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Locally quasi-convex topologies on the group of the integers

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Resumen

La topología más natural en el grupo $\mathbb{Z}$ de los enteros, es la discreta. Son también muy conocidas las topologías p-ádicas en $\mathbb{Z}$, para cualquier número primo p. Otra topología de grupo importante es la topología débil asociada al grupo de los homomorfismos de $\mathbb{Z}$ en el círculo unidad del plano complejo; es decir, la definida por los caracteres y que se conoce por "toplogía de Bohr" en $\mathbb{Z}$.

En [10], se comprueba que tomando como base de entornos de cero los conjuntos (W_n), donde $W_n := \{k \in \mathbb{Z} \mid \forall x \in S, k \cdot x \in [-\frac{1}{4n}, \frac{1}{4n}] + \mathbb{Z}\}$ con S una sucesión quasi-convexa en $\mathbb{T}$, se obtiene una toplogía de grupo sobre los enteros, τ_S. Sabemos que la topología τ_S es metrizable y localmente cuasi-convexa.

En este trabajo caracterizamos las sucesiones convergentes en τ_S, para ciertos subconjuntos $S \subset \mathbb{T}$. Damos condiciones sobre los elementos de $\mathbb{Z}$ para pertenecer a un cierto entorno W_n. Tomando como referencia una sucesión (b_n) de números naturales, con ciertas restricciones, hemos considerado la "topología lineal asociada" cuya base de entornos de 0 es $\{b_n\mathbb{Z} \mid n \in \mathbb{N}\}$ y la de "la convergencia uniforme" en $S = \{\frac{1}{b_n} + \mathbb{Z} \mid n \in \mathbb{N}\} \subset \mathbb{T}$. Hacemos un estudio comparativo entre ambas clases de topologías.

Palabras clave: Topología de grupo, subconjunto quasi-convexo, polar, dualidad, topología lineal, carácter, topologías compatibles, topología de la convergencia uniforme, topología compacto-abierta.

Abstract

The most natural group topology on $\mathbb{Z}$ is the discrete one. There are other well-known group topologies on $\mathbb{Z}$, like the p-adic, defined for any prime number p. It is also an important group topology the weak topology with respect to the group of homomorphisms from $\mathbb{Z}$ to the unit circle of the complex plane; that is, the one defined by the characters and which is known as "the Bohr topology" on $\mathbb{Z}$.

In [10], it is proved that taking as a neighbourhood basis at 0 the subsets $\{W_n \mid n \in \mathbb{N}\}$, defined by $W_n := \{k \in \mathbb{Z} \mid \forall x \in S, k \cdot x \in [-\frac{1}{4n}, \frac{1}{4n}] + \mathbb{Z}\}$, where S is a quasi-convex sequence in $\mathbb{T}$, a group topology on $\mathbb{Z}$ is obtained, τ_S. We know that the topology τ_S is metrizable and locally quasi-convex.

In this monograph we characterize convergent sequences in τ_S, for some $S \subset \mathbb{T}$. We give sufficient conditions on the elements of $\mathbb{Z}$ in order that they belong to a neighbourhood W_n. For a fixed sequence (b_n) of natural numbers, restricted to mild conditions, we have considered the "linear topology associated" whose neighbourhood basis at 0 is $\{b_n\mathbb{Z} \mid n \in \mathbb{N}\}$ and the "topology of uniform convergence" on $S = \{\frac{1}{b_n} + \mathbb{Z} \mid n \in \mathbb{N}\} \subset \mathbb{T}$. We make a comparative study between both classes of topologies.

Key words: Group topology, neighbourhood basis for a group topology, quasi-convex subset, polar, duality, linear topology, character, compatible topologies, topology of the uniform convergence, compact-open topology.

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Introduction

The idea of a general theroy of continuous groups is due to Sophus Lie, who developed his theory in the decade 1874-1884. Lie's work is the origin of both modern theory of Lie groups and the general theory of topological groups. However, the topological considerations that are nowadays essential in both theories, are not part of his work.
A topological point of view in the theory of continuous groups was first introduced by Hilbert. Precisely, in his famous list of 23 problems, presented in the International Congress of Mathematicians of 1900, held in Paris, the Fifth Problem boosted investigations on topological groups.
In modern language, the Fifth Problem asked if any locally euclidean topological group can be endowed with a structure of analytic variety in such a way that it becomes a Lie group.
In 1929 von Neumann using integration on compact general groups, introduced by himself, solved the problem for compact groups. In 1934, Pontryagin solved it for locally compact abelian groups, using the character theory introduced by himself.
The final resolution, at least in this interpretation of what Hilbert meant, came with the work of A. Gleason, D. Montgomery and L. Zippin in the 1950s.
In 1953, Hidehiko Yamabe obtained the final answer to Hilberts Fifth Problem: A connected locally compact group G is a projective limit of a sequence of Lie groups, and if G "has no small subgroups", then G is a Lie group.
In 1932, Stefan Banach, defined in Thèorie des Opèrations Linèaires, the spaces that would be named after him as special cases of topological groups. Since then, both theories have been developed in a different, but somehow parallel, way.
Several theorems for Banach, or even for locally convex spaces have been reformulated for abelian topological groups [3], [4], [7], [8], [11]. The main obstacle for this task is the lack of the notion of convexity for topological groups. However, Vilenkin gave the definition of quasi-convex subset for a topological abelian group inspired in the Hahn-Banach theorem. In order to deal with it, many convenient tools had to be developed and this opened the possibility of a fruitful treatment of topological

groups. The notion of quasiconvexity depends on the topology, in contrast with convexity, which is a purely algebraic notion.

After this, with quasi-convex subsets at hand, it was quite natural to define locally quasi-convex groups, which was also done by Vilenkin in [15].

Duality theory for locally convex spaces was mainly developed in the mid of twentieth century, and by now is a well-known rich theory.

There is a very natural way to extend it to locally quasi-convex abelian groups. From now on, we only speak of abelian groups although not explicitly mentioned. Fix first the dualizing object as the unit circle of the complex plane $\mathbb{T}$. The continuous homomorphisms from a topological group into $\mathbb{T}$ play the role of the continuous linear forms and they will be named continuous characters.

The set of continuous characters defined on a group G has a natural group structure provided by the group structure of $\mathbb{T}$. Thus, we can speak of the group of characters of G. If it is endowed with the compact-open topology, the topological group obtained is called the dual group of G.

The research we have done in this memory deals with the following problem: Let (G, τ) be an abelian topological group and let $G^\wedge$ be its dual group. Consider the set of all locally quasi-convex group topologies in G whose dual group coincides with $G^\wedge$. They will be called compatible topologies.

It is known that there is a minimum for this set, which is the weak topology induced by $G^\wedge$.

However, it is not known if this set has a maximum element; whenever it exists, it will be called the Mackey topology for (G, τ). The problem of finding the largest compatible locally quasi-convex topology for a given topological group G, with this degree of generality (i.e. in the framework of locally quasi-convex groups) was first settled in 1999 in [6]. Previously in 1964 Varopoulos [14] had studied the question for the class of locally precompact group topologies.

The current state of the question is as follows: in general, it is not known if there is a maximum for the set of all locally quasi-convex topologies in G. Partial answers are the following: if G is complete and metrizable, then there exists the Mackey topology [6]. If G is metrizable (but not complete) the original topology may not coincide with the Mackey topology [2].

It is known that there is always at least one locally quasi-convex compatible topology, namely the Bohr topology. This happens to be minimum. If G is a locally compact abelian group, then the original topology coincides with the Mackey topology. Furthermore, the set of all locally quasiconvex compatible group topologies has cardinal greater or equal to 3 [2]. In our future work we are looking forward to finding the general solution of existence (or non-existence) of the Mackey topology, as posed in [6]. The conjecture is that it does not exist, in general, and the solution will pass through describing some topologies on $\mathbb{Z}$, the group of the integers, which may give us the same dual, but they may have no maximum (or the maximum generated by them, may not be compatible). So we start in this monograph by studying thoroughly different topologies on $\mathbb{Z}$.

We now explain the contents of the memory: In the first three chapters we introduce the fundamental notion of a topological group, give a survey of the main known result concerning topological groups, duality and reflexivity. Chapters 4 and 5 contain the **new** results obtained.

We summarize the contens of each chapter.

The basic definitions of topological groups, as well as general results on them and standard notation on duality are given in **chapter 1**.

Chapter 2 deals with the relationship between duality and quasi-convex groups. It is important to observe that any dual group is locally quasi-convex, and, hence any reflexive group is locally quasi-convex.

In section 5, we provide some results on embeddings of locally quasi-convex groups in a product of locally quasi-convex metrizable groups, which were stablished by Lydia Aussenhofer in [1].

In **chapter 3**, we tackle the group of the integers with different group topologies. First, we study the 2-adic topology, denoted ussually by τ_2, as a particular case of p-adic topologies.

Then, we study topologies of uniform convergence on a **fixed** subset S of $\mathbb{T}$. Here $\mathbb{T}$ is considered as the dual group of the integers. Clearly, the mentioned topology (we shall denote it by τ_S) depends on the set S. We have started by taking S to be a sequence. We have in mind to continue with other types of subsets S.

In this chapter, we describe the convergent sequences in τ_2 and τ_S (for particular sets S), and as a consequence of the criterion found we can claim that the topologies τ_S and τ_2 are related.

In **chapter 4**, we try to obtain the dual group of $\mathbb{Z}$ endowed with τ_S (for particular cases).

We remark the result obtained in 4.1.1. It provides us with a development of an integer number as a sum pivoted by a particular -previously fixed- sequence of natural numbers. This tool, which is close to Number Theory, gives a constructive way to obtain the coefficients for this sum, and it is important is the sequel.

We characterize null sequences in τ_S in terms of the mentioned development. We also try to characterize the elements of a fixed neighbourhood at 0 for τ_S through their development as a series pivoted by the elements in S.

In theorem 4.6.3, we prove that for our particular type of set S, the topology τ_S does not coincide with the 2-adic topology.

In **chapter 5**, we consider linear topologies on $\mathbb{Z}$, which are those whose neighbourhood basis at 0 consists of subgroups of $\mathbb{Z}$.

For a fixed sequence (b_n) of natural numbers we assign two different topologies, namely the linear topology whose neighbourhood basis is $\{b_n\mathbb{Z} \mid n \in \mathbb{N}\}$, denoted by $\tau_{(b_n\mathbb{Z})}$, and the "associated S-topology generated by (b_n)" which is the topology of the uniform convergence on $\{\frac{1}{b_n} + \mathbb{Z} \mid n \in \mathbb{N}\}$, where $\mathbb{T}$ is considered to be the quotient $\mathbb{R}/\mathbb{Z}$. The latter will be denoted (as usual) by τ_S.
We characterize the null sequences of τ_S and the elements of a fixed 0-neighbourhood.
We do also characterize the null sequences in $\tau_{(b_n\mathbb{Z})}$.
Finally, we prove that for any (b_n), the linear topology and the associated S-topology generated by (b_n) are different.
These linear group topologies are in certain sense the natural extensions of the p-adic topology. In this line, the dual of $\mathbb{Z}$ endowed with these topologies is a natural extension of the Prüfer's group.

Contents

Chapter 1

Preliminaries

1.1 General definitions and results on topological groups

Definition 1.1.1 Let G be the supporting set of a group and of a topological structure. Suppose that:

(i) the mapping $(x, y) \mapsto xy$ of $G \times G \to G$ is a continuous mapping.

(ii) the mapping $x \mapsto x^{-1}$ of $G \to G$ is continuous.

Then G is called a topological group.

We will deal only with abelian groups.

For a topological group G, the traslation mapping $t_a : G \to G$ defined by $t_a(x) = x \cdot a$ (for $a \in G$) is a homeomorphism (See proposition 1 in [12]).

As a consequence of this fact, the topology of a topological group can be defined just giving a neighbourhood basis at the neutral element.

Lemma 1.1.2 *Let G be a topological group and let $\mathcal{U}$ be a non-empty subset of the power set of G satisfying:*

(i) $e \in U$ for all $U \in \mathcal{U}$.

(ii) For all $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $-V \subseteq U$.

(iii) For all $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $V + V \subseteq U$.

(iv) For every pair $U, V \in \mathcal{U}$, there exists $W \in \mathcal{U}$ such that $W \subseteq U \cap V$.

