

## On strongly reflexive topological groups

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**ABSTRACT.** An Abelian topological group  $G$  is *strongly reflexive* if every closed subgroup and every Hausdorff quotient of  $G$  and of its dual group  $G^\wedge$ , is reflexive.

In this paper we prove the following: the annihilator of a closed subgroup of an almost metrizable group is topologically isomorphic to the dual of the corresponding Hausdorff quotient, and an analogous statement holds for the character group of the starting group. As a consequence of this perfect duality, an almost metrizable group is strongly reflexive just if its Hausdorff quotients, as well as the Hausdorff quotients of its dual, are reflexive. The simplification obtained may be significant from an operative point of view.

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### INTRODUCTION AND PRELIMINARY CONCEPTS

One of the first questions when dealing with the reflexivity of some classes of objects (say topological vector spaces, or Abelian topological groups), is to know whether closed subobjects, or quotients by those, behave the same as the initial objects. It is known that, for locally convex spaces in general, reflexivity is not inherited by closed subspaces, nor by Hausdorff quotients. Under additional conditions the situation improves. For instance, all closed subspaces of a reflexive Frechet space are reflexive, but the property fails for quotients. In this context Banach spaces are in the optimal situation: closed subspaces and quotients by closed subspaces of a reflexive Banach space are reflexive.

For a topological Abelian group  $G$ , the symbol  $G^\wedge$  denotes the group of all continuous characters (i.e. homomorphisms from  $G$  into  $\mathbb{T}$ , the multiplicative

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group of complex numbers with modulus 1), with multiplication defined pointwise, endowed with the compact open topology. It is a Hausdorff topological Abelian group called the *dual group* of  $G$ . The bidual group  $G^{\wedge\wedge}$  is defined as  $(G^\wedge)^\wedge$ , and  $\alpha_G : G \rightarrow G^{\wedge\wedge}$  stands for the canonical embedding. If  $\alpha_G$  is a topological isomorphism, the group  $G$  is said to be *reflexive*. Examples of reflexive groups are: locally compact Abelian groups (LCA-groups) (Pontryagin, Van Kampen, 1935), the additive groups of Banach spaces and the additive groups of reflexive vector spaces (Smith, 1952), the products of reflexive groups (Kaplan, 1948), etc.

In the framework of Abelian topological groups the subclass of locally compact ones (LCA) behaves nicest. Pontryagin duality theorem establishes the reflexivity of LCA groups, and obviously closed subgroups and Hausdorff quotients of a LCA group are again LCA groups. A Banach space considered as an additive topological group is always reflexive, as mentioned above; but neither the closed subgroups nor the Hausdorff quotients by closed subgroups are reflexive in general.

An enforcement of the concept of reflexivity for groups was done for the first time in a remarkable paper by Brown, Higgins, Morris (1975) [5], where they considered "strong duality". After some evolution, the following definition is given in [3]: An Abelian topological group  $G$  is *strongly reflexive* if every closed subgroup and every Hausdorff quotient of  $G$  and of  $G^\wedge$  is reflexive.

The countable products and sums of real lines and circles were the first examples of non locally compact strongly reflexive groups [5]. In a natural way Banaszczyk extended this result proving that all countable products and sums of LCA groups are strongly reflexive [4]. He observed that all these examples were included in a larger class of groups, defined by him and thoroughly studied in [3]. The class of the so called nuclear groups contains the locally convex nuclear vector spaces and the locally compact Abelian groups, and it is closed under forming products, subgroups, Hausdorff quotients and countable sums. Finally, Aussenhofer proved that all Čech-complete nuclear groups are strongly reflexive [1], extending thus the same property obtained in [3] for complete metrizable nuclear groups. As a matter of fact, all the strongly reflexive groups so far mentioned are Čech-complete nuclear groups, and we do not know any example out of the class formed by nuclear groups and their duals.

All groups considered are Abelian. In the sequel we will omit this word. A subgroup  $H$  of a topological group  $G$  is said to be *dually closed* if, for every element  $x$  of  $G \setminus H$ , there is a continuous character  $\varphi$  in  $G^\wedge$  such that  $\varphi(H) = 1$  and  $\varphi(x) \neq 1$ . The subgroup  $H$  is said to be *dually embedded* if every continuous character defined on  $H$  can be extended to a continuous character on  $G$ . The *annihilator* of  $H$  is defined as the subgroup  $H^\circ := \{\varphi \in G^\wedge : \varphi(H) = 1\}$ .

