Bol. Soc. Mat. Mexicana (3) Vol. 14, 2008

A CHARACTERIZATION OF K-ANALYTICITY OF GROUPS OF CONTINUOUS HOMOMORPHISMS

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ABSTRACT. For an abelian locally compact group X let $X_p^\wedge$ be the group of continuous homomorphisms from X into the unit circle $\mathbb{T}$ of the complex plane endowed with the pointwise convergence topology. It is proved that X is metrizable iff $X_p^\wedge$ is K-analytic iff X endowed with its Bohr topology $\sigma(X, X^\wedge)$ has countable tightness. Using this result, we establish a large class of topological groups with countable tightness which are not sequential, so neither Fréchet-Urysohn.

1. Introduction

For abelian topological groups X and Y we denote by $\mathrm{Hom}_p(X, Y)$ and $\mathrm{Hom}_c(X, Y)$ the set $\mathrm{Hom}(X, Y)$ of all continuous homomorphisms from X into Y endowed with the pointwise and compact-open topology, respectively. Set $X_p^\wedge =: \mathrm{Hom}_p(X, \mathbb{T})$, $X_c^\wedge =: \mathrm{Hom}_c(X, \mathbb{T})$, where $\mathbb{T}$ denotes the unit circle of the complex plane. For every $x \in X$ the function $x^\wedge : X^\wedge \to \mathbb{T}$, defined by $x^\wedge(f) := f(x)$ for $f \in X^\wedge$, is a continuous homomorphism on $X_c^\wedge$ and $\{x^\wedge : x \in X\} \subset (X_c^\wedge)^\wedge$. By Pontryagin-van Kampen's Theorem (see [10], Theorem 24.8), *if X is a locally compact abelian group, the mapping $\alpha : x \mapsto x^\wedge$ is a topological isomorphism between X and $(X_c^\wedge)_c^\wedge$*. If X is an abelian locally compact group, $X_c^\wedge$ is also locally compact and abelian, and by Peter-Weyl-van Kampen's Theorem $X_c^\wedge$ is *dual separating*, i.e. for different $x, y \in X$, there exists $f \in X^\wedge$ such that $f(x) \neq f(y)$, see [14], Theorem 21. For an abelian group X the set of all homomorphisms from X into $\mathbb{T}$ endowed with the pointwise convergence topology is a compact abelian group, as it is a closed subgroup of the product $\mathbb{T}^X$, see [11], Proposition 1.16. For a metrizable abelian topological group X the group $X_c^\wedge$ is always an abelian complete Hausdorff hemicompact group which is also a k-space, see [1], Corollary 4.7 and [5].

The main result of this paper is the following theorem:

THEOREM (1.1). *Let X be a locally compact abelian group. The following assertions are equivalent: (1) X is metrizable. (2) $X_p^\wedge$ is σ-compact. (3) $X_p^\wedge$ is K-analytic. (4) $(X, \sigma(X, X^\wedge))$ has countable tightness. Moreover, if X is*

2000 *Mathematics Subject Classification*: Primary 22A05, 43A40, 54H11.

Keywords and phrases: Locally compact groups, compact groups, angelic spaces, Lindelöf spaces.

The projects MTM2005 - 01182 and MTM 2006-03036 of the Spanish Ministery of Education and Science, co-financed by the European Community (Feder projects) supported the first and the third named authors and the second one respectively. Also the first named author was supported by Ministry of Science and Higher Education, Poland, grant no. N 201 2740 33, the second named author by Proyectos de Investigación Santander-Complutense PR 27/05-14039 and the third named author by Primeros Proyectos de Investigación UPV PAID-06-06 grant no. 20070317.

Lindelöf, then each of the above conditions is equivalent to (5) $X_c^\wedge$ is metric complete and separable.

We believe this theorem is interesting since it relates two different fields of research. It was motivated by several similar results concerning the spaces $C_p(X)$ of all continuous real-valued functions on a completely regular space X endowed with the pointwise convergence topology. (Calbrix [4], Theorem 2.3.1) showed that if for a Tychonoff space X the space $C_p(X)$ is analytic, then X must be σ-compact and analytic (cf. also [7], Theorem 3.7). If X is locally compact, then X is a polish space iff $C_p(X)$ is analytic, [13], Corollary 5.7.6. In [9] Corson proved that a locally compact topological group X is metrizable iff the Banach space $C_0(X)$ of continuous, complex valued functions which vanish at infinity is weakly Lindelöf. On the other hand, $C_p(X)$ is Lindelöf provided X is second countable. The converse fails in general but very recently we have shown [12] that for a locally compact topological group X the space $C_p(X)$ *is Lindelöf* iff X is metrizable and σ-compact; in particular, $C_p(X)$ is Lindelöf iff X is second countable. Recall that a (Hausdorff) topological space X is said to be K-analytic [16] if there is an upper semi-continuous set-valued map from the polish space $\mathbb{N}^\mathbb{N}$ with compact values in X whose union is X. Notice that analytic $\Rightarrow$ K-analytic $\Rightarrow$ Lindelöf. A topological space X is said to have *countable tightness* if for each set $A \subset X$ and any $x \in \overline{A}$ (the closure of A) there exists a countable subset $B \subset A$ whose closure contains x. For a topological abelian group X the coarsest group topology on X for which all elements of $X^\wedge$ are continuous is called the *Bohr topology*; we denote this topology by $\sigma(X, X^\wedge)$. A topological space X is said to be *hemicompact* if X is covered by a fundamental sequence of compact sets, i.e. there is a sequence $(K_n)_n$ of compact subsets of X such that each compact set in X is contained in some K_n. Finally by $\mathbb{R}$ and $\mathbb{C}$ we denote the sets of real and complex numbers, respectively.