Then there exists a unique group topology $\mathcal{O}$ on G such that $\mathcal{U}$ is a neighbourhood basis at the unit element. $(G, \mathcal{O})$ is a Hausdorff space if and only if $\bigcap_{U \in \mathcal{U}} U = \{e\}$.

*P*roof: See chapter III, §1.2, proposition 1 in [5]. QED

Lemma 1.1.3 *Let G be a topological group. If a subgroup $H \subset G$ contains a neighbourhood of the neutral element, then H is open.*

Definition 1.1.4 If G is a group, and $S \subset G$, we denote by $\langle S \rangle$ the subgroup generated by S.

Definition 1.1.5 A group topology is called linear if it has a neighbourhood basis at 0 consisting of subgroups.

1.2 Basic definitions on duality

In this section we describe the dual group of a topological group. We omit the proofs which are simple verifications.

Notation 1.2.1 *The unit circle of the complex plane is denoted by $\mathbb{T} := \{z \in \mathbb{C} :| z |= 1\}$ and let $\mathbb{T}_+ := \mathbb{T} \cap \{Re z \geq 0\}$*

Remark 1.2.2 We can also consider $\mathbb{T}$ as the quotient of $\mathbb{R}$ over $\mathbb{Z}$. In this way we shall consider $\mathbb{T} \approx [-\frac{1}{2}, \frac{1}{2})$. We define $\mathbb{T}_+ = [-\frac{1}{4}, \frac{1}{4}]$ and $\mathbb{T}_m = [-\frac{1}{4m}, \frac{1}{4m}]$ for any integer m.

Notation 1.2.3 *Let G be a topological group. The set of all continuous homomorphisms from G to $\mathbb{T}$ will be denoted by $CHom(G, \mathbb{T})$.*

Clearly CHom$(G, \mathbb{T})$ endowed with the natural multiplication: $\varphi_1\varphi_2 : x \mapsto \varphi_1(x)\varphi_2(x)$ is an abelian group.

Definition 1.2.4 *The dual of a topological group G is the group $CHom(G, \tau)$ equipped with the compact open topology. We shall denote it by $G^\wedge$ (or by $(G, \tau)^\wedge$ it we want to stress the topology on G.)*

Definition 1.2.5 Let (G, τ) a topological group. The polar of a subset $S \subset G$ is defined as $S^\triangleright := \{\chi \in G^\wedge \mid \chi(S) \subset \mathbb{T}_+\}$.

Lemma 1.2.6 *Let (G, τ) be a topological group. The compact-open topology in $G^\wedge$ can be described as the family of sets $\mathcal{U}_{G^\wedge}(1) = \{K^\triangleright \mid K \subset G$ is compact $\}$ taken as a neighborhood basis at 0.*

It is straightforward to prove that the compact open topology on $G^\wedge$ is in fact a group topology.

Definition 1.2.7 For a dual group $G^\wedge$, the inverse polar of a subset $S \subset G^\wedge$ is defined as $S^\triangleleft := \{x \in G \mid \chi(x) \in \mathbb{T}_+ \text{ for all } \chi \in S\}$.

Definition 1.2.8 Let (G, τ) be a topological group. We define $G^{\wedge\wedge} = (G^\wedge)^\wedge$ and endow it with the corresponding compact-open topology. That is, the topology is defined by $\mathcal{U}_{G^{\wedge\wedge}}(1) = \{K^\triangleright \mid K \subset G^\wedge \text{ compact}\}$.

Chapter 2

Duality and locally quasi-convex groups

Duality and locally quasi-convexity are two related notions. As we will see, for any topological group (G, τ), its dual is locally quasi-convex.

2.1 General results on duality

We now state the main definitions and elementary results concerning duality of topological groups. Here, instead of considering $\mathbb{T}$ as a subgroup of $\mathbb{C}$ we are going to consider $\mathbb{T}$ as the quotient of $\mathbb{R}$ over $\mathbb{Z}$.

The torsion subgroup of $\mathbb{T}$ is $\mathbb{T}_t = \{\xi_n^k \mid k \in \mathbb{N}_0, n \in \mathbb{N}\} = \{e^{2\pi i \frac{k}{n}} \mid n \in \mathbb{N}, k \in \{0, \dots n-1\}\} = \{e^{2\pi i q} \mid q \in \mathbb{Q}\} \approx \mathbb{Q}/\mathbb{Z}$.

Proposition 2.1.1 *If* $\beta \in \mathbb{R} \setminus \mathbb{Q}$ *then* $L = \langle e^{2\pi i\beta} \rangle$ *is a dense subgroup in* $\mathbb{T}$

*P*roof: Suppose that L is a non dense subgroup of $\mathbb{T}$.

Consider the following mapping $\varphi : \mathbb{Z} \to \mathbb{T}$, where $n \mapsto e^{2\pi i n\beta}$.

First, we show that φ is injective. Suppose there exist $n, m \in \mathbb{N}$ such that $\varphi(n) = \varphi(m)$. This means that $e^{2\pi i n\beta} = e^{2\pi i m\beta}$. This implies that $n\beta - m\beta \in \mathbb{Z}$. Since $\beta \notin \mathbb{Q}$, we get that $n\beta - m\beta = 0$; or equivalently, $n = m$.

Since, φ is injective L is an infinite subset of $\mathbb{T}$, and $p^{-1}(L)$ is not dense.

We consider now $p : \mathbb{R} \to \mathbb{T}$. $p^{-1}(L) = gp\langle \beta, 1 \rangle$. We know that $p^{-1}(L)$ is closed (proposition 19 in [12]).

Choose $a, b \in \mathbb{R}$. We see, now, that $gp\rangle a, b\langle$ is closed if and only if a and b are rationally dependent.

The only closed subgroups in $\mathbb{R}$ are $\mathbb{R}$, the empty set and those isomorphic to $\mathbb{Z}$ (proposition 20 in [12]). Hence $gp\langle a,b\rangle$ is closed implies $gp\langle a,b\rangle \approx c\mathbb{Z}$. Obviously, $a = z_1 c$ and $b = z_2 c$ and they are rationally dependent.
Conversely, suppose that $a = \frac{m}{n} b$, where $\text{mcd}(m,n) = 1$. Hence therer exist p, q such that $mp+nq = 1$. Multiplying by a, we get $amp + anq = a$. Equivalently, $amp + mbq = a = m(ap + bq)$. Multiplying $mp + nq = 1$ by b we get $b = n(ap + bq)$. Hence $gp\langle a,b\rangle = gp\langle ap + bq\rangle \approx \mathbb{Z}$, and as consequence closed.
Since our $p^{-1}(L) = gp\langle\beta, 1\rangle$ is closed 1 and β are rationally dependent. Then $\beta \in \mathbb{Q}$, which contradicts the hypothesis.

QED

Another easy result is:

Proposition 2.1.2 *If G has order m, then $G^\wedge$ has order m.*

*P*roof: $\chi^m(x) = \chi(x) \overset{(m)}{\cdots} \chi(x) = \chi(mx) = \chi(0) = 1$ for all $\chi \in G^\wedge$. QED

In $G^\wedge$ we consider the compact-open topology, τ_{CO}. We define the compact-open topology in the following way: $V_{K,U} = \{\varphi \in \text{CHom}(G,\mathbb{T}) \mid \varphi(K) \subset U\}$. We take as subbasis for the compact-open topology the family $V_{K,U}$ where $K \subset G$ is compact and U is an open subset of $\mathbb{T}$.
It can be easily seen that the family $\{K^\triangleright \mid K \subset G \text{ is compact}\}$ is a neighbourhood basis at 1 for τ_{CO}.
In particular for $G = \mathbb{R}_u$, every compact $K \subset \mathbb{R}$ is contained in $[-n,n]$ for a suitable natural number, in addition, $[-n,n]^\triangleright \subset K^\triangleright$. Therefore a neighbourhood basis at the null character $1 \in \mathbb{R}^\wedge$ is given by the sets: $[-n,n]^\triangleright = \{\chi_t \mid \chi_t(x) \in \mathbb{T}_+ \;\forall \mid x \mid\leq n\} = \{\chi_t \mid e^{2\pi itx} \in \mathbb{T}_+ \;\forall \mid x \mid\leq n\} = [-\frac{1}{4n}, \frac{1}{4n}]$.
Thus, $\mathbb{R}^\wedge$ can be algebraical and topologically identified with $(\mathbb{R}, \tau_u)$
We also know that $\mathbb{T}^\wedge = \mathbb{Z}$ and $\mathbb{Z}^\wedge = \mathbb{T}$
Clearly: a topological group is discrete if and only if $\{0\}$ is open.
The first statement about duality is:

Proposition 2.1.3 *Let K be a compact abelian topological group. Then $K^\wedge$ is discrete.*

*P*roof: $K^\triangleright = \{\chi \in K^\wedge \mid \chi(K) \subseteq \mathbb{T}_+\}$. Since $\chi(K)$ is a subgroup of $\mathbb{T}$, $\chi(K) \subset \mathbb{T}_+ \iff \chi(K) = \{1\}$. Hence $K^\triangleright = \{\chi \in K^\wedge \mid \chi(K) = \{1\}\}$. Thus, $K^\triangleright$ consists only of the null character, 1. Hence $\{1\}$ is open, and $K^\wedge$ is discrete. QED

Proposition 2.1.4 *Let (G, τ_{dis}) be a discrete topological group. Then Hom$(G, \mathbb{T})$ is a compact subgroup of $\mathbb{T}^G$.*

*P*roof: By Tychonoff theorem, we know that $\mathbb{T}^G$ is compact. Therefore, it suffices to prove that Hom(G, τ) is a closed subgroup of $\mathbb{T}^G$.
In order to prove that Hom$(G, \mathbb{T})$ is closed in $\mathbb{T}^G$, we take $(\chi_j)_{j\in J}$ a net in Hom$(G, \mathbb{T})$ converging to $\chi : G \to \mathbb{T}$. We must see that χ is a homomorphism.
Let $x, y \in G$. It suffices to show that $\chi(x + y) = \chi(x)\chi(y)$.
$\chi(x + y) = \lim_j \chi_j(x + y) = \lim_j(\chi_j(x)\chi_j(y)) = \lim_j \chi_j(x) \lim_j \chi_j(y) = \chi(x)\chi(y)$
Hence Hom$(G, \mathbb{T})$ is closed.

QED

Corollary 2.1.5 *Let (G, τ) be a discrete group. Then $(G, \tau)^\wedge$ is compact.*

*P*roof: Since Hom$(G, \tau) = (G, \tau)^\wedge$ the result follows.

QED

Definition 2.1.6 The natural embedding $\alpha_G : G \to G^{\wedge\wedge}$, is defined by $x \mapsto \alpha_G(x) : G^\wedge \to \mathbb{T}$, where $\chi \mapsto \chi(x)$

Proposition 2.1.7 *α_G is a homomorphism.*

*P*roof: $\alpha_G(x + y)(\chi) = \chi(x + y) = \chi(x)\chi(y) = \alpha_G(x)(\chi)\alpha_G(y)(\chi) = (\alpha_G(x)\alpha_G(y))(\chi)$, for all $\chi \in G^\wedge$.
Thus, $\alpha_G(x + y) = \alpha_G(x)\alpha_G(y)$

QED

Definition 2.1.8 A topological group (G, τ) is reflexive if α_G is a topological isomorphism

Proposition 2.1.9 *The canonical mapping α_G is injective if and only if $G^\wedge$ separates points; that is, for every $x \neq 0$, there exists $\chi \in G^\wedge$ such that $\chi(x) \neq 1$.*

The following highly non-trivial assertion was proved by Weyl and is the corner-stone for the duality Theorem of Pontryagin-Van Kampen.

Theorem 2.1.10 *Let G be a compact abelian topological group, then α_G is injective.*

An important tool in our subsequent work is the Pontryagin-Van Kampen theorem, which states the following.

Theorem 2.1.11 (Pontryagin-Van Kampen) *Let (G,τ) be a locally compact abelian topological group. Then G is reflexive.*

2.2 On the continuity of α_G and equicontinuity

The general definition of equicontinuity in the framework of uniform spaces and continuous mappings can be restated in a simpler form in the particular case of topological groups and continuous homomorphisms as we do next:

Definition 2.2.1 Let $S \subset G^\wedge$, S is equicontinuous if for every $W \in \mathcal{U}_{\mathbb{T}}(0)$ there exists $V \in \mathcal{U}_G(0)$ such that $\varphi(V) \subset W$ for every $\varphi \in S$.

