuous characters* and $G^\wedge$ itself is *the (topological) dual* of G. Accordingly, for a group G and for a group topology τ in G we denote by $(G, \tau)^\wedge$ the dual of the topological group (G, τ), i.e.,

$$(G, \tau)^\wedge = \{\chi \in \mathrm{Hom}(G, \mathbb{S}) : \chi \text{ is } \tau - \text{continuous}\}\,.$$

A topological group G is called *maximally almost periodic*, for short a MAP-group, if $G^\wedge$ is separating. It is known that every locally precompact Hausdorff topological group is a MAP-group (this follows from the highly non-trivial statement that every locally compact Hausdorff topological group is a MAP-group

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[13, Theorem 22.17]). It is relatively easy to show that every discrete group (i.e. a group endowed with the discrete topology) is MAP; in this case the cardinality of the dual group can be calculated as follows.

Fact 1.1. ([14]; see also [13, (24.47)]) *Let D be an infinite discrete group. Then $|D^\wedge| = 2^{|D|}$.*

Let us recall the definition of a locally quasi-convex group.

Definition 1.2. [19] *A subset A of a* **topological** *group G is called* **quasi-convex** *if for every $x \in G\setminus A$ there exists $\chi \in G^\wedge$ such that*

$$\chi(A) \subset \mathbb{S}_+, \quad but\ \chi(x) \notin \mathbb{S}_+ .$$

A topological group G is called **locally quasi-convex** *if $\mathcal{N}(G)$ admits a basis consisting of quasi-convex subsets of G.*

Similar concepts (without a reference to [19]) were defined later in [17], where the terms *polar set* and *locally polar group* are used instead of 'quasi-convex set' and 'locally quasi-convex group'. It is easy to see that every locally quasi-convex Hausdorff topological abelian group is a MAP-group. For more information about locally quasi-convex topological groups we refer [3, 4, 6].

The locally precompact groups are a prominent class of locally quasi-convex groups.

If G is a compact Hausdorff topological abelian group, then $w(G) = \mathrm{Card}(G^\wedge)$ [13, Theorem 24.15]. In particular, an infinite compact Hausdorff topological abelian group G is metrizable iff $\mathrm{Card}(G^\wedge) = \aleph_0$. The following statement characterizes the locally precompact groups with countable dual.

Proposition 1.3. [8] *For an infinite locally precompact Hausdorff topological abelian group G TFAE:*

(i) *G is precompact metrizable.*
(ii) *$G^\wedge$ is countably infinite.*

The implication $(ii) \Longrightarrow (i)$ of Proposition 1.3 may fail if G is a locally quasi-convex Hausdorff group, which is not locally precompact (see Proposition 2.6).

In Section 2 we introduce the group of sequences $X^{\mathbb{N}}$ for a topological group X and its uniform topology $\mathfrak{u}$. The following statement is the main result of this note.

Theorem 1.4. *Let $X \neq \{0\}$ be a compact Hausdorff topological abelian group and $G = (X^{\mathbb{N}}, \mathfrak{u})$. Then*

(a) *$|\mathrm{CHom}(G, X)| \geq 2^{\mathfrak{c}}$.*
(b) *If $|X| \leq 2^{\mathfrak{c}}$ (in particular, if X is separable), then $|\mathrm{CHom}(G, X)| = 2^{\mathfrak{c}}$.*
(c) *If $X = \mathbb{S}$, then $|G^\wedge| = 2^{\mathfrak{c}}$.*

The proof of this theorem is given in Section 3.

Remark 1.5. In connection with Theorem 1.4 we note that in [11] it is presented the first example of a Banach space G over $\mathbb{R}$ with the following properties: $|G| = \mathfrak{c}$ and $|\mathrm{CHom}(G, \mathbb{R})| = 2^{\mathfrak{c}}$.

2. Two groups of sequences and their uniform topology

Let X be a set. As usual, $X^{\mathbb{N}}$ will denote the set of all sequences $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ of elements of X.

If X is a group with the neutral element θ, then

$X^{(\mathbb{N})}$ will stand for the subgroup of $X^{\mathbb{N}}$ consisting of all sequences from $X^{\mathbb{N}}$, which eventually equal to θ. If X is a topological group with the neutral element θ, then

$$c_0(X) := \{(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}} : \lim_n x_n = \theta\}.$$

Clearly $c_0(X)$ is a subgroup of $X^{\mathbb{N}}$, $X^{(\mathbb{N})} \subseteq c_0(X)$ and $X^{(\mathbb{N})} = c_0(X)$ iff X has only trivial convergent sequences.