2. Proof of the Theorem

The proof of the theorem is derived from the following facts:

FACT (1). *A locally compact Lindelöf topological group X is hemicompact.*

Proof. Take an open neighbourhood of the neutral element U whose closure $\overline{U}$ is compact. Since $X = \bigcup_{x\in X} xU$ and X is Lindelöf, there exists a sequence $(x_n)_n$ such that $X = \bigcup_n x_n\overline{U}$. Set $K_n := \bigcup_{i=1}^n x_i\overline{U}$. Then $(K_n)_n$ is a fundamental sequence of compact subsets of X, so X is hemicompact. □

FACT (2). *If X is a metrizable abelian group, $X_c^\wedge$ is a hemicompact k-space.*

Proof. Let $(U_n)_{n\in\mathbb{N}}$ be a decreasing basis of neighbourhoods of zero in X. Then $U_n^\triangleright := \{\phi \in X_c^\wedge : \phi(U_n) \subseteq \mathbb{T}_+\}$ is compact in the compact-open topology, where $\mathbb{T}_+ := \{z \in \mathbb{C} : |z| = 1, \operatorname{Re} z \geqslant 0\}$. But $X^\wedge = \bigcup_n U_n^\triangleright$. In fact, if $\phi \in X^\wedge$, then $\phi^{-1}(\mathbb{T}_+)$ is a neighbourhood of zero, so there exists $m \in \mathbb{N}$ such that $U_m \subseteq \phi^{-1}(\mathbb{T}_+)$. Therefore $\phi \in U_m^\triangleright$. If K is a compact set in $X_c^\wedge$, then $K^\triangleright \subset (X_c^\wedge)^\wedge$ is a neighbourhood of zero. The canonical mapping α is continuous, therefore $\alpha^{-1}(K^\triangleright) = \{x \in X : \operatorname{Re} \phi(x) \geqslant 0, \forall\phi \in K\} =: K^\triangleleft$, is a neighborhood of zero in X. Thus, $K^\triangleleft \supseteq U_m$ for some $m \in \mathbb{N}$, and we have $K \subseteq K^{\triangleright\triangleleft} \subseteq U_m^\triangleright$.

The proof of the fact that $X_c^\wedge$ is a k-space is harder. It was given independently in [1], Corollary 4.7 and [5]. □

FACT (3) (See [1], Proposition 2.8). *If a topological abelian group X is hemicompact, then $X_c^\wedge$ is metrizable.*

FACT (4). *Let X be a Tychonoff space and assume that $C_p(X,\mathbb{R})$ has countable tightness. Then $C_p(X,Y)$ also has countable tightness for any metric space (Y,d).*

Proof. Let $A \subset C_p(X,Y)$ and assume that $f \in \overline{A}$ (the closure in $C_p(X,Y)$). Define a continuous map $T\colon C_p(X,Y) \to C_p(X,\mathbb{R})$ by $T(g)(x) := d(g(x), f(x))$, where $g \in C_p(X,Y)$ and $x \in X$. Note that $0 = T(f) \in T(\overline{A}) \subset \overline{T(A)}$. By assumption there exists a countable subset $B \subset A$ such that $T(f) \in \overline{T(B)}$; hence, as easily seen from the definition of the pointwise convergence topology in $C(X,\mathbb{R})$ and in $C(X,Y)$, $f \in \overline{B}$. □

Proof of the Theorem. (1) ⇒ (2): By Fact (2) the group $X_c^\wedge$ is hemicompact. So $X_p^\wedge$ is σ-compact.

(2) ⇒ (3): If $(B_n)_n$ is an increasing sequence of compact sets covering $X_p^\wedge$, set $T(\alpha) := B_{n_1}$ for $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$. It is clear that T is upper semi-continuous, with compact values which cover $X_p^\wedge$.

(3) ⇒ (4): Since $X_p^\wedge$ is K-analytic, then any finite product $(X_p^\wedge)^n$ is Lindelöf. By [2], Theorem II.1.1, the space $C_p(X_p^\wedge,\mathbb{R})$ has countable tightness. Now Fact (4) applies to deduce that the space $C_p(X_p^\wedge,\mathbb{C})$ also has countable tightness. Therefore $(X,\sigma(X,X^\wedge))$ (as a subspace of $C_p(X_p^\wedge,\mathbb{C})$) has countable tightness.