α_G is continuous if and only if for every compact $K \subset G^\wedge$, $\alpha_G^{-1}(K^\triangleright)$ is a neighbourhood of 0 in G.
$\alpha_G^{-1}(K^\triangleright) = \{x \in G \mid \alpha_G(x) \in K^\triangleright\} = \{x \in G \mid \alpha_G(x)(\chi) \in \mathbb{T}_+,\ \forall\chi \in K\} = \{x \in G \mid \chi(x) \in \mathbb{T}_+,\ \forall\chi \in K\}$.
We have that α_G is continuous if and only if for every $K \subset G^\wedge$ compact, there exists $U \in \mathcal{U}_G(0)$ such that $U \subset \{x \mid \chi(x) \in \mathbb{T}_+,\ \forall\chi \in K\}$
That is to say, every compact of $G^\wedge$ is equicontinuous.
A further simplification is given in the following proposition.

Proposition 2.2.2 *$S \subset G^\wedge$ is equicontinuous if there exists $U \in \mathcal{U}_G(0)$ such that for every $x \in U$ and for every $\chi \in S$, $\chi(x) \in \mathbb{T}_+$, that is $S \subset U^\triangleright$.*

*P*roof: Let $[-\frac{1}{4n}, \frac{1}{4n}] + \mathbb{Z}$ be a neighbourhood of $0 + \mathbb{Z}$ in $\mathbb{T}$.
By 1.1.2 (iii), there exists V where $V + \overset{(n)}{\cdots} + V \subset U$. Hence, $\chi(V + \overset{(n)}{\cdots} + V) \subset \chi(U) \subset \mathbb{T}_+$.
Choose $x \in V$; $\chi(x), \chi(2x), \ldots, \chi(nx) \in \mathbb{T}_+ \Rightarrow \chi(x) \in [-\frac{1}{4n}, \frac{1}{4n}] + \mathbb{Z} \Rightarrow \chi(V) \subset [-\frac{1}{4n}, \frac{1}{4n}] + \mathbb{Z}$.
$U^\triangleright \supset S \iff \forall n,\ \exists V$ such that $\forall\chi \in S,\ \forall x \in V\ \chi(x) \in [-\frac{1}{4n}, \frac{1}{4n}] + \mathbb{Z}$

QED

Proposition 2.2.3 *If G is metrizable, then α_G is continuous.*

2.3 Introduction to locally quasi-convex groups

The notion of convexity is one of the most fruitful tools in Mathematics. A convex set is defined only in the context of vector spaces in a pure algebraic mode.
For a topological group G the notion of convexity is not available, since there is no scalar multiplication. However an analogous definition could be obtained modeled in the separation theorem, which is obtained as a corollary to Hahn-Banach theorem, which states the following: let E be a locally convex topological vector space. A subset $M \subset E$ is convex if for every $x \in E \setminus M$, there exists a continuous linear mapping φ such that $\varphi(x) > \varphi(y)$ for every $y \in M$.
For a topological group (G, τ) the analogous definition would rely on continuous homomorphisms, instead of linear forms.
However, the dualizing object cannot be $\mathbb{R}$ as we make evident in the following paragraph:
Let G be a compact group and let $\varphi : G \to \mathbb{R}$ be a continuous group homomorphism. Since G is compact and φ is a continuous homomorphism, $\varphi(G)$ is a compact subgroup of $\mathbb{R}$. Hence $\varphi(G) = \{0\}$.
We must choose another group instead of $\mathbb{R}$. In 25.36 of [9] it is proved that for results on duality, this group must be $\mathbb{T}$. Hence we pick characters (that is continuous group homomorphisms into $\mathbb{T}$) $\chi : G \to \mathbb{T}$, instead of continuous linear forms.
Now, there exists no order in $\mathbb{T}$, so it has no sense to state $\chi(x) < \chi(y)$. The "modified definition" given by Vilenkin in the 50's, see [15], goes as follows:

Definition 2.3.1 Let (G, τ) be a topological group. A subset S is quasi-convex if for every $x \in G \setminus S$ there exists $\chi \in G^\wedge$ such that $\chi(S) \subset \mathbb{T}_+$ and $\chi(x) \notin \mathbb{T}_+$, where $G^\wedge = \{\chi \mid \chi : G \to \mathbb{T} \text{ continuous}\}$ and $\mathbb{T}_+ = \mathbb{T} \cap \{Re(z) \geq 0\}$.

Definition 2.3.2 A topological group G is said to be locally quasi-convex if the neutral element e_G has a neighbourhood basis formed by quasi-convex sets.

2.4 Properties of quasi-convex sets

In this section we prove some properties of quasi-convex sets, and also we prove that the dual of any topological group is locally quasi-convex. This fact will be a easy prove of the fact that any reflexive toplogical group is locally quasi-convex.

Properties 2.4.1 *Let (G, τ) be a topological group and $A \subset G$ be a quasi-convex set, then:*
(i) A is symmetric.

(ii) $0 \in A$.
(iii) $A = \bigcap_{\chi \in A^{\triangleright}} \chi^{-1}(\mathbb{T}_+) = A^{\triangleright\triangleleft}$
(iv) A is closed.

Proposition 2.4.2 *Let (G, τ) be a topological group and let $M \subset G$. Then $M^{\triangleright}$ is quasi-convex*

*P*roof: Take $\chi_0 \in G^\wedge \setminus M^{\triangleright}$; that is, $\exists x \in M$ such that $\chi_0(x) \notin \mathbb{T}_+$. $\alpha_G(x) \in G^{\wedge\wedge}$, and for all $\chi \in M^{\triangleright}$ we have $\alpha_G(x)(\chi) = \chi(x) \in \mathbb{T}_+$ but $\alpha_G(x)(\chi_0) = \chi_0(x) \notin \mathbb{T}_+$

QED

Proposition 2.4.3 *Let (G, τ) be a topological group; then $(G^\wedge, \tau_{CO})$ is locally quasi-convex.*

*P*roof: We remember that the family $\mathcal{K} = \{K^{\triangleright} \mid K \text{ is compact}\}$ is a neighbourhood basis for τ_{CO}.
We have just proved that the polar of any subset is quasi-convex, in particular $\mathcal{K}$ is a neighbourhood basis formed by quasi-convex sets.
Hence, $(G^\wedge, \tau_{CO})$ is locally quasi-convex.

QED

Example 2.4.4 $\mathbb{R}$ is locally quasi-convex, since $[-\frac{1}{n}, \frac{1}{n}]$ is quasi-convex for all n.
$\mathbb{T}$ is locally quasi-convex, since $\{[-\frac{1}{4n}, \frac{1}{4n}] + \mathbb{Z} \mid n \in \mathbb{N}\}$ is a quasi-convex neighbourhood basis for 0 in $\mathbb{T}$. In particular, $\mathbb{T}_+$ is quasi-convex.

Proposition 2.4.5 *Let $\varphi : (G, \tau) \to (H, \sigma)$ be a continuous homomorphism and $A \subset H$ quasi-convex; then $\varphi^{-1}(A) \subset G$ is quasi-convex.*

*P*roof: Let $x \notin \varphi^{-1}(A)$, then $\varphi(x) \notin A$.
Since A is quasi-convex, there exists $\chi \in H^\wedge$ such that $\chi\varphi(x) \notin \mathbb{T}_+$ and $\chi(A) \subset \mathbb{T}_+$. Consider $\Psi = \chi \circ \varphi : G \to \mathbb{T}$.
$\Psi(\varphi^{-1}(A)) = \chi \circ \varphi \circ \varphi^{-1}(A) \subset \chi(A) \subset \mathbb{T}_+$.
$\Psi(x) = \chi(\varphi(x)) \notin \mathbb{T}_+$.

QED

Proposition 2.4.6 *Let $(A_i)_{i \in I} \subset G$ be a family of quasi-convex sets. Then $\cap_{i \in I} A_i$ is quasi-convex.*

*P*roof: $x \notin \bigcap_{i\in I} A_i \Rightarrow \exists i_0$ such that $x \notin A_{i_0}$.
Since A_{i_0} is quasi-convex, there exists $\varphi \in G^\wedge$ such that $\varphi(x) \notin \mathbb{T}_+$ and $\varphi(A_{i_0}) \subset \mathbb{T}_+$.
Since $\bigcap_{i\in I} A_i \subset A_{i_0}$, $\varphi(\bigcap_{i\in I} A_i) \subset \varphi(A_{i_0}) \subset \mathbb{T}_+$.

QED

Proposition 2.4.7 *$S \subset G^\wedge$. Then $S^{\triangleleft} := \bigcap_{\chi\in S} \chi^{-1}(\mathbb{T}_+)$ is quasi-convex.*

*P*roof: $\mathbb{T}_+$ is quasi-convex $\Rightarrow \chi^{-1}(\mathbb{T}_+)$ is quasi-convex $\Rightarrow \bigcap_{\chi\in S} \chi^{-1}(\mathbb{T}_+)$ is quasi-convex. QED

2.5 Embedding of a locally quasi-convex group into a product of metrizable locally quasi-convex groups

Our next aim is to embed a Hausdorff locally quasi-convex topological group into a product of groups with "good" properties.
A deeper result was obtained by Lydia Aussenhofer in [1], but on the other hand it is much more difficult to prove because it also deals with dually closed images. We shall omit the result that the resulting group are complete, getting a weaker result.
Furthermore, this proof tries to be self-contained using only results proved in this section

Remark 2.5.1 In this section we shall consider (G, τ) a topological group as usual and U to be a quasi-convex neighbourhood at 0 for τ. We must note that some of the results of this section are also true for a generic subset.

Definition 2.5.2 *Let $X \subset G$. Define $\left(\frac{1}{n}\right)X = \{x \in G|\ x, 2x, \ldots, nx \in X\}$ and $X_\infty = \bigcap_{n\geq 1}\left(\frac{1}{n}\right)X$. In particular, $\left(\frac{1}{n}\right)\mathbb{T}_+ = \left[-\frac{1}{4n}, \frac{1}{4n}\right] + \mathbb{Z} = \mathbb{T}_n$ and $\mathbb{T}_\infty = \{0\} + \mathbb{Z}$.*

Lemma 2.5.3 *Let (G, τ) be a topological group and $U \subset G$ be a quasi-convex neighbourhood, then $\left(\frac{1}{n}\right)U = \bigcap_{\chi\in U^{\triangleright}} \chi^{-1}(\mathbb{T}_n)$*

*P*roof: $x \in \left(\frac{1}{n}\right)U \iff kx \in U, 1 \leq k \leq n \iff \forall\chi \in U^{\triangleright}, 1 \leq k \leq n, \chi(kx) \in \mathbb{T}_+ \iff \forall\chi \in U^{\triangleright}, 1 \leq k \leq n, k\chi(x) \in \mathbb{T}_+ \iff \forall\chi \in U^{\triangleright}, \chi(x) \in \mathbb{T}_n \iff x \in \bigcap_{\chi\in U^{\triangleright}} \chi^{-1}(\mathbb{T}_n)$

QED

Lemma 2.5.4 *Let (G, τ) be a topological group, and $U \subset G$ a quasi-convex neighbourhood. Then $\left(\left(\frac{1}{n}\right)U\right)_{n\geq 1}$ is a neighbourhood basis at 0 for a group topology.*

*P*roof: We check conditions set in 1.1.2 . Conditions (i) and (iv) are trivial from the choice we have done. (ii) is obtained as a direct consequence of the fact that U_n is symmetric. In order to prove (iii), we will see that $\left(\frac{1}{2n}\right)U + \left(\frac{1}{2n}\right)U \subset \left(\frac{1}{n}\right)U$. Let $x \in \left(\frac{1}{2n}\right)U$ for all $\chi \in U^{\triangleright}$. By the previous characterization, we have $\chi(x) \in \mathbb{T}_{2n}, \ \forall \chi \in U^{\triangleright}$. Let $x, y \in \left(\frac{1}{2n}\right)U$, therefore $\chi(x+y) = \chi(x) + \chi(y) \in \mathbb{T}_{2n} + \mathbb{T}_{2n} = \mathbb{T}_n, \ \forall \chi \in U^{\triangleright}$. Equivalently, $x + y \in \left(\frac{1}{n}\right)U$. QED

Then, we have that $\{(\frac{1}{n})U \mid n \in \mathbb{N}\}$ is a neighbourhood basis at e for a group topology which will be denoted by $\mathfrak{T}_U$.