Let  $f : G \rightarrow E$  be a continuous homomorphism of topological groups. The *dual mapping*  $f^\wedge : E^\wedge \rightarrow G^\wedge$  defined by  $(f^\wedge(\chi))(g) := (\chi \circ f)(g)$  is a continuous homomorphism. If  $f$  is onto, then  $f^\wedge$  is injective. For a closed subgroup  $H$  of a topological group  $G$ , denote by  $p : G \rightarrow G/H$  the canonical projection and by  $i : H \rightarrow G$  the inclusion. The dual mappings  $p^\wedge$  and  $i^\wedge$  give rise to

the natural continuous homomorphisms  $\varphi : (G/H)^\wedge \rightarrow H^\circ$  and  $\psi : G^\wedge/H^\circ \rightarrow H^\wedge$ . Observe that if  $H$  is dually embedded,  $\psi$  is onto. It is easy to prove that, a closed subgroup  $H$  of a topological group  $G$  is dually closed if and only if the quotient group  $G/H$  has sufficiently many continuous characters, or equivalently, if  $\alpha_G^{-1}(H^{\circ\circ}) = H$ . Here  $H^{\circ\circ}$  denotes the subgroup  $(H^\circ)^\circ$  of  $G^\wedge$ .

Reflexive groups are locally quasi-convex, a property introduced by Vilenkin in [12], which we now explain. A subset  $A$  of a topological group  $G$  is called *quasi-convex* if for every  $g \in G \setminus A$ , there is some  $\chi \in A^\circ := \{\chi \in G^\wedge : \operatorname{Re}\chi(z) \geq 0, \forall z \in A\}$ , such that  $\operatorname{Re}\chi(g) < 0$ . In particular, for subgroups quasi-convex means dually closed. An Abelian topological group  $G$  is called *locally quasi-convex* if it has a neighborhood basis of the neutral element  $e_G$  given by quasi-convex sets. The additive group of a topological vector space  $E$  is locally quasi-convex if and only if the vector space  $E$  is locally convex [3], therefore local quasi-convexity is an extension for groups of the notion of local convexity in vector spaces.

If  $G$  is locally quasi-convex,  $\alpha_G$  is relatively open. Furthermore, under the additional assumption that  $G$  is Hausdorff,  $G^\wedge$  has enough characters to separate points of  $G$ ; thus,  $\alpha_G$  is also one to one. A dual group  $G^\wedge$  is locally quasi-convex since a neighborhood basis of the neutral element  $e_{G^\wedge}$  is given by the sets  $K^\circ$ , where  $K \subset G$  is compact. Therefore local quasi-convexity is a necessary condition in order that a group be reflexive.

Local quasi-convexity is inherited by subgroups but not by Hausdorff quotients. It can be proved that quotients of locally quasi-convex groups by compact subgroups are locally quasi-convex. However, in the class of locally compact groups, and in the more general class of nuclear groups, every Hausdorff quotient is locally quasi-convex, see [3, (7.5)].

## 1. SPECIAL FEATURES OF ALMOST METRIZABLE GROUPS

The class of almost metrizable groups is interesting since it contains the classes of metrizable groups and of Čech-complete groups, therefore, in particular, that of locally compact groups. From the point of view of the Pontryagin duality theory, almost metrizable groups have some good properties which we study in this section.

Recall that a topological space  $X$  is called almost metrizable if every  $x \in X$  is contained in a compact subset having a countable neighbourhood basis in  $X$ . A Hausdorff topological group  $G$  is almost metrizable if and only if it has a compact subgroup  $K$  such that  $G/K$  is metrizable [11]. The completion of an almost metrizable group is a Čech-complete group. In the next Lemma we collect, for further reference, some known facts.

**Lemma 1.1.** *Let  $G$  be an almost metrizable topological group. Then,*

- i)  $G$  is a  $k$ -space.
- ii) The dual group  $G^\wedge$  is a  $k$ -space and  $G^{\wedge\wedge}$  is Čech-complete.
- iii) The canonical homomorphisms  $\alpha_G$  and  $\alpha_{G^\wedge}$  are continuous.

- iv) If  $G$  is furthermore reflexive, every dually closed and dually embedded subgroup of  $G$  is reflexive.