In what follows X will be a fixed Hausdorff topological group.

We denote by $\mathfrak{p}_X$ the product topology in $X^{\mathbb{N}}$ and by $\mathfrak{b}_X$ the box topology in $X^{\mathbb{N}}$. It is easily verified that the collection

$$\{V^{\mathbb{N}} : V \in \mathcal{N}(X)\}$$

is a basis at $e := (\theta, \theta, \dots)$ for a group topology in $X^{\mathbb{N}}$ which we denote by $\mathfrak{u}_X$. In all three cases we shall omit the subscript $_X$ when no confusion is likely.

The topology $\mathfrak{u}$ in $X^{\mathbb{N}}$ is nothing else but the topology of uniform convergence on $\mathbb{N}$ when the elements of $X^{\mathbb{N}}$ are viewed as functions from $\mathbb{N}$ to X and X is considered as a uniform space with respect to its left (=right) uniformity [5]. So it will be called *the uniform topology.* Since it plays an important role in the sequel, we give in the next proposition an account of its main properties.

We write:

$$\mathfrak{p}_0 := \mathfrak{p}|_{c_0(X)} \ , \ \mathfrak{b}_0 := \mathfrak{b}|_{c_0(X)} \quad \text{and} \quad \mathfrak{u}_0 := \mathfrak{u}|_{c_0(X)}.$$

Proposition 2.1. [8] *Let $(X,+)$ be a Hausdorff topological abelian group.*

(a) *The uniform topology $\mathfrak{u}$ is a Hausdorff group topology in $X^{\mathbb{N}}$ with $\mathfrak{p} \leq \mathfrak{u} \leq \mathfrak{b}$. Moreover,*
 - (a_1) $\mathfrak{p}|_{X^{(\mathbb{N})}} = \mathfrak{u}|_{X^{(\mathbb{N})}} \iff X = \{0\}$.
 - (a_2) $\mathfrak{u}|_{X^{(\mathbb{N})}} = \mathfrak{b}|_{X^{(\mathbb{N})}} \implies X$ *is a* P-group $\implies \mathfrak{u} = \mathfrak{b}$*; in particular, if X is metrizable and* $\mathfrak{u}|_{X^{(\mathbb{N})}} = \mathfrak{b}|_{X^{(\mathbb{N})}}$*, then X is discrete.*

(b) *The passage from X to $(X^{\mathbb{N}},\mathfrak{u})$ preserves (sequential) completeness, metrizability, MAP and local quasi-convexity.*

(c) *If $X \neq \{0\}$ and $G := (X^{\mathbb{N}},\mathfrak{u})$, then:*
 - (c_1) $c(G) \geq \mathfrak{c}$*, in particular G is not separable;*
 - (c_2) $(X^{(\mathbb{N})},\mathfrak{u}|_{X^{(\mathbb{N})}})$ *is not precompact and hence, $(c_0(X),\mathfrak{u}_0)$ and $(X^{\mathbb{N}},\mathfrak{u})$ are not precompact.*

Remark 2.2. The topology of $(\mathbb{S}^{\mathbb{N}},\mathfrak{u})$ can be induced by the invariant metric ρ defined by the equality

$$\rho(\mathbf{x},\mathbf{y}) = \sup_{n\in\mathbb{N}} |x_n - y_n|, \quad \mathbf{x},\mathbf{y} \in \mathbb{S}^{\mathbb{N}}\,.$$

The metric group $G := (\mathbb{S}^{\mathbb{N}},\rho)$ was considered earlier in [7, Example 4.2], where it was noted that G is not precompact, but every uniformly continuous $f : G \to \mathbb{R}$ is bounded.

Proposition 2.3. [8] *Let X be a Hausdorff topological abelian group.*

(a) $(c_0(X),\mathfrak{u}_0)$ *is a Hausdorff topological group having as a basis at zero the collection* $\{V^{\mathbb{N}} \cap c_0(X) : V \in \mathcal{N}(X)\}$.