Lemma 2.5.5 *The set U_∞ is a subgroup of G for any U*

*P*roof: $U_\infty = \bigcap_n \left(\frac{1}{n}\right)U = \bigcap_n \bigcap_{\chi \in U^{\triangleright}} \chi^{-1}(\mathbb{T}_n) = \bigcap_{\chi \in U^{\triangleright}} \chi^{-1}(\cap_n \mathbb{T}_n) = \bigcap_{\chi \in U^{\triangleright}} \chi^{-1}(\{0 + \mathbb{Z}\}) = \bigcap_{\chi \in U^{\triangleright}} \ker(\chi) < G$.

QED

Lemma 2.5.6 *Let (G, τ) be a topological group. Then $\overline{\{0\}}^{\tau} = \bigcap_{V \in \mathcal{U}_\tau(0)} V$.*

*P*roof: $G \setminus \overline{\{0\}} = \{x \in G | \ \exists W \in \mathcal{U}(x) : \ 0 \notin W\}$. We can take $W = xV$ where $V \in \mathcal{U}(0)$ is symmetric. Now, $0 \notin xV \iff x^{-1} \notin V \iff x \notin V$. Then, $\{x \in G | \ \exists W \in \mathcal{U}(x), \ 0 \notin W\} = \{x | \ \exists V \in \mathcal{U}(0), \ x \notin V\} = G \setminus (\bigcap_{V \in \mathcal{U}_\tau(0)} V)$.

QED

Lemma 2.5.7 *Let (G, τ) be a topological group. Then $\overline{\{0\}}^{\mathfrak{T}_U} = U_\infty$*

*P*roof: $\overline{\{0\}}^{\mathfrak{T}_U} = \bigcap_n \left(\left(\frac{1}{n}\right)U\right) = U_\infty$. It is enough to pick a neighbourhood basis, because any other neighbourhood contains a basic one. QED

Corollary 2.5.8 *Given a topological group (G, τ), U_∞ is a closed subgroup of G.*

By the characterization of $\left(\frac{1}{n}\right)U$ of 2.5.3, we can define $\mathfrak{T}_U$ in another way. We choose as neighbourhoods: $V_{U^\triangleright,n} = \{x \in G \mid \forall \chi \in U^\triangleright, \chi(x) \in \mathbb{T}_n\} = \bigcap_{\chi \in U^\triangleright} \chi^{-1}(\mathbb{T}_n) = \left(\frac{1}{n}\right)U.$

Then, we have two possibilities to reach the same topology. Given U a quasi-convex neighbourhood of 0, we can take either the topology generated by $\left(\frac{1}{n}\right)U$, or the $U^\triangleright$-topology (or topology of uniform convergence on $U^\triangleright$).

We now consider the mapping

$$\text{id} : (G, \tau) \to (G, \mathfrak{T}_U)$$

Since $\{(1/n)U \mid n \in \mathbb{N}\}$ is a neighbourhood basis at 0 in $(G, \mathfrak{T}_U)$ and $(1/n)U$ is a neighbourhood of 0 in τ, the mapping is continuous.

We can now define

$$\varphi_U : (G, \mathfrak{T}_U) \to (G/U_\infty, \mathfrak{T}_U/U_\infty)$$

as the canonical projection. Since $U_\infty < G$, the quotient is well-defined. Considering now the composition of both mappings, we get:

$$\varphi_U \circ \text{ id} : (G, \tau) \to (G/U_\infty, \mathfrak{T}_U/U_\infty)$$

We define $(H_U, \sigma_U) := (G/U_\infty, \mathfrak{T}_U/U_\infty)$.

We consider $\chi \in U^\triangleright$, we know $\chi(U_\infty) \subset \chi(U) \subset \mathbb{T}_+$. Since $\chi(U_\infty)$ is a subgroup which is contained in $\mathbb{T}_+$, it must be $\chi(U_\infty) = \{0\}$. Consequently, χ factorizes in a unique way described in the diagram:

$$\begin{array}{ccc} G & \xrightarrow{\chi} & \mathbb{T} \\ \varphi_U \searrow & & \nearrow \overline{\chi} \\ & G/U_\infty & \end{array}$$

Since $\chi\left(\left(\frac{1}{n}\right)U + U_\infty\right) = \chi\left(\left(\frac{1}{n}\right)U\right) \subset \mathbb{T}_n$ holds, we get the continuity of $\bar{\chi}$. Furthermore, we can consider $U^\triangleright$ as a subset in $(H_U, \sigma_U)^\triangleright$.

We choose $V_{U^\triangleright,n} = \{x + U_\infty \mid \forall \chi \in U^\triangleright\ \bar{\chi}(x + U_\infty) = \chi(x) \in \mathbb{T}_n\} = \left(\frac{1}{n}\right)U + U_\infty$ as neighbourhood basis in $\mathfrak{T}_U/U_\infty = \sigma_U$.

We want that $\sigma_U = \mathfrak{T}_S$, for some adequate S. $\varphi_U(U)$ is a neighbourhood at 0 in (H, σ_U) because U is a neighbourhood at 0 in $(G, \mathfrak{T}_U)$ and the quotient mapping is always open.

The proof of the main result of this section is supported by the following two lemmas:

Lemma 2.5.9 $U + ker(\varphi_U) = U$

*P*roof: Clearly $U + \ker(\varphi_U) \supset U$. We now prove the converse. Choose $x \in U, y \in U_\infty$; $\chi(x + y) = \chi(x) + \chi(y) = \chi(x) \in \mathbb{T}_+$ for every $\chi \in U^\triangleright$.

QED

Lemma 2.5.10 *Let (G, τ) be a Hausdorff locally quasi-convex topological group and let $U \in \mathcal{U}_\tau(0)$ be a quasi-convex set. Then, there exists $\varphi_U : (G, \tau) \to (H_U, \sigma_U)$ continuous, $\varphi_U(U)$ is an open neighbourhood in (H_U, σ_U); furthermore $\sigma_U = \mathfrak{T}_S$ for $S = U^\triangleright$ and (H_U, σ_U) is locally quasi-convex.*

*P*roof: It only remains to see that (H_U, σ_U) defined above is locally quasi-convex.
Let φ_U be the canonical projection; we will see that $\{\varphi_U$
$frac1nU \mid n \in \mathbb{N}\}$ is a neighbourhood basis for the topology we are interested in. For that it is enough to see that: $\varphi_U((\frac{1}{n})U) = V_{U^\triangleright,n}$. $V_{U^\triangleright,n} = \{x + U_\infty \mid \forall \chi \in U^\triangleright\ \overline{\chi}(x + U_\infty) \in \mathbb{T}_n\} = \{x + U_\infty \mid x \in (\frac{1}{n})U\} = \varphi_U((\frac{1}{n})U)$

QED

We can now state and prove the main result:

Theorem 2.5.11 *Let (G, τ) be a Hausdorff locally quasi-convex group. Then*

$$\Psi : G \to \prod_{U \in \mathcal{U}(0)\ q.c} (H_U, \sigma_U)$$

where $x \mapsto (\varphi_U(x))_{U \in \mathcal{U}(0)}$ is a topological embedding. Furthermore (H_U, σ_U) are metrizable groups.

*P*roof: By the previous constructions (H_U, σ_U) is Hausdorff, since it is a quotient by a closed subgroup. It is metrizable, since $\{(\frac{1}{n})U \mid n \in \mathbb{N}\}$ is a countable neighbourhood basis at e.
We will see that Ψ is an embedding:
First we prove it is one-to-one. $x \in ker(\Psi)$; that is: $\varphi_U(x) = 0\ \forall U \Rightarrow x \in \bigcap_U ker(\varphi_U) \subset \bigcap_{U \in \mathcal{U}\ q.c} U = \overline{\{0\}}^\tau$. Since τ is Hausdorff, this implies that $x = 0$. Hence Ψ is one-to-one.
The continuity of each φ_U is determined in 2.5.10. Since Ψ is continuous in each factor (of a product), in particular, it is continuous.
Finally, we must see that Ψ is open respect to its image. It suffices to prove that for each quasi-convex neighbourhood U, $\Psi(U)$ is a neighbourhood in $\Psi(G)$.
$\Psi(G) \cap (\varphi_U(U) \times \prod_{V \neq U} H_V) = \{\Psi(x) \mid \varphi_U(x) \in \varphi_U(U)\} = \{\Psi(x) \mid x \in U + \ker(\varphi_U)\}$. By 2.5.9, the last term equals $\{\Psi(x) \mid x \in U\} = \Psi(U)$.

QED

Chapter 3

Different topologies on the group of the integers

3.1 The 2-adic topology

The most natural non-discrete topologies on $\mathbb{Z}$ are the p-adic topologies. We first study the 2-adic topology in $\mathbb{Z}$, which is usually denoted by τ_2.

The 2-adic topology is defined by the following neighbourhood basis at 0: $\mathcal{U} = \{2^n\mathbb{Z} \mid n \in \mathbb{N}\}$. We will denote $2^n\mathbb{Z}$ by U_n. It is easy to check that is a neighbourhood basis at 0 for a group topology.

Convergent sequences in $(\mathbb{Z}, \tau_2)$

We try to find a characterization of sequences that converge to 0 in τ_2. Choose $l_j \to 0$ in τ_2; by definition of convergence, for each neighbourhood $U \in \mathcal{U}(0)$ there exists $N_0 \in \mathbb{N}$ such that $l_j \in U$ for all $j \geq N_0$. Now, $l_j \in U_{n_0}$ if $l_j = 2^{n_0}k$ where $k \in \mathbb{Z}$. Hence, for each $n \in \mathbb{N}$ there must exist j_n such that for every $j \geq j_n$, $l_j \in 2^n\mathbb{Z}$, or, equivalently $2^n \mid l_j$. Then, we have found that:

$$l_j \to 0 \text{ in } \tau_2 \iff \forall n \in \mathbb{N}\ \exists j_n \text{ such that for all } j \geq j_n,\ 2^n \mid l_j$$

The dual of $(\mathbb{Z}, \tau_2)$.

Let $\chi \in (\mathbb{Z}, \tau_2)^\wedge$.Since χ is continuous, there exists a neighbourhood $U = 2^n\mathbb{Z}$ such that $\chi(U) = \chi(2^n\mathbb{Z}) \subset \mathbb{T}_+$. On the other hand $2^n\mathbb{Z}$ is a subgroup of $\mathbb{Z}$, hence, its image by a homomorphism will be again a subgroup in $\mathbb{T}$. The only subgroup contained in $\mathbb{T}_+$ is $\{0\}$, therefore, $\chi(2^n\mathbb{Z}) = \{0 + \mathbb{Z}\}$. Let $x + \mathbb{Z} = \chi(1)$. Hence, $\chi(2^n) = 2^n x + \mathbb{Z} = 0 + \mathbb{Z}$. This implies that $2^n x \in \mathbb{Z}$.

We remember that $x \in \mathbb{R}$, and $x = \frac{k}{2^n} + \mathbb{Z} \in \mathbb{T}$. This x represents the character χ we have chosen at the beginnning. Hence:

$$(\mathbb{Z}, \tau_2)^\wedge \subset \{\tfrac{k}{2^n} + \mathbb{Z} \mid\ k \in \mathbb{Z}, n \in \mathbb{Z}\}.$$

Conversely, $(\mathbb{Z}, \tau_2)^\wedge \supset \{\frac{k}{2^n} + \mathbb{Z} \mid\ k \in \mathbb{Z}, n \in \mathbb{Z}\}$:

Let $k \in \mathbb{Z}, n \in \mathbb{N}$; we want to prove that $\frac{k}{2^n} + \mathbb{Z} \in (\mathbb{Z}, \tau_2)^\wedge$.

$\ker(\frac{k}{2^n} + \mathbb{Z}) = \{j \in \mathbb{Z} \mid j\frac{k}{2^n} + \mathbb{Z} = 0 + \mathbb{Z}\} \supset 2^n\mathbb{Z}$. Since any subgroup containing a neighbourhood is open, in particular, $\ker(\frac{k}{2^n} + \mathbb{Z})$ is open.