*Proof.* i) See [11] or [1, (1.24)] . ii) see [7] and [1, (5.20)] . iii) see [10]. iv) Theorem (3.1) of [10] can be applied.  $\square$

The projection of a Čech-complete group onto any of its Hausdorff quotients is known to be compact covering, and therefore the corresponding dual mapping is a topological embedding. On this line, with strictly weaker assumptions, we obtain the following results:

**Proposition 1.2.** *Let  $G$  be a topological group with  $\alpha_G$  continuous, let  $H$  be a closed subgroup of  $G$  and let  $p : G \rightarrow G/H$ ,  $q : G^\wedge \rightarrow G^\wedge/H^\circ$  be the canonical projections. Then,*

- i)  $p^\wedge : (G/H)^\wedge \rightarrow H^\circ$  is compact covering, therefore  $p^{\wedge\wedge} : (H^\circ)^\wedge \rightarrow (G/H)^{\wedge\wedge}$  is an embedding.
- ii) If furthermore  $H^\circ$  is a  $k$ -space, then  $p^\wedge : (G/H)^\wedge \rightarrow G^\wedge$  is an embedding.
- iii) If  $G$  is almost metrizable, then  $p^\wedge : (G/H)^\wedge \rightarrow G^\wedge$  and  $q^\wedge : (G^\wedge/H^\circ)^\wedge \rightarrow G^{\wedge\wedge}$  are embeddings.

*Proof.* i) If  $K \subset H^\circ$  is compact,  $K^\circ$  is a neighborhood of  $e_{G^{\wedge\wedge}}$  and  $V := (\alpha_G)^{-1}(K^\circ)$  is a neighborhood of  $e_G$  in  $G$ . It is straightforward to check that  $K \subset V^\circ$ , therefore  $(p^\wedge)^{-1}(K) \subset (p^\wedge)^{-1}(V^\circ) = (p(V))^\circ$ . The latter, being the polar of a zero-neighborhood, is compact and  $(p^\wedge)^{-1}(K)$  is a closed subset, thus also compact. The second claim can be easily checked.

ii) The dual mapping of  $p$  considered onto its image  $p^\wedge : (G/H)^\wedge \rightarrow H^\circ$  is a continuous isomorphism. In order to prove that its inverse  $(p^\wedge)^{-1}$  is also continuous, we must prove that  $(p^\wedge)^{-1} \Big|_K$  is continuous for all compact  $K \subset H^\circ$ . By i)  $(p^\wedge)^{-1}(K)$  is compact, therefore  $p^\wedge : (p^\wedge)^{-1}(K) \rightarrow K$  is a topological isomorphism. Thus,  $(p^\wedge)^{-1} \Big|_K$  is continuous.

- iii) The claim follows from ii) and Lemma 1.1, ii).  $\square$

**Remark 1.3.** The infinite sum of countably many copies of  $\mathbb{R}$ ,  $G := \omega\mathbb{R}$ , is a topological group with  $\alpha_G$  continuous and such that  $H^\circ$  is a  $k$ -space for every closed subgroup  $H \subset G$ . However, it is not almost metrizable. This shows that Proposition 1.2 ii) is more general than the same claim for almost metrizable groups.

In [5] the following two properties are defined: a topological group  $G$  has the property X1 if every closed subgroup of  $G$  is dually closed, and it has the property X2 if every closed subgroup of  $G$  is dually embedded.

Strongly reflexive groups and nuclear groups verify X1 and X2 [3]. If a complete metrizable vector space, considered as a topological group, has X1 and X2, then it must be nuclear and hence strongly reflexive. This is a deep result contained in [2] and [3, (17.3)].

We search now conditions under which a closed subgroup must be dually closed or dually embedded. It is straightforward to prove that a closed subgroup  $H$  of an Abelian topological group  $G$  is dually closed if and only if  $\alpha_{G/H}$  is injective. We now state similar conditions in order that  $H$  be dually embedded.

**Theorem 1.4.** *Let  $G$  be a topological group and let  $H$  be a closed subgroup of  $G$ . Then,*

- i)  $\alpha_{G/H}$  surjective implies  $H^\circ$  dually embedded.
- ii) If  $\alpha_G$  is surjective and continuous and the subgroup  $H^\circ$  is  $k$ -space, the converse of i) also holds.
- iii) If  $G$  is reflexive and  $H$  a dually closed subgroup,  $\alpha_{G/H^\circ}$  surjective implies  $H$  dually embedded. If furthermore  $H$  is a  $k$ -space, the converse statement also holds.

*Proof.* Let  $\kappa : H^\circ \rightarrow \mathbb{T}$  be a continuous homomorphism and let  $p : G \rightarrow G/H$  be the canonical projection. Since  $\kappa p^\wedge$  is continuous, it belongs to  $(G/H)^\wedge$ . But being  $\alpha_{G/H}$  surjective, there is some  $x \in G$  such that  $\alpha_{G/H}(p(x)) = \kappa p^\wedge$ . We now check that  $\alpha_G(x)$  is an extension of  $\kappa$ . Take an element  $\phi \in H^\circ$  and  $\tau \in (G/H)^\wedge$  such that  $p^\wedge(\tau) = \tau p = \phi$ . Then  $\alpha_G(x)(\phi) = \phi(x) = \tau(p(x))$  and  $\kappa(\phi) = \kappa(p^\wedge(\tau)) = \alpha_{G/H}(p(x))(\tau) = \tau(p(x))$ .