(4) ⇒ (1): Since X is a locally compact group, there exist a compact subgroup G of X, $n \in \mathbb{N}\cup\{0\}$, and a discrete subset $D \subset X$ such that X is homeomorphic to the product $\mathbb{R}^n \times D \times G$, see [8], Theorem 1, Remark(ii). Therefore the induced topology $\sigma(X,X^\wedge)|G$ coincides with the original one of G. Hence G has countable tightness. Since a compact group with countable tightness is metrizable, (see e.g. [12], Theorem 2); so X is metrizable as well.

For the last statement observe that under the assumption that X is metrizable and Lindelöf, Facts (1) and (3) apply to deduce that $X_c^\wedge$ is a separable metric space. It is complete since the dual group of a locally compact abelian group is also locally compact. The result follows now from the fact that $X_p^\wedge$ is the continuous image of $X_c^\wedge$ under the identity mapping. □

Recall that a topological space is sequential if every sequentially closed subset of X is closed. In [6], Theorem 2.1, it was shown that for a metrizable topological group X the dual group $X_c^\wedge$ is Fréchet-Urysohn iff $X_c^\wedge$ is locally compact and metrizable. This result provides a large class of complete strictly angelic hemicompact sequential non Fréchet-Urysohn groups [6], Theorem 2.3. We supplement this result by the following

COROLLARY (2.1). *If X is a metrizable locally compact non-compact abelian group, then the group $(X,\sigma(X,X^\wedge))$ has countable tightness but it is not sequential, nor Fréchet-Urysohn.*

Proof. From Glicksberg's Theorem it follows that $(X,\sigma(X,X^\wedge))$ has the same compact subsets as X. Since X is metrizable, it is already a k-space, and X

does not admit a weaker k-space topology with the same compact subsets. Therefore $(X, \sigma(X, X^\wedge))$ is not a k-space, and in particular it is not sequential neither Fréchet-Urysohn, which are stronger properties. $\square$

Remark (2.2). If X is a hereditarily separable topological group (in particular if it is metrizable and separable), then $(X, \sigma(X, X^\wedge))$ has countable tightness.

This derives from the more general fact, brought to our attention by L. Aussenhofer: *If (X, τ) is a hereditarily separable topological space, then any weaker topology ξ on X is hereditarily separable and hence (as easily seen) has countable tightness*.

We complete the note with the following observation about continuous functions defined on a topological abelian group which are not homomorphisms.

PROPOSITION (2.3). *Let $f: X \to \mathbb{T}$ be a continuous functions defined on an abelian topological group X which is not a homomorphism. Then there exist a finite collection $\{z_1, \ldots, z_n\}$ of integer numbers and a finite set $\{x_1, \ldots, x_n\}$ in X such that for all $\phi \in X^\wedge$ one has $\phi(x_1)^{z_1}\phi(x_2)^{z_2} \cdots \phi(x_n)^{z_n} = 1$ and $\mathrm{Re}(f(x_1)^{z_1} \cdots f(x_n)^{z_n}) < 0$, where Re stands for the real part.*

Proof. Clearly $X_p^\wedge$ is a closed subgroup of $C_p(X, \mathbb{T})$ which can be considered as a subgroup of the topological group $\mathbb{T}^X$. Thus, $C_p(X, \mathbb{T})$ is precompact and so $X_p^\wedge$ is dually closed as a subgroup of $C_p(X, \mathbb{T})$, see [3], (8.6). This means that every element of $C_p(X, \mathbb{T}) \setminus X^\wedge$ can be separated from $X^\wedge$ by means of a continuous character of $C_p(X, \mathbb{T})$, which in particular is the restriction of a character on $\mathbb{T}^X$. On the other hand, it is well-known that the character group of a product of topological abelian groups is the direct sum of the corresponding dual groups. So the direct sum $\mathbb{Z}^{(X)}$ is the character group of $\mathbb{T}^X$. Hence for $f \in C_p(X, \mathbb{T}) \setminus X^\wedge$ there exists a character ξ which can be written as $z_{x_1} + z_{x_2} + \cdots + z_{x_n}$ with $z_{x_i} \in \mathbb{Z}^{(X)}$ such that $\xi(\phi) = \phi(x_1)^{z_{x_1}}\phi(x_2)^{z_{x_2}} . \cdots . \phi(x_n)^{z_{x_n}} = 1$ for all $\phi \in X^\wedge$ and $\mathrm{Re}(\xi(f)) = f(x_1)^{z_{x_1}} f(x_2)^{z_{x_2}} . \cdots . f(x_n)^{z_{x_n}} < 0$. $\square$

Acknowledgements

The authors are grateful to the referees, for their thorough reading of the paper.

Received May 22, 2007

Final version received January 17, 2008

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