(b) $\mathfrak{p}_0 \leq \mathfrak{u}_0 \leq \mathfrak{b}_0$. *Moreover,* $\mathfrak{p}_0 = \mathfrak{u}_0 \iff X = \{0\}$*; if X is metrizable and* $\mathfrak{u}_0 = \mathfrak{b}_0$*, then X is discrete.*

(c) *The passage from X to $(c_0(X),\mathfrak{u}_0)$ preserves (sequential) completeness, metrizability, separability, MAP, local quasi-convexity, non-discreteness, and connectedness.*

Remark 2.4. Let X be the additive group $\mathbb{R}$ with the usual topology.

(1) By Proposition 2.1 $(\mathbb{R}^{\mathbb{N}},\mathfrak{u})$ is a complete metrizable topological abelian group. Note that although $\mathbb{R}^{\mathbb{N}}$ is a vector space over $\mathbb{R}$, $(\mathbb{R}^{\mathbb{N}},\mathfrak{u})$ **is not** a topological vector space over $\mathbb{R}$. The group $(\mathbb{R}^{\mathbb{N}},\mathfrak{u})$ is not connected; the connected component of the null element coincides with l_∞ and the topology $\mathfrak{u}|_{l_\infty}$ is the usual Banach-space topology of l_∞.

(2) By Proposition 2.3 $(c_0(\mathbb{R}),\mathfrak{u}_0)$ is a complete separable metrizable topological abelian group. Note that $c_0(\mathbb{R})$ is a vector space over $\mathbb{R}$ and $(c_0(\mathbb{R}),\mathfrak{u}_0)$ **is** a topological vector space over $\mathbb{R}$. The topology $\mathfrak{u}_0$ is the usual Banach-space topology of c_0.

(3) It is easy to see that $\mathbb{Z}^{(\mathbb{N})}$ is a closed subgroup of $(c_0(\mathbb{R}),\mathfrak{u}_0)$ and the quotient group

$$(c_0(\mathbb{R}),u_0)/\mathbb{Z}^{(\mathbb{N})}$$

is topologically isomorphic with $(c_0(\mathbb{S}),\mathfrak{u}_0)$.

Remark 2.5. The topology of $(c_0(\mathbb{S}),\mathfrak{u}_0)$ can be induced by the invariant metric ρ_0 defined by the equality

$$\rho_0(\mathbf{x},\mathbf{y}) = \sup_{n\in\mathbb{N}} |x_n - y_n|, \quad \mathbf{x},\mathbf{y} \in c_0(\mathbb{S})\,.$$

It seems that the metric group $(c_0(\mathbb{S}),\rho_0)$ was first considered by Rolewicz in [16], where he proves that it is a monothetic group. As he underlines, a monothetic and completely metrizable group need not be compact or discrete, a fact which breaks the dichotomy existing in the class of LCA-groups: namely, a monothetic LCA-group must be either compact or discrete ([18, Lemme 26.2 (p. 96)]; see also [2, Remark 5], where a construction of a different example of a complete metrizable monothetic non-locally compact group is indicated). In [15] it is observed that $|(c_0(\mathbb{S}),d_0)^\wedge| = \aleph_0$.
A proof of the fact that $(c_0(\mathbb{S}),\rho_0)$ is monothetic and $|(c_0(\mathbb{S}),\rho_0)^\wedge| = \aleph_0$ is contained also in [9, pp. 20–21]. In [12] it is shown further that $(c_0(\mathbb{S}),\rho_0)$ is a Pontryagin reflexive group.

The following statement provides, in particular, a wide class of non-compact Polish locally quasi-convex topological abelian groups with countable dual .

Proposition 2.6. [8] *For an infinite locally compact Hausdorff topological abelian group X TFAE:*
(i) *X is compact connected and metrizable.*
(ii) $|(c_0(X), \mathfrak{u}_0)^\wedge| = \aleph_0$.

Our Theorem 1.4 shows, in particular, that in the implication $(i) \implies (ii)$ of Proposition 2.6 the group $(c_0(X), \mathfrak{u}_0)$ cannot be replaced by the group $(X^{\mathbb{N}}, \mathfrak{u})$.

3. Auxiliary statements and proof of Theorem 1.4

We will need the following refinement of item (c_1) of Proposition 2.1.

Proposition 3.1. *Let X be a Hausdorff topological group and $G := (X^{\mathbb{N}}, \mathfrak{u})$. Then $d(G) \leq d(X)^{\aleph_0}$.*

Proof. Let D be a dense subset of X of size $d(X)$. It suffices to show that $D^{\mathbb{N}}$ is dense in G. This follows immediately from the definitions. □

We will also use later the following known statement.