In order to prove the second assertion, suppose that  $H^\circ$  is dually embedded and a  $k$ -space and that  $\alpha_G$  is surjective and continuous. By Proposition 1.2 ii) we obtain that  $p^\wedge : (G/H)^\wedge \rightarrow H^\circ$  is a topological isomorphism.

Let  $\tau : (G/H)^\wedge \rightarrow \mathbb{T}$  be a continuous homomorphism. Then  $\kappa = \tau(p^\wedge)^{-1} \in (H^\circ)^\wedge$  and by the assumption it has some extension, say  $\tilde{\kappa}$ , to  $G^\wedge$ . Clearly  $\tilde{\kappa} \in G^{\wedge\wedge} = \alpha_G(G)$ , and so, there is some  $x \in G$  such that  $\tilde{\kappa} = \alpha_G(x)$ . Let us now check that  $\alpha_{G/H}(p(x)) = \tau$ . If  $\phi \in (G/H)^\wedge$ ,  $\alpha_{G/H}(p(x))(\phi) = \phi(p(x)) = \alpha_G(x)(\phi p) = \tilde{\kappa}(\phi p) = \tilde{\kappa}(p^\wedge(\phi)) = \tau(p^\wedge)^{-1} p^\wedge(\phi) = \tau(\phi)$ .

In order to prove iii), apply i) to the group  $G^\wedge/H^\circ$ . We obtain that  $H^\circ$  is dually embedded in  $G^{\wedge\wedge}$ . By the assumptions in iii), these groups can be identified to  $H$  and  $G$  respectively.  $\square$

**Example.** By means of the previous theorem we study the nonreflexivity of a class of groups, analogous in some respects to the Banach sequence spaces  $\ell_p$ , with  $1 < p < +\infty$ .

For a fixed real number  $p$ ,  $1 < p < +\infty$  denote by  $\ell_p(\mathbb{N}, \mathbb{T})$  the set of sequences  $t : \mathbb{N} \rightarrow \mathbb{T}$  with the property  $\sum_n |t(n) - 1|^p < +\infty$ . This set under pointwise multiplication is a subgroup of the multiplicative group  $\mathbb{T}^{\mathbb{N}}$ . We consider in  $\ell_p(\mathbb{N}, \mathbb{T})$  the topology generated by the distance

$$d(t, s) = \left( \sum_n |t(n) - s(n)|^p \right)^{\frac{1}{p}}, \quad t, s \in \ell_p(\mathbb{N}, \mathbb{T}).$$

With respect to this topology,  $\ell_p(\mathbb{N}, \mathbb{T})$  is a complete separable metrizable topological Abelian group and the characters defined by  $t \rightarrow t(n)$ ,  $n \in \mathbb{N}$ , separate the points of  $\ell_p(\mathbb{N}, \mathbb{T})$ .

The groups thus introduced are closely related to the Banach spaces  $\ell_p$ ,  $1 < p < \infty$ . In fact, they are topologically isomorphic to quotient groups of the corresponding spaces. For consider the homomorphism  $X : \mathbb{R}^{\mathbb{N}} \rightarrow \mathbb{T}^{\mathbb{N}}$  defined by  $X(x)(n) = \exp 2\pi i x(n)$ ,  $x \in \mathbb{R}^{\mathbb{N}}$ ,  $n \in \mathbb{N}$ . Clearly  $X$  is surjective and continuous with  $\ker X = \mathbb{Z}^{\mathbb{N}}$ , therefore the quotient homomorphism  $\tilde{X} : \frac{\mathbb{R}^{\mathbb{N}}}{\mathbb{Z}^{\mathbb{N}}} \rightarrow \mathbb{T}^{\mathbb{N}}$  is a topological isomorphism.