Furthermore, since $(\frac{k}{2^n} + \mathbb{Z})^{-1}(\mathbb{T}_n) \supset \ker(\frac{k}{2^n} + \mathbb{Z})$ our homomorphism is continuous. Hence we have the equality between both groups.

This group is called Prüfer's group and it is denoted by $\mathbb{Z}(2^\infty)$.

3.2 Definition of S-topologies

Let G be a topological group. Our aim is to define topologies on G. We choose $S \subset \mathrm{Hom}(G, \mathbb{T})$ and set $\mathcal{U} = \{\bigcap_{\chi \in S} \chi^{-1}(\mathbb{T}_n) \mid n \in \mathbb{N}\}$. It would be nice that these neighbourhoods were a neighbourhood basis at 0. We should verify conditions in 1.1.2; we will denote $U_n = \bigcap_{\chi \in S} \chi^{-1}(\mathbb{T}_n)$:

(i) $0 \in U_n, \forall n \in \mathbb{N}$.

Trivial, since $\chi(0) = 0 + \mathbb{Z}, \forall \chi \in \mathrm{Hom}(G, \mathbb{T})$.

(ii) $\forall U \in \mathcal{U}, \exists V \in \mathcal{U}$ such that $-V \subset U$.

We will see that U_n is symmetric for every n. $\mathbb{T}_n$ is symmetric for every n. We also know that the inverse image by a continuous function of a symmetric subset is again a symmetric subset; hence $\chi^{-1}(\mathbb{T}_n)$ is symmetric. Since intersection of symmetric sets is again symmetric, we can assure that U_n is symmetric.Thus, if $U \supset U_n$, then $V = U_n$ suits our purposes.

(iii) $\forall U \in \mathcal{U}, \exists V \in \mathcal{U}$ such that $V + V \subset U$.

Fix $U = U_n$. We will see that we can choose $V = U_{2n}$. Let $x, y \in U_{2n} = \{z \mid \chi(z) \in \mathbb{T}_{2n}, \forall \chi \in S\}$ and let $\chi \in S$. We compute $\chi(x + y) = \chi(x) + \chi(y) \in \mathbb{T}_{2n} + \mathbb{T}_{2n} = \mathbb{T}_n$. Then $x + y \in U_n$, as desired.

(iv) $\forall U, V \in \mathcal{U}, \exists W \in \mathcal{U}$ such that $W \subset U \cap V$.

Let $U = U_n$ and $V = U_m$ where $n, m \in \mathbb{N}$. Let $k = \max(n, m)$ and $W = U_k$. By the definition of the neighbourhoods we have $W = U \cap V$.

By 1.1.2, there exists a unique group topology τ_S for which $\mathcal{U}$ is a neighbourhood basis at 0.

Now, we will see that $S \subset U_1^{\triangleright} = (\bigcap_{\chi \in S} \chi^{-1}(\mathbb{T}_+))^{\triangleright}$. Choose $\varphi \in S$ and $x \in \bigcap_{\chi \in S} \chi^{-1}(\mathbb{T}_+) \subset \varphi^{-1}(\mathbb{T}_+)$. Hence, $\varphi(x) \in \mathbb{T}_+$.

3.3 Introduction to S-topologies in $\mathbb{Z}$

Since we are interested in topologies on $\mathbb{Z}$, we should choose $S \subset \mathbb{T} = \mathrm{Hom}(\mathbb{Z}, \mathbb{T})$. Here $U_n = \bigcap_{\chi \in S} \chi^{-1}(\mathbb{T}_n) = \{k \in \mathbb{Z}$ such that $\forall z + \mathbb{Z} \in S, z \cdot k + \mathbb{Z} \in \mathbb{T}_n\}$. Related to these questions we can find this theorem in [2]:

Theorem 3.3.1 *Let $\{x_n\}$ be a strictly decreasing sequence in $(0, \frac{1}{2}]$, where $x_n \to 0$, such that $(\frac{x_n}{x_{n+1}})_{n\in\mathbb{N}}$ is bounded. Let $S = \{x_n + \mathbb{Z} \mid n \in \mathbb{N}\}$. Then τ_S is discrete.*

*P*roof: Since $(\frac{x_n}{x_{n+1}})$ is bounded, there exists $m \in \mathbb{N}$, with $m > 1$ such that $\frac{x_n}{x_{n+1}} \leq m, \ \forall n \in \mathbb{N}$. Choose $k \in \mathbb{Z}$ where $k \in \bigcap_{\chi\in S} \chi^{-1}(\mathbb{T}_m)$. Since $\mathbb{T}_m$ is symmetric we can pick $k \geq 0$.
First, we prove $k \leq \frac{1}{4x_1}$. By contradiction, suppose that $k > \frac{1}{4x_1}$. Since $(x_n)_{n\in\mathbb{N}}$ is strictly decreasing, there exists a unique $n \in \mathbb{N}$ such that $\frac{1}{4x_n} < k \leq \frac{1}{4x_{n+1}}$. Multiplying by x_{n+1} we obtain that $\frac{1}{4}\frac{x_{n+1}}{x_n} < kx_{n+1} \leq \frac{1}{4}$. Since m is a bound for $\frac{x_n}{x_{n+1}}$ we get $\frac{1}{4m} < kx_{n+1} \leq \frac{1}{4}$. Hence $kx_{n+1} + \mathbb{Z} \notin \mathbb{T}_m$. Choosing $\chi : k \mapsto k \cdot x_{n+1} + \mathbb{Z}$ we get that $\chi(k) \notin \mathbb{T}_m$. Which contradicts $k \in \bigcap_{\chi\in S} \chi^{-1}(\mathbb{T}_m)$.
Hence, $\bigcap_{\chi\in S} \chi^{-1}(\mathbb{T}_m) \subset \mathbb{Z} \cap [-\dfrac{1}{4x_1}, \dfrac{1}{4x_1}]$, which is a finite subset. Now, fix $l \in \mathbb{N}$ such that $4lx_1 > 1$. Considering $j \in \bigcap_{\chi\in S} \chi^{-1}(\mathbb{T}_{lm})$ we get $j, 2j, \ldots, lj \in \bigcap_{\chi\in S} \chi^{-1}(\mathbb{T}_m) \subset [-\dfrac{1}{4}, \dfrac{1}{4}] \cap \mathbb{Z}$. Equivalently: $j \in [-\frac{1}{4lx_1}, \frac{1}{4lx_1}] \cap \mathbb{Z}$, by the choice of l this means that $j = 0$. $\bigcap_{\chi\in S} \chi^{-1}(\mathbb{T}_{lm}) = \{0\}$. Hence τ_S is discrete.

QED

However, this proposition does not give any information if $(\frac{x_n}{x_{n+1}})$ is an unbounded sequence. In the following pages, we will study some S-topologies where $S = \{x_n \mid n \in \mathbb{N}\}$ with $(\frac{x_n}{x_{n+1}})$ unbounded.

3.4 The S-topology corresponding to $S = \{2^{-n^2} + \mathbb{Z} \mid n \in \mathbb{N}\} \subset \mathbb{T}$

We remember that in order to get a S-topology in $\mathbb{Z}$, we should choose $S \subset \mathrm{Hom}(\mathbb{Z}, \mathbb{T}) \approx \mathbb{T}$. We will consider, in particular sequences $(a_n)_{n\in\mathbb{N}} \subset \mathbb{N}$ strictly increasing and we choose $S = \{2^{-a_n} + \mathbb{Z} \mid n \in \mathbb{N}\}$.
We have already seen that if $| \frac{2^{-a_n}}{2^{-a_{n+1}}} | = | \frac{2^{a_{n+1}}}{2^{a_n}} | = 2^{a_{n+1}-a_n}$ is bounded, then τ_S is discrete.
Then, τ_S has then as neighbourhood basis at 0 : $V_{S,n} = \{k \in \mathbb{Z} \mid \forall z + \mathbb{Z} \in S, \ z \cdot k + \mathbb{Z} \in \mathbb{T}_n\}$.
Our objective will be to find $(\mathbb{Z}, \tau_S)^\wedge$ where $(a_{n+1} - a_n)_{n\in\mathbb{N}}$ is an unbounded sequence. For example, we could consider $S_1 = \{2^{-2^n} + \mathbb{Z} \mid n \in \mathbb{N}\}$, $S_2 = \{2^{-n^2} + \mathbb{Z} \mid n \in \mathbb{N}\}$, $S_3 = \{2^{-n!} + \mathbb{Z} \mid n \in \mathbb{N}\}$. We will focus our attention in the case $S = \{2^{-n^2} + \mathbb{Z} \mid n \in \mathbb{N}\}$.
In order to study the topology, we will study its convergent sequences. It will also be useful to find $(\mathbb{Z}, \tau_S)^\wedge$, since $\chi \in (\mathbb{Z}, \tau_S)^\wedge \iff \forall (k_n) \subset \mathbb{Z}$ where $k_n \to 0$ in τ_S, then $\chi(k_n) \to 0 + \mathbb{Z}$.

Fix now $S = \{2^{-n^2} + \mathbb{Z} \mid n \in \mathbb{N}\}$. We want to find sequences converging to zero in τ_S. For that purpose, we will find a characterization of those sequences.
$l_j \to 0$ in $\tau_S \iff \forall m \in \mathbb{N}, \exists j_m$ such that $l_j \in V_{S,m}$ for all $j \geq j_m$. What does this mean? $l_j \in V_{S,m} = \{k \in \mathbb{Z} \mid \forall x + \mathbb{Z} \in S,\ kx + \mathbb{Z} \in \mathbb{T}_m\} = \{k \in \mathbb{Z} \mid \frac{1}{2^{n^2}}k + \mathbb{Z} \in \mathbb{T}_m, \forall n \in \mathbb{N}\}$. This means, $\frac{l_j}{2^{n^2}} + \mathbb{Z} \in \mathbb{T}_m\ \forall n \in \mathbb{N}$. Hence

$$l_j \to 0 \text{ in } \tau_S \iff \forall m \in \mathbb{N}\ \exists j_m, \text{ such that } \forall n \in \mathbb{N}, \frac{l_j}{2^{n^2}} + \mathbb{Z} \in \mathbb{T}_m \text{ for all } j \geq j_m.$$

Now, as we have different criteria for convergence in τ_S and τ_2 we will see the realtionship between them.

Proposition 3.4.1 *$l_j \to 0$ in $\tau_S \Rightarrow l_j \to 0$ in τ_2. Consequently: $\tau_2 \leq \tau_S$*

*P*roof: Let $n_0 \in \mathbb{N}$. We will choose $m = 2^{n_0^2}$. By hypothesis, there exists j_m such that $\frac{l_j}{2^{n^2}} + \mathbb{Z} \in \mathbb{T}_m, \forall n \in \mathbb{N}$, if $j \geq j_m$. Equivalently, $\frac{l_j}{2^{n^2}} + \mathbb{Z}, \frac{2l_j}{2^{n^2}} + \mathbb{Z}, \ldots \frac{ml_j}{2^{n^2}} + \mathbb{Z} \in \mathbb{T}_+$. Which means, $\frac{l_j}{2^{n^2}} + \mathbb{Z}, \frac{2l_j}{2^{n^2}} + \mathbb{Z}, \ldots, \frac{2^{n_0^2}l_j}{2^{n^2}} + \mathbb{Z} \in \mathbb{T}_+; \forall n \in \mathbb{N}$. In particular, this is true for $n = n_0$, then $\frac{l_j}{2^{n_0^2}} + \mathbb{Z}, \frac{2l_j}{2^{n_0^2}} + \mathbb{Z}, \ldots, \frac{ml_j}{2^{n_0^2}} + \mathbb{Z} \in \mathbb{T}_+$. But, $\{\frac{l_j}{2^{n_0^2}} + \mathbb{Z}, \frac{2l_j}{2^{n_0^2}} + \mathbb{Z}, \ldots, \frac{ml_j}{2^{n_0^2}} + \mathbb{Z}\} = \langle \frac{l_j}{2^{n_0^2}} + \mathbb{Z} \rangle$. Since it is a subgroup contained in $\mathbb{T}_+$, it must be $\frac{l_j}{2^{n_0^2}} + \mathbb{Z} = 0 + \mathbb{Z}$. Therefore, $\frac{l_j}{2^{n_0^2}} \in \mathbb{Z}$; or, equivalently $2^{n_0^2} | l_j$. This was our convergence criterion in τ_2.