Proposition 3.2. *Let X be a compact Hausdorff topological group.*
(a) *If $|X| \leq 2^{\mathfrak{c}}$, then $d(X) \leq \mathfrak{c}$.*
(b) *If X is infinite, then $|X| = 2^{w(X)}$.*

Proof. (a). From $w(X) \leq |X|$ (see [10, (3.1.21)]) and $|X| \leq 2^{\mathfrak{c}}$ we have: $w(X) \leq 2^{\mathfrak{c}}$. From the last inequality according to Ivanovskii-Kuzminov theorem [1, Theorem 4.1.7 (p. 222)] we get the existence of a continuous surjection $f : \{0,1\}^{2^{\mathfrak{c}}} \to X$. By [10, (2.3.15)] $d(\{0,1\}^{2^{\mathfrak{c}}}) \leq \mathfrak{c}$. Consequently, $d(X) = d(f(\{0,1\}^{2^{\mathfrak{c}}})) \leq d(\{0,1\}^{2^{\mathfrak{c}}}) \leq \mathfrak{c}$.

(b) Let D be the group $X^\wedge$ endowed with the compact-open topology. Then it is not hard to see that D is a discrete group. Endow $D^\wedge$ with the compact-open topology (which, in this case, coincides with the topology of point-wise convergence). It follows easily from Tikhonov-product theorem that $D^\wedge$ is compact Hausdorff topological group. The powerful Pontryagin duality theorem implies that the compact groups $D^\wedge$ and X are topologically isomorphic: $D^\wedge \cong X$. By [13, (24.15)] we have: $w(X) = |D|$. Since X is infinite, D is infinite as well; from this and $X \cong D^\wedge$ by Fact 1.1 we have: $|X| = 2^{|D|}$. From this equality, as $w(X) = |D|$, we get (b). □

In the sequel we deal with cardinals larger than $\mathfrak{c}$ and even $2^{\mathfrak{c}}$. We need to recall several standard definitions on cardinals.

Definition 3.3. *Let κ be a cardinal.*
- *κ^+ will denote the successor of κ.*
- *κ is called a* strong limit cardinal*, if $2^\lambda < \kappa$ for all $\lambda < \kappa$.*

Clearly, a strong limit cardinal κ is a limit cardinal (i.e., κ is not of the form λ^+ for any cardinal λ). Obviously, $\aleph_0$ is a strong limit cardinal. To obtain the next strong limit cardinal one has to go a long way. To this end let

$$\beth_0 = \aleph_0 \quad \text{and} \quad \beth_{n+1} = 2^{\beth_n} \quad \text{for all} \quad n \in \mathbb{N}.$$

Then $\beth_\omega := \sup_{n\in\mathbb{N}} \beth_n$ is the smallest uncountable strong limit cardinal. From Proposition 3.2(b) one can easily deduce that the cardinality of an infinite compact group is never a strong limit cardinal. In particular, if X is a compact Hausdorff topological group, then $|X| \neq \beth_\omega$.

Proposition 3.4. *Let X be a Hausdorff topological group and $G := (X^{\mathbb{N}}, \mathfrak{u})$. Then:*
(a) $|\mathrm{CHom}(G, X)| \leq |X|^{d(X)^{\aleph_0}}$.
(b) *If X is compact and $|X| \leq 2^{\mathfrak{c}}$, then $|\mathrm{CHom}(G, X)| \leq 2^{\mathfrak{c}}$.*
(c) *If X is compact and $2^{\mathfrak{c}} < |X| < \beth_\omega$, then under the assumption of the Generalized Continuum Hypothesis (GCH) we have: $|\mathrm{CHom}(G, X)| \leq |X|$.*

Proof. (a) Denote by $\mathrm{C}(G,X)$ the set of all continuous mappings $f:G\to X$; it is easy to see that $|\mathrm{C}(G,X)|\le |X|^{d(G)}$. From this inequality and Proposition 3.1 we get: $|\mathrm{C}(G,X)|\le |X|^{d(X)^{\aleph_0}}$. Consequently,

$$|\mathrm{CHom}(G,X)|\le |\mathrm{C}(G,X)|\le |X|^{d(X)^{\aleph_0}}.$$

(b) From $|X|\le 2^{\mathfrak{c}}$ by Proposition 3.2(a) we have: $d(X)\le\mathfrak{c}$. From the inequalities $|X|\le 2^{\mathfrak{c}}$, $d(X)\le\mathfrak{c}$ and from (a) we obtain: $|\mathrm{CHom}((G,X)|\le |X|^{d(X)^{\aleph_0}}\le (2^{\mathfrak{c}})^{\mathfrak{c}^{\aleph_0}}=2^{\mathfrak{c}}$.