Let  $X_p$  be the restriction of  $X$  to  $\ell_p$ ,  $1 < p < \infty$ . It is easy to see that  $X_p(\ell_p) = \ell_p(\mathbb{N}, \mathbb{T})$ . Then, if  $1 < p < \infty$ ,  $\ker X_p = \mathbb{Z}_0^{\mathbb{N}}$  is the subgroup of  $\mathbb{Z}^{\mathbb{N}}$  of finitary non null sequences. The corresponding quotient maps  $\tilde{X}_p : \frac{\ell_p}{\mathbb{Z}_0^{\mathbb{N}}} \rightarrow \ell_p(\mathbb{N}, \mathbb{T})$  are topological isomorphisms. Observe that  $\mathbb{Z}_0^{\mathbb{N}}$ , as a subgroup of  $\ell_p$ ,  $1 < p < \infty$ , is dually closed and discrete.

The dual group  $(\ell_p)^\wedge$  is topologically isomorphic to the additive group of the space  $\ell_q$  with  $1/p + 1/q = 1$ , endowed with the topology  $\tau$  of the uniform convergence on compact sets. The annihilator  $(\mathbb{Z}_0^{\mathbb{N}})^\circ$  of the subgroup  $\mathbb{Z}_0^{\mathbb{N}}$  is again  $\mathbb{Z}_0^{\mathbb{N}}$ . Take now some  $x \in c_o \setminus \ell_p$ . The homomorphism  $\phi : \mathbb{Z}_0^{\mathbb{N}} \rightarrow \mathbb{T}$  defined by  $y \rightarrow \exp 2\pi i \langle x, y \rangle$  is continuous but it does not have a continuous extension to  $(\ell_p)^\wedge$ . Consequently the subgroup  $(\mathbb{Z}_0^{\mathbb{N}})^\circ$  of  $(\ell_p)^\wedge$  is not dually embedded and by 1.2, Theorem 1.4 ii),  $\alpha_{\ell_p/\mathbb{Z}_0^{\mathbb{N}}}$  is not surjective. Thus,  $\ell_p(\mathbb{N}, \mathbb{T})$  is not a reflexive group.

## 2. STRONG REFLEXIVITY OF ALMOST METRIZABLE GROUPS

We prove now that the requirements for strong reflexivity can be weakened in the framework of almost metrizable groups.

**Theorem 2.1.** *For an almost metrizable reflexive topological group, the following assertions are equivalent.*

- i)  $G$  is strongly reflexive.
- ii) Closed subgroups of  $G$  and  $G^\wedge$  are dually embedded and Hausdorff quotients of  $G$  and of  $G^\wedge$  are locally quasi-convex.

*Proof.* The fact that i) implies ii) is clear from all the preceding comments.

Conversely, assume that the Hausdorff quotients of  $G$  and of  $G^\wedge$  are locally quasi-convex. Then, the closed subgroups of both must be dually closed. By the assumption they are also dually embedded, and applying Lemma 1.1 iv), we obtain that all closed subgroups of  $G$  and of  $G^\wedge$  are reflexive.

Moreover, for any closed subgroup  $H$ , the quotient  $G/H$  is locally quasi-convex, therefore  $\alpha_{G/H}$  is injective and open into its image. By the previous theorem  $\alpha_{G/H}$  is surjective. Finally  $\alpha_{G/H}$  is continuous because  $\alpha_G$  is continuous [3, (14.7)]. Thus,  $G/H$  is reflexive and the same arguments show that  $G^\wedge/H^\circ$  is reflexive. On the other hand, being  $\alpha_G$  onto, every dually closed subgroup  $L$  of  $G^\wedge$  can be written as an annihilator  $L = H^\circ$ , where  $H = \alpha_G^{-1}(L^\circ)$ . Therefore all Hausdorff quotients of  $G^\wedge$  are reflexive, and this ends the proof of i).  $\square$

The theorem does not hold without the assumption of almost metrizability, as the next example shows.

**Example.** The group  $G := \mathbb{R}^\omega \times \omega\mathbb{R}$ , where  $\mathbb{R}^\omega$  and  $\omega\mathbb{R}$  are the countable product of real lines and the countable direct sum respectively, is a reflexive selfdual topological group. Since  $G$  is also nuclear its closed subgroups are dually embedded, and the Hausdorff quotients are locally quasi-convex, see [3, p.82 and 83]. Therefore  $G$  verifies *ii*) of Theorem 2.1. However,  $G$  is not strongly reflexive since it has a nonreflexive Hausdorff quotient, see [3, p.155].

**Corollary 2.2.** *For an almost metrizable topological group, the following assertions are equivalent:*

- i)  $G$  is strongly reflexive.*
- ii) Hausdorff quotients of  $G$  and  $G^\wedge$  are reflexive.*

*Proof.* Use the above two theorems. □

We do not know if Hausdorff quotients could be replaced in Corollary 1.5 *ii*), by closed subgroups.

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