QED

By this proposition, we get that if a sequence does not converge in τ_2 it will neither converge in τ_S, which is a useful fact, because the convergence criterion in τ_2 is much easier than the one in τ_S.
From here it is also derived the continuity of

$$\text{id} : (\mathbb{Z}, \tau_S) \to (\mathbb{Z}, \tau_2).$$

We now focus on homomorphisms between dual groups induced by continuous homomorphisms between the original groups.

Definition 3.4.2 *Let $\varphi : G \to H$ be a continuous homomorphism between Hausdorff topological groups. Define $\varphi^\wedge : H^\wedge \to G^\wedge$ as $\chi \mapsto \chi \circ \varphi$. It is called the dual homomorphism.*

Proposition 3.4.3 *Let $\varphi : G \to H$ be a continuous homomorphism. Then $\varphi^\wedge$ is continuous*

*P*roof: As usual, we choose a neighbourhood of $G^\wedge$, compute its inverse image and see if it is a neighbourhood of $H^\wedge$. Let $K \subset G$ be a compact subset, $K^\triangleright \subset \mathcal{U}_{G^\wedge}(0)$.
In order to see that $(\varphi^\wedge)^{-1}(K^\triangleright) \in \mathcal{U}_{H^\wedge}(0)$, we compute $\varphi(K)^\triangleright$ and try to verify $\varphi(K)^\triangleright \subset (\varphi^\wedge)^{-1}(K^\triangleright)$.
$\varphi(K)$ is compact and, hence $\varphi(K)^\triangleright$ is a neighbourhood of 0 in $H^\wedge$

Let $\psi \in \varphi(K)^{\triangleright} \iff \psi(\varphi(K)) \subset \mathbb{T}_+$.
We need that $\varphi^\wedge(\psi) \in K^{\triangleright}$. We fix $x \in K$ and compute $\varphi^\wedge(\psi(x)) = \psi(\varphi(x)) \in \psi(\varphi(K)) \subset \mathbb{T}_+$. And the result follows.

QED

In our case, the identity function id : $(\mathbb{Z}, \tau_S) \to (\mathbb{Z}, \tau_2)$ induces a dual homomorphism:

$$\mathrm{id}^\wedge : (\mathbb{Z}, \tau_2)^\wedge = \mathbb{Z}(2^\infty) \to (\mathbb{Z}, \tau_S)^\wedge$$

What else can we say about id$^\wedge$? We will prove an easy but useful result:

Proposition 3.4.4 *If $\varphi : G \to H$ is onto, then $\varphi^\wedge : H^\wedge \to G^\wedge$ is one-to-one.*

*P*roof: Take $\chi \in \ker\varphi_{G^\wedge}$. Observe that: $0_{G^\wedge} = \varphi^\wedge(\chi) = \chi o \varphi \iff \forall x \in G : \chi(\varphi(x)) = 0 + \mathbb{Z}$.
Since φ is onto, we obtain $\forall h \in H, \chi(h) = 0 + \mathbb{Z}$. That means, $\chi = 0_{H^\wedge}$.
Hence $\varphi^\wedge$ is one-to-one.

QED

Since id : $(\mathbb{Z}, \tau_S) \to (\mathbb{Z}, \tau_2)$ is onto, $\mathrm{id}^\wedge : (\mathbb{Z}, \tau_2)^\wedge \to (\mathbb{Z}, \tau_S)^\wedge$ is one-to-one. Hence, we have proved, for our S, that the Prüfer's group $\mathbb{Z}(2^\infty)$ can be embedded in $(\mathbb{Z}, \tau_S)^\wedge$. For any other S we would only need to prove the continuity of the corresponding identity function, as we will see in next section.
With this new information we keep on seeking sequences converging to 0 in τ_S.
We will take advantage on $l_j \to 0$ in $\tau_S \Rightarrow l_j \to 0$ in τ_2. Hence, $l_j \to 0$ implies that $\forall n \in \mathbb{N}\ \exists j_n,\ 2^{n^2} | l_j$ for all $j \geq j_n$. Choose j_n minimal, and such that $2^{n^2} | l_j\ j \geq j_n$. We get an increasing sequence $j_1 \leq j_2 \leq j_3 \cdots$
Define:
$M_0 := \{1, 2, \cdots, j_1 - 1\}$.

$$M_n = \begin{cases} \{j_n\} & : \quad j_n = j_{n+1} \\ \{j_n, j_n + 1, \cdots j_{n+1} - 1\} & : \quad j_n < j_{n+1} \end{cases}$$

$$S_n = \max_{j \in M_n} \frac{|l_j|}{2^{(n+1)^2}} \in \mathbb{R}$$

Proposition 3.4.5 *$S_n \to 0$ in $\mathbb{R} \Rightarrow l_j \to 0$ in τ_S.*

A similar result can be found in page 115 of [10], but it is not correct; in fact we have a counterexample below.

*P*roof: Let $m \in \mathbb{N}$. By hypothesis, $\exists n_0$ such that $S_n < \frac{1}{4m}$ where $n \geq n_0$. Let $j \in M_n$ for some $n \geq n_0$.
We want to see if $l_j \in V_{S,m}$. For that, it must be $\frac{l_j}{2^{k^2}} + \mathbb{Z} \in \mathbb{T}_m$.
If $k \leq n$ then $2^{k^2} | 2^{n^2} | l_j \Longrightarrow \frac{l_j}{2^{k^2}} + \mathbb{Z} = 0 + \mathbb{Z} \in \mathbb{T}_m$.
If $k \geq n+1$ we have that $0 \leq \frac{|l_j|}{2^{k^2}} \leq \frac{|l_j|}{2^{(n+1)^2}} \leq S_n < \frac{1}{4m} \Longrightarrow \frac{l_j}{2^{(n+1)^2}} \in \mathbb{T}_m$

QED

Although the result mencioned above is not true, we can prove the following result:

Proposition 3.4.6 *Let:*

(i) $S_n \to 0$ *in* $\mathbb{R}$

(ii) $l_j \to 0$ *in* τ_S

(iii) $S_n + \mathbb{Z} \to 0 + \mathbb{Z}$ *in* $\mathbb{T}$

Then (i) $\Longrightarrow$ *(ii) and (ii)* $\Longrightarrow$ *(iii)*

Obviously, (iii)$\not\Rightarrow$(i), hence the three conditions are not equivalent, now we are interested in what happens with (ii)$\Rightarrow$(i). In fact we have two examples in which we can see that (ii)$\not\Rightarrow$(i).
The ideas and examples up to the end of the section were obtained in collaboration with Lydia Aussenhofer during her stay at the UCM.
We are seeking examples of (ii) $\not\Rightarrow$ (i). For that we will construct sequences in the form $l_j = 2^{j^2} \cdot a_j$ where a_j will be an odd number for every j. In this way we will get that $M_n = \{n\}$; $j_n = \{n\}$; $y\ S_n = \frac{l_n}{2^{(n+1)^2}}$.

Example 3.4.7 $(S_n)_{n\in\mathbb{N}}$ bounded.
We want that $S_n \to 1$. We know that $S_n = \frac{l_n}{2^{(n+1)^2}} = \frac{2^{n^2} \cdot a_n}{2^{n^2+2n+1}} = \frac{a_n}{2^{2n+1}} \approx 1$. So, we fix $a_n = 2^{2n+1} - 1$ (remember that it must be an odd number), so $l_j = 2^{(j+1)^2} - 2^j$.
$S_n + \mathbb{Z} = \frac{2^{2n+1}-1}{2^{2n+1}} + \mathbb{Z} = 1 - \frac{1}{2^{2n+1}} + \mathbb{Z} \to 0 + \mathbb{Z}$ in $\mathbb{T}$. (With this example we do not know if (iii)$\Longrightarrow$(ii) is false or not). But $S_n \nrightarrow 0$ in $\mathbb{R}$. (with this we will get (ii) $\not\Rightarrow$ (i))
We will see that, in fact, $l_j \to 0$ in τ_S: Let $m \in \mathbb{N}$. For $j_0 = m$ and $j \geq j_0$ we have $2^{-2j-1} + \mathbb{Z} \in \mathbb{T}_m$.
As before, for $n \leq j$ the proof is completely trivial.
We have in fact $n = j + k; k \geq 1$

$\frac{l_j}{2^{(j+k)^2}} + \mathbb{Z} \in \mathbb{T}_m$?

$\frac{l_j}{2^{(k+j)^2}} = \frac{2^{(j+1)^2}}{2^{(j+k)^2}} - \frac{2^{j^2}}{2^{(j+k)^2}} = \frac{2^{j^2+2j+1}}{2^{j^2+2jk+k^2}} - \frac{2^{j^2}}{2^{j^2+2jk+k^2}} = 2^{-2j(k-1)-k^2+1} - 2^{-2jk-k^2}$

If $k = 1$, $2^{-2j(k-1)-k^2+1} - 2^{-2jk-k^2} = 2^0 - 2^{-2j-1}$. But, $2^0 - 2^{-2j-1} + \mathbb{Z} = -2^{-2j-1} + \mathbb{Z} \in \mathbb{T}_m$.

If $k \geq 2$

$$\begin{cases} -2j(k-1) - k^2 + 1 << -2j \Longrightarrow 2^{-2j(k-1)-k^2+1} + \mathbb{Z} \in \mathbb{T}_{2m} \\ y \\ -2jk - k^2 << -2j \Longrightarrow -2^{-2jk-k^2} + \mathbb{Z} \in \mathbb{T}_{2m} \end{cases}$$

Hence, $2^{-2j(k-1)-k^2+1} - 2^{-2jk-k^2} + \mathbb{Z} \in \mathbb{T}_{2m} + \mathbb{T}_{2m} = \mathbb{T}_m$.

Example 3.4.8 $(S_n)_{n\in\mathbb{N}}$ unbounded.
We will fix S_n increasing as n; $S_n = \frac{2^{n^2}\cdot a_n}{2^{(n+1)^2}} = \frac{a_n}{2^{2n+1}} \approx n \Longrightarrow a_n \approx n \cdot 2^{2n+1}$.
As we want that a_n be an odd number we get $a_n = n2^{2n+1} - 1$. Hence $l_j = 2^{j^2}(j2^{2j+1} - 1) = j2^{(j+1)^2} - 2^{j^2}$.
Obviously $S_n \nrightarrow 0$ in $\mathbb{R}$.
We will see $l_j \to 0$ in τ_S. As before, we want that $\frac{l_j}{2^{n^2}} + \mathbb{Z} \in \mathbb{T}_m$; and, also as before, the case $n \leq j$ does not matter. Now, we consider m_1 such that $m_1 2^{-2m_1} < \frac{1}{8m}$. Let $j_0 = \max(m, m_1)$ and $j \geq j_0$.
$\frac{l_j}{2^{(k+j)^2}} = \frac{j2^{(j+1)^2}}{2^{(j+k)^2}} - \frac{2^{j^2}}{2^{(j+k)^2}} = j2^{-2j(k-1)-k^2+1} - 2^{-2jk-k^2}$
If $k = 1$ then $j2^{-2j(k-1)-k^2+1} - 2^{-2jk-k^2} = j - 2^{-2j-1}$ and $j - 2^{-2j-1} + \mathbb{Z} = -2^{-2j-1} + \mathbb{Z} \in \mathbb{T}_m$.
If $k > 1$,
$-2j(k-1) - k^2 + 1 << -2j$; hence $j2^{-2j(k-1)-k^2+1} \leq j2^{-2j} \leq \frac{1}{8m}$. Thus, $j2^{-2j(k-1)-k^2+1} + \mathbb{Z} \in \mathbb{T}_{2m}$.
$-2jk - k^2 << -2j$. Then, $-2^{-2jk-k^2} + \mathbb{Z} \in \mathbb{T}_{2m}$.
$j2^{-2j(k-1)-k^2+1} - 2^{-2jk-k^2} + \mathbb{Z} \in \mathbb{T}_{2m} + \mathbb{T}_{2m} = \mathbb{T}_m$.
Thus, $l_j \to 0$ in τ_S.

The following example shows that (iii)$\not\Rightarrow$(ii).