(c) Let us recall that according to GCH, one has $2^\kappa=\kappa^+$ for every infinite cardinal κ. In particular, every uncountable $\kappa<\beth_\omega$ has the form $\kappa=\beth_m$ for some $m\ge 1$. In particular, the hypothesis $2^{\mathfrak{c}}<|X|<\beth_\omega$ and the fact that $2^{\mathfrak{c}}=\beth_2$ imply that $|X|=\beth_m$ for some $m>2$. By Proposition 3.2(b), it follows that $w(X)=\beth_{m-1}$, with $m-1>1$. So $w(X)=2^{\beth_{m-2}}$ and consequently,

$$w(X)^{\aleph_0}=(2^{\beth_{m-2}})^{\aleph_0}=2^{\beth_{m-2}}=\beth_{m-1}\quad\text{and}\quad d(X)^{\aleph_0}\le w(X)^{\aleph_0}=\beth_{m-1}.$$

Using the latter inequality and item (a), we get

$$|\mathrm{CHom}(G,X)|\le |X|^{d(X)^{\aleph_0}}\le \beth_m^{\beth_{m-2}}=(2^{\beth_{m-1}})^{\beth_{m-2}}=2^{\beth_{m-1}\cdot\beth_{m-2}}=2^{\beth_{m-1}}=\beth_m=|X|.$$

□

We prove in Theorem 1.4 that in Proposition 3.4(b) is practically an equality.
The groups X for which in Proposition 3.4(c) can occur an equality will be treated elsewhere.

Proof of Theorem 1.4.
(a). Denote by $\mathfrak{F}$ the set of all ultrafilters on $\mathbb{N}$.
It is known that

$$\mathrm{Card}(\mathfrak{F})=2^{\mathfrak{c}}. \tag{3.1}$$

For a filter $\mathcal{F}$ on $\mathbb{N}$, $(x_n)_{n\in\mathbb{N}}\in X^{\mathbb{N}}$ and $x\in X$ we write:

$$\lim_{n,\mathcal{F}} x_n=x$$

if for every $W\in\mathcal{N}(X)$ there is $F\in\mathcal{F}$ such that $x_n-x\in W$, $\forall n\in F$.

Since X is compact Hausdorff, it follows that for every $\mathcal{F}\in\mathfrak{F}$ and $(x_n)_{n\in\mathbb{N}}\in X^{\mathbb{N}}$ there exists a unique $x\in X$ such that $\lim_{n,\mathcal{F}}x_n=x$. This follows from the fact that the sets $A_F:=\{x_n:n\in F\}$, when F runs over $\mathcal{F}$, give rise to a filter base on X that generates an ultrafilter $\mathcal{F}^*$ on X that has a unique limit point x.

For a filter $\mathcal{F}\in\mathfrak{F}$ define the mapping $\chi_{\mathcal{F}}:X^{\mathbb{N}}\to X$ by equality:

$$\chi_{\mathcal{F}}(\mathbf{x})=\lim_{n,\mathcal{F}}x_n,\quad \forall \mathbf{x}=(x_n)_{n\in\mathbb{N}}\in X^{\mathbb{N}}.$$

Then

$$\chi_{\mathcal{F}}\in \mathrm{CHom}(G,X)\quad \forall\mathcal{F}\in\mathfrak{F}. \tag{3.2}$$

When $\mathcal{F}$ is the principal ultrafilter on $\mathbb{N}$ generated by a fixed $n\in\mathbb{N}$, then $\chi_{\mathcal{F}}$ is simply the projection $p_n:X^{\mathbb{N}}\to X$ on the n-th coordinate, so (3.2) is obvious. To verify (3.2) in the general case, fix $\mathcal{F}\in\mathfrak{F}$. As

$$\chi_{\mathcal{F}}(\mathbf{x}+\mathbf{y})=\lim_{n,\mathcal{F}}(x_n+y_n)=\lim_{n,\mathcal{F}}x_n+\lim_{n,\mathcal{F}}y_n=\chi_{\mathcal{F}}(\mathbf{x})+\chi_{\mathcal{F}}(\mathbf{y}),\quad \forall\mathbf{x},\mathbf{y}\in X^{\mathbb{N}},$$

we conclude that $\chi_{\mathcal{F}}\in \mathrm{Hom}(X^{\mathbb{N}},X)$. To see that χ is continuous on $(X^{\mathbb{N}},\mathfrak{u})$, fix a closed $W\in\mathcal{N}(X)$. Since W is closed, for $\mathbf{x}=(x_n)_{n\in\mathbb{N}}\in W^{\mathbb{N}}$ we shall have

$$\chi_{\mathcal{F}}(\mathbf{x})=\lim_{n,\mathcal{F}}x_n\in W.$$

Consequently, $\chi_{\mathcal{F}}(W^{\mathbb{N}})\subset W$. From this relation, as $W^{\mathbb{N}}\in\mathcal{N}(X^{\mathbb{N}},\mathfrak{u})$, we get that $\chi_{\mathcal{F}}$ is continuous on $(X^{\mathbb{N}},\mathfrak{u})$ and (3.2) is proved.

We also have:

$$\mathcal{F}_1\in\mathfrak{F},\ \mathcal{F}_2\in\mathfrak{F},\ \mathcal{F}_1\ne\mathcal{F}_2\implies \chi_{\mathcal{F}_1}\ne\chi_{\mathcal{F}_2} \tag{3.3}$$

In fact, as $\mathcal{F}_1$ and $\mathcal{F}_2$ are *distinct* ultrafilters, there is $F \in \mathcal{F}_1$ such that $F \notin \mathcal{F}_2$. Let $\mathbf{x} = (x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ be defined by conditions: $x_n = 0$ if $n \in F$ and $x_n = a \neq 0$ if $n \in \mathbb{N} \setminus F$. Then $\chi_{\mathcal{F}_1}(\mathbf{x}) = 0$ and $\chi_{\mathcal{F}_2}(\mathbf{x}) = a$. Therefore, $\chi_{\mathcal{F}_1} \neq \chi_{\mathcal{F}_2}$ and (3.3) is proved.

Clearly (3.1),(3.2) and (3.3) imply that $|\mathrm{CHom}(G, X)| \geq 2^{\mathfrak{c}}$.

(b) follows from (a) and from Proposition 3.4(b).

(c) follows immediately from (b). □

Remark 3.5. In notation of Theorem 1.4 and its proof let

$$\Gamma := \{\chi_{\mathcal{F}} : \mathcal{F} \in \mathfrak{F}\}\,.$$

Denote by $\langle\Gamma\rangle$ the subgroup of $\mathrm{CHom}(G, X)$ generated by the set Γ. As $|\Gamma| = 2^{\mathfrak{c}}$, we have also that $|\langle\Gamma\rangle| = 2^{\mathfrak{c}}$; so, in view of Theorem 1.4(b), for X with $|X| \leq 2^{\mathfrak{c}}$ we have the equality: $|\langle\Gamma\rangle| = 2^{\mathfrak{c}} = |\mathrm{CHom}(G, X)|$.

We do not know whether for X with $|X| \leq 2^{\mathfrak{c}}$ we have the equality $\langle\Gamma\rangle = \mathrm{CHom}(G, X)$ as well.

Remark 3.6. It follows from Proposition 2.1(a_2) that on $\mathbb{S}^N$ the box topology $\mathfrak{b}$ is strictly finer than the uniform topology $\mathfrak{u}$. This implies that we have the set-theoretic inclusion

$$\left(\mathbb{S}^N, \mathfrak{u}\right)^\wedge \subset \left(\mathbb{S}^N, \mathfrak{b}\right)^\wedge\,. \tag{3.4}$$

From Fact 1.1 we get:

$$|\mathrm{Hom}(\mathbb{S}^N, \mathbb{S})| = 2^{\mathfrak{c}}\,. \tag{3.5}$$

From (3.4), (3.5) and Theorem 1.4(c) we obtain:

$$|\left(\mathbb{S}^N, \mathfrak{u}\right)^\wedge| = 2^{\mathfrak{c}} = |\left(\mathbb{S}^N, \mathfrak{b}\right)^\wedge|\,. \tag{3.6}$$

This equality shows that from the pure cardinality arguments it is not possible to conclude that in (3.4) we have the strict inclusion. Nevertheless, we conjecture that the inclusion in (3.4) is strict, i.e., $\left(\mathbb{S}^N, \mathfrak{u}\right)^\wedge \neq \left(\mathbb{S}^N, \mathfrak{b}\right)^\wedge$.

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