Example 3.4.9 Define:

$$l_j = \begin{cases} 2^{(n+1)^2} & : \quad j = n^2 - 2 \\ 2^j & : \quad otherwise \end{cases}$$

$j_1 = 1,\ j_2 = 4,\ j_3 = 9, \ldots, j_n = n^2$.
$M_1 = \{1, 2, 3\}$, $M_2 = \{4, 5, 6, 7, 8\}$, $M_3 = \{9, \ldots, 15\}, \cdots, M_n = \{n^2, \ldots, (n+1)^2 - 1\}$.
$S_1 = \frac{l_2}{2^{2^2}} = 1$, $S_2 = \frac{l_7}{2^{2^3}} = 1$, $S_n = \frac{l_{n^2-2}}{2^{(n+1)^2}} = 1$. Hence $S_n + \mathbb{Z} \to 0 + \mathbb{Z}$.
It remains to see that $l_j \nrightarrow 0$ in τ_S.
We observed that, $l_j \to 0$ in $\tau_S \Longleftrightarrow \forall m \in \mathbb{N}\, \exists j_m$ such that $\forall n\ \frac{l_j}{2^{n^2}} + \mathbb{Z} \notin \mathbb{T}_m$ for all $j \geq j_m$.
Let $m = 1$. It suffices to show that $A = \{j_m \in \mathbb{N} \mid \exists n$ such that $\frac{l_j}{2^{n^2}} + \mathbb{Z} \in \mathbb{T}_+$ for all $j \geq j_m\}$ is cofinal in $\mathbb{N}$.
Fix n. Find j_n such that $\frac{l_{j_n}}{2^{n^2}} = \frac{1}{2} \Longleftrightarrow l_{j_n} = 2^{n^2-1} \Longleftrightarrow j_n = n^2 - 1$.

Thus, $A = \{n^2 - 1 \mid n > 1\}$ and it is obviously cofinal.
Hence $l_j \nrightarrow 0$ in τ_S

3.5 General case $S = \{2^{-a_n} + \mathbb{Z} \mid n \in \mathbb{N}\}$

We must impose certain restrictions to the sequences (a_n). For example, $a_n \in \mathbb{N},\ \forall n$, furthermore, we want (a_n) to be a strictly increasing sequence, where $a_n \to \infty$ and $(a_{n+1} - a_n)_{n\in\mathbb{N}}$ be an unbounded sequence.
In this case: $V_{S,m} = \{k \in \mathbb{Z} | \ \forall x + \mathbb{Z} \in S,\ x \cdot k + \mathbb{Z} \in \mathbb{T}_m\}$.
We seek a convergence criterion, for our new τ_S: $l_j \to 0$ in $\tau_S \iff \forall m \in \mathbb{N}\ \exists j_m$ such that $l_j \in V_{S,m}$ for all $j \geq j_m \iff \forall m \in \mathbb{N},\ \exists j_m$ such that $\frac{l_j}{2^{a_n}} + \mathbb{Z} \in \mathbb{T}_m$ if $j \geq j_m,\ \forall n \in \mathbb{N}$
We want to see the sequencial continuity of $\mathrm{id} : (\mathbb{Z}, \tau_S) \to (\mathbb{Z}, \tau_2)$. Since τ_S is a metrizable topology, continuity and sequencial continuity are equivalent. For that, we must prove the following result:

Proposition 3.5.1 *$l_j \to 0$ in $\tau_S \Rightarrow l_j \to 0$ in τ_2.*

For this result it is only necessary that $(a_n)_{n\in\mathbb{N}}$ is a sequence of natural numbers. As we want (a_n) to be an increasing sequence, we can also include this hypothesis.
*P*roof: Let $n_0 \in \mathbb{N}$. We fix $m = 2^{a_{n_0}}$. By hypothesis there exists j_m such that $\frac{l_j}{2^{a_n}} + \mathbb{Z} \in \mathbb{T}_m\ \forall n$ for all $j \geq j_m$. This means, $\frac{l_j}{2^{a_n}} + \mathbb{Z}, \frac{2l_j}{2^{a_n}} + \mathbb{Z}, \ldots, \frac{ml_j}{2^{a_n}} + \mathbb{Z} \in \mathbb{T}_+\ \forall n \iff \frac{l_j}{2^{a_n}} + \mathbb{Z}, \frac{2l_j}{2^{a_n}} + \mathbb{Z}, \ldots, \frac{2^{a_{n_0}} l_j}{2^{a_n}} + \mathbb{Z} \in \mathbb{T}_+\ \forall n$. In particular for $n = n_0$, we have $\frac{l_j}{2^{a_{n_0}}} + \mathbb{Z}, \frac{2l_j}{2^{a_{n_0}}} + \mathbb{Z}, \ldots, \frac{2^{a_{n_0}} l_j}{2^{a_{n_0}}} + \mathbb{Z} \in \mathbb{T}_+$. But $\frac{l_j}{2^{a_{n_0}}} + \mathbb{Z}, \frac{2l_j}{2^{a_{n_0}}} + \mathbb{Z}, \ldots, \frac{2^{a_{n_0}} l_j}{2^{a_{n_0}}} + \mathbb{Z} = \langle \frac{l_j}{2^{a_{n_0}}} + \mathbb{Z} \rangle \subset \mathbb{T}_+$. Thus, $\frac{l_j}{2^{a_{n_0}}} + \mathbb{Z} = 0 + \mathbb{Z}$. This means, $\frac{l_j}{2^{a_{n_0}}} \in \mathbb{Z}$; or, $2^{a_{n_0}} \mid l_j$. Hence, $l_j \to 0$ in τ_2.
QED

Remark 3.5.2 The previous proposition implies that

$$\mathrm{id} : (\mathbb{Z}, \tau_S) \to (\mathbb{Z}, \tau_2)$$

is continuous. Since it is also surjective, the mapping:

$$\mathrm{id}^\wedge : (\mathbb{Z}, \tau_2)^\wedge \to (\mathbb{Z}, \tau_S)^\wedge$$

is injective, and $(\mathbb{Z}, \tau_2)^\wedge \subseteq (\mathbb{Z}, \tau_S)^\wedge$

In the same line of the previous results for $S = \{2^{-n^2} \mid n \in \mathbb{N}\}$, we define j_n minimal, such that $2^{a_n} \mid l_j$ for all $j \geq j_n$. Define M_n as in the previous section and $S_n = \max\{\frac{|l_j|}{2^{a_{n+1}}} | j \in M_n\}$

Proposition 3.5.3 *$S_n \to 0$ in $\mathbb{R} \Rightarrow l_j \to 0$ in τ_S.*

Proof: Let $m \in \mathbb{N}$. Since $S_n \to 0$ in $\mathbb{R}$, $\exists n_0$ such that $S_n < \frac{1}{4m}$ where $n \geq n_0$. Choose $j \in M_n$ where $n > n_0$.

We want to see that $l_j \in V_{S,m}$. For that, $\frac{l_j}{2^{a_k}} + \mathbb{Z} \in \mathbb{T}_m$, $\forall k$.

If $k \leq n$, then $a_k \leq a_n$; and hence, $2^{a_k} \mid 2^{a_n} \mid l_j$. That is, $\frac{l_j}{2^{a_k}} + \mathbb{Z} = 0 + \mathbb{Z} \in \mathbb{T}_m$.

If $k \geq n+1$, we have $0 \leq \frac{|l_j|}{2^{a_k}} \leq \frac{|l_j|}{2^{a_{n+1}}} \leq S_n < \frac{1}{4m}$. Hence, $\frac{l_j}{2^{a_k}} \in \mathbb{T}_m$.

QED

We will see that $l_j \to 0$ in $\tau_S \nRightarrow S_n \to 0$ in $\mathbb{R}$. Choose $l_j = 2^{a_j} b_j$ where b_j is an odd number. Thus, $j_n = \{n\}$, $M_n = \{n\}$ and $S_n = \frac{l_n}{2^{a_{n+1}}}$.

Example 3.5.4 $S_n \to 1$

Now we will suppose that $(a_{n+1} - a_n)_{n \in \mathbb{N}}$ is a strictly increasing sequence. If we delete this hypothesis, the example is not valid as counter example of the implication we are studying, remainig as an open question.

$S_n = \frac{2^{a_n} b_n}{2^{a_{n+1}}} \approx 1$. Then, $b_n \approx 2^{a_{n+1} - a_n}$. Now, since b_n must be odd, we choose $b_n = 2^{a_{n+1} - a_n} - 1$. Thus, $l_j = 2^{a_{j+1}} - 2^{a_j}$.

We will see now that $l_j \to 0$ in τ_S. For that, we fix m, and choose $j_0 = 2m$. Let $j \geq j_0$

We must see that $\frac{l_j}{2^{a_n}} + \mathbb{Z} \in \mathbb{T}_m$. That means, $\frac{2^{a_{j+1}} - 2^{a_j}}{2^{a_n}} + \mathbb{Z} \in \mathbb{T}_m$ $\forall n$.

If $n \leq j$, then $2^{a_n} \mid 2^{a_{j+1}}$ and $2^{a_n} \mid 2^{a_j}$, hence $\frac{2^{a_{j+1}} - 2^{a_j}}{2^{a_n}} + \mathbb{Z} = 0 + \mathbb{Z} \in \mathbb{T}_m$.

We fix now $n = j + k$. We must see $\frac{2^{a_{j+1}}}{2^{a_{j+k}}} - \frac{2^{a_j}}{2^{a_{j+k}}} + \mathbb{Z} \in \mathbb{T}_m$.

If k=1 $\frac{2^{a_{j+1}}}{2^{a_{j+k}}} - \frac{2^{a_j}}{2^{a_{j+k}}} + \mathbb{Z} = -\frac{2^{a_j}}{2^{a_{j+1}}} + \mathbb{Z}$. By the choice of j_0 we have $-\frac{2^{a_j}}{2^{a_{j+1}}} + \mathbb{Z} \in \mathbb{T}_m$.

If $k > 1$, we must see $2^{a_{j+1} - a_{j+k}} - 2^{a_j - a_{j+k}} + \mathbb{Z} \in \mathbb{T}_m$. By the choice of j_0 we have that $2^{a_{j+1} - a_{j+k}} \in \mathbb{T}_{2m}$ and $2^{a_j - a_{j+k}} \in \mathbb{T}_{2m}$. Thus, $2^{a_{j+1} - a_{j+k}} - 2^{a_j - a_{j+k}} + \mathbb{Z} \in \mathbb{T}_m$.

Example 3.5.5 S_n unbounded

As in the previous example we will need $(a_{n+1} - a_n)_{n \in \mathbb{N}}$ be a strictly increasing sequence.

Fix $S_n = \frac{2^{a_n} b_n}{2^{a_{n+1}}} \approx n$. Choose $b_n = n 2^{a_{n+1} - a_n} - 1$. Thus, $l_j = j 2^{a_{j+1}} - 2^{a_j}$.

We want to see that $l_j \to 0$ in τ_S; which means, $\frac{l_j}{2^{a_n}} + \mathbb{Z} \in \mathbb{T}_m$ $\forall n$.

We fix $m \in \mathbb{N}$. Let m_1 such that $m_1 2^{a_{m_1+1} - a_{m_1}} \leq \frac{1}{8m}$. Let $j_0 = \max\{m_1, 2m\}$. Let $j \geq j_0$.

If, $n \leq j$, as before, the result follows trivially.

We choose, then, $\frac{j 2^{a_{j+1}} - 2^{a_j}}{2^{a_{j+k}}} + \mathbb{Z} = j 2^{a_{j+1} - a_{j+k}} - 2^{a_j - a_{j+k}} + \mathbb{Z}$.

If $k = 1$, the expression turns into $j - 2^{a_{j+1} - a_j} + \mathbb{Z} \in \mathbb{T}_m$, by the choice of j_0.

If $k > 1$:

By the choice of j_0 we have $j 2^{a_{j+1} - a_{j+k}} \leq \frac{1}{8m}$. That is, $j 2^{a_{j+1} - a_{j+k}} + \mathbb{Z} \in \mathbb{T}_{2m}$.

Also, $-2^{a_j - a_{j+k}} + \mathbb{Z} \in \mathbb{T}_{2m}$.
Hence, $j2^{a_{j+1} - a_{j+k}} - 2^{a_j - a_{j+k}} + \mathbb{Z} \in \mathbb{T}_m$.

Chapter 4

An approach to the dual of $(\mathbb{Z}, \tau_S)$ for $S = \{2^{-a_n} + \mathbb{Z} \mid n \in \mathbb{N}\}$

4.1 Writing of a natural number as series

We produce a device in order to express a natural number as a sum pivoted by a particular sequence of natural numbers.

Proposition 4.1.1 *Let $(b_n)_{n\in\mathbb{N}_0}$ a sequence of natural numbers such that $b_0 = 1$, $b_n \neq b_{n+1}$ and $b_n \mid b_{n+1}$, for all $n \in \mathbb{N}$. Then, for each natural number $l \in \mathbb{N}$, there exists a natural number $N(l)$, and integers $k_0, \dots, k_{N(l)}$, such that $l = \sum_{i=0}^{N(l)} k_i b_i$ and such that $\mid k_n \mid \leq \frac{b_{n+1}}{2b_n}$, for $0 \leq n \leq N(l)$. Also, $\mid \sum_{i=0}^{n} k_i b_i \mid \leq \frac{b_{n+1}}{2}$ for all n.*

*P*roof: Consider $rd(r)$ as the closest integer to r. In case $r = k + 0.5$, where $k \in \mathbb{Z}$ choose $rd(r)$ as the closest integer to 0 between k and $k + 1$. It is obvious that $|rd(x)| = rd(|x|)$
Fix $l \in \mathbb{Z}$.
Let N be the minimum natural number such that $b_N \geq\mid l \mid$.
Put $k_n = 0$ for $n > N$. Let us define recursively the coefficients $k_N, \dots, k_1, k_0$ in the following way: define $k_N := rd(\frac{l}{b_N})$. Once defined $k_N, \dots, k_{n+1}$, define $k_n := rd(\frac{l-\sum_{i=n+1}^{N} k_i b_i}{b_n})$. As a consequence, we obtain $k_0 = rd(\frac{l-\sum_{i=1}^{N} k_i b_i}{b_0}) = rd(l - \sum_{i=1}^{N} k_i b_i) = l - \sum_{i=1}^{N} k_i b_i$ and hence $l = \sum_{i=0}^{N} k_i b_i$.
Now let us see that $\mid \sum_{i=0}^{n} k_i b_i \mid \leq \frac{b_{n+1}}{2}$ for any l, $0 \leq n \leq N$.
$\mid \sum_{i=0}^{n} k_i b_i \mid = \mid (l - \sum_{i=n+2}^{N} k_i b_i) - k_{n+1} b_{n+1} \mid = \mid b_{n+1}(\frac{l-\sum_{i=n+2}^{N} k_i b_i}{b_{n+1}}) - b_{n+1} rd(\frac{l-\sum_{i=n+2}^{N} k_i b_i}{b_{n+1}}) \mid = b_{n+1} \mid \frac{l-\sum_{i=n+2}^{\infty} k_i b_i}{b_{n+1}} - rd(\frac{l-\sum_{i=n+2}^{N} k_i b_i}{b_{n+1}}) \mid \leq \frac{b_{n+1}}{2}$.
Finally we want to show that $\mid k_n \mid \leq \frac{b_{n+1}}{2b_n}$.

$\mid k_n \mid = \mid rd(\frac{l-\sum_{i=n+1}^{N} k_i b_i}{b_n}) \mid = rd(\frac{|l-\sum_{i=n+1}^{N} k_i b_i|}{b_n}) = rd(\frac{|\sum_{i=0}^{n} k_i b_i|}{b_n})$.
Since the round function is non-decreasing and $\mid \sum_{i=0}^{n} k_i b_i \mid \leq \frac{b_{n+1}}{2}$, $rd(\frac{|\sum_{i=0}^{n} k_i b_i|}{b_n}) \leq rd(\frac{b_{n+1}}{2b_n})$.
If $\frac{b_{n+1}}{b_n}$ is even, then $rd(\frac{b_{n+1}}{2b_n}) = \frac{b_{n+1}}{2b_n}$. If $\frac{b_{n+1}}{b_n}$ is odd, then $rd(\frac{b_{n+1}}{2b_n}) = \frac{b_{n+1}}{2b_n} - 0.5 \leq \frac{b_{n+1}}{2b_n}$.
In both cases $\mid k_n \mid \leq \frac{b_{n+1}}{2b_n}$

QED

4.2 Characterization of convergent sequences in τ_S. Particular case $S = \{2^{-n^2} + \mathbb{Z} \mid n \in \mathbb{N}\}$

Let $(l_j)_{j\in\mathbb{N}}$ be a sequence in $\mathbb{Z}$ converging to 0 in τ_S. By the proposition in the previous section, we can write $l_j = \sum_q k_{j,q} 2^{a_q}$, where $(a_n)_{n\in\mathbb{N}}$ a strictly increasing sequence (at some point of the chapter, we may put more conditions on (a_n)).
Our objective is to find necessary and/or sufficient conditions on the coefficients $k_{j,q}$ in order that the sequence (l_j) converges to 0.
We begin with our special case $S = \{2^{-n^2} + \mathbb{Z} \mid n \in \mathbb{N}\}$. We now put $l_j = \sum_q k_{j,q} 2^{q^2}$.
In the previous chapter, we saw that $l_j \to 0 \iff \forall m \in \mathbb{N},\ \exists j_m$ such that $\forall n \in \mathbb{N},\ \frac{l_j}{2^{n^2}} + \mathbb{Z} \in \mathbb{T}_m$ for all $j \geq j_m$.
Since $l_j = \sum_q k_{j,q} 2^{q^2}$, we can write $\frac{l_j}{2^{n^2}} = \sum_q k_{j,q} \frac{2^{q^2}}{2^{n^2}}$.
We observe that $\sum_q k_{j,q} \frac{2^{q^2}}{2^{n^2}} + \mathbb{Z} = \sum_{q=0}^{n-1} k_{j,q} \frac{2^{q^2}}{2^{n^2}} + \mathbb{Z},\ \forall n \in \mathbb{N}$.
What do these conditions mean?
For $n = 1$, $k_{j,0}\frac{1}{2} + \mathbb{Z} \in \mathbb{T}_m$.
For $n = 2$, $k_{j,0}\frac{1}{2^4} + k_{j,1}\frac{2}{2^4} + \mathbb{Z} \in \mathbb{T}_m$
For $n = 3$, $k_{j,0}\frac{1}{2^9} + k_{j,1}\frac{2}{2^9} + k_{j,2}\frac{2^4}{2^9} + \mathbb{Z} \in \mathbb{T}_m$
For $n = 4$, $k_{j,0}\frac{1}{2^{16}} + k_{j,1}\frac{2}{2^{16}} + k_{j,2}\frac{2^4}{2^{16}} + k_{j,3}\frac{2^9}{2^{16}} + \mathbb{Z} \in \mathbb{T}_m$
We study the case $n = 1$, $k_{j,0} \in \{0, 1\}$, then $\frac{k_{j,0}}{2} + \mathbb{Z} \in \{0 + \mathbb{Z}, \frac{1}{2} + \mathbb{Z}\}$. Since $\frac{k_{j,0}}{2} + \mathbb{Z} \in \mathbb{T}_m$, we get that $k_{j,0} = 0$ for all $j > j_1$.
For $n = 2$, we choose $j > j_1$, then the condition turns into $\frac{k_{j,1}}{8} + \mathbb{Z} \in \mathbb{T}_m$. For $m \geq 4$, and $j \geq j_4$, $k_{j,1} = 0$.
Repeating the same arguments for each q, we choose $m = (q+1)^2$; for all $j \geq j_{(q+1)^2}$, we will get that $k_{j,q} = 0$.

4.3 Characterization of $V_{S,m}$. Particular case $S = \{2^{-n^2} + \mathbb{Z} \mid n \in \mathbb{N}\}$

In this section we try to find some conditions on an integer k to be in $V_{S,m}$.

In this case, we will have that $k = \sum_{i=0}^{N(k)} 2^{i^2} k_i$ and $\mid k_i \mid \leq \frac{2^{(i+1)^2}}{2 \cdot 2^{i^2}} = 2^{2i}$.
For our purposes we will use this technical lemma:

Lemma 4.3.1 $\mid \frac{k_1}{2^{(N+1)^2-1}} + \frac{k_2}{2^{(N+1)^2-4}} + \cdots + \frac{k_N}{2^{(N+1)^2-N^2}} \mid \leq \frac{2}{3}$, *for every N.*

*P*roof: $\mid \frac{k_1}{2^{(N+1)^2-1}} + \frac{k_2}{2^{(N+1)^2-4}} + \cdots + \frac{k_N}{2^{(N+1)^2-N^2}} \mid \leq \frac{|k_1|}{2^{(N+1)^2-1}} + \frac{|k_2|}{2^{(N+1)^2-4}} + \cdots + \frac{k_N}{2^{(N+1)^2-N^2}} = \frac{|k_1|+|k_2|2^3+\cdots|k_N|2^{N^2-1}}{2^{(N+1)^2-1}} \leq \frac{2^2+2^3 2^4+\cdots+2^{(N+1)^2-2}}{2^{(N+1)^2-1}} \leq \frac{\sum_{j=1}^{\infty} 2^{(N+1)^2-2j}}{2^{(N+1)^2-1}} = \frac{\frac{2^{(N+1)^2-2}}{\frac{3}{4}}}{2^{(N+1)^2-1}} = \frac{2}{3}$

QED

Remark 4.3.2 This lemma has an important consequence: whenever we have $\frac{k_1}{2^{(N+1)^2}-1} + \frac{k_2}{2^{(N+1)^2}-4} + \cdots + \frac{k_N}{2^{(N+1)^2-N^2}} + \mathbb{Z} \in \mathbb{T}_m$ then equivalently $\frac{k_1}{2^{(N+1)^2}-1} + \frac{k_2}{2^{(N+1)^2}-4} + \cdots + \frac{k_N}{2^{(N+1)^2-N^2}} \in [-\frac{1}{4m}, \frac{1}{4m}]$.

In the previous chapter we observed that $V_{S,m} = \{k \in \mathbb{Z} \mid \frac{k}{2^{n^2}} + \mathbb{Z} \in \mathbb{T}_m \ \forall n \in \mathbb{N}\}$.
Obviously, the necessary and sufficient conditions for an integer $k = \sum_{j=0}^{N} k_j 2^{j^2}$ to be in $V_{S,m}$ are $\frac{\sum_{i=0}^{n-1} k_i 2^{i^2}}{2^{n^2}} + \mathbb{Z} \in \mathbb{T}_m \ \forall n \in \mathbb{N}$.
As we can observe, the admissible values k_l depend on $k_0, \ldots, k_{l-1}$. Hence, it is too difficult to find necessary and sufficient conditions for each coefficient.
We put $k = k_0 + k_1 2^{1^2} + k_2 2^{2^2} + \cdots$, where $-\frac{2^{(n+1)^2}}{2 \cdot 2^{n^2}} \leq k_n \leq \frac{2^{(n+1)^2}}{2 \cdot 2^{n^2}}$; or equivalently, $\mid k_n \mid \leq 2^{2n}$.
We start studying the case $n = 1$: $\frac{k}{2^{1^2}} + \mathbb{Z} = \frac{k_0}{2} + \mathbb{Z} \in \mathbb{T}_m$ if and only if $k_0 = 0$.
We now study what happens when $n = 2$: $\frac{k}{2^{2^2}} + \mathbb{Z} = k_0 \frac{1}{2^{2^2}} + k_1 \frac{2^{1^2}}{2^{2^2}} + \mathbb{Z} = \frac{k_1}{2^3} + \mathbb{Z} \in \mathbb{T}_m$. We know that $\mid k_1 \mid \leq 2^2$. Hence, the condition we were searching is that $\frac{k_1}{2^3} + \mathbb{Z} \in \mathbb{T}_m$.
We move directly to $n = 3$. We will obtain a condition for k_2.
$\frac{k}{2^{3^2}} + \mathbb{Z} = k_1 \frac{2}{2^9} + k_2 \frac{2^4}{2^9} + \mathbb{Z} \in \mathbb{T}_m$. We must find k_2 such that the previous holds. $\frac{k_1}{2^8} + \frac{k_2}{2^5} + \mathbb{Z} = \frac{1}{2^5}(\frac{k_1}{2^3} + k_2) + \mathbb{Z}$.
At this point we remember that by 4.3.1 we have $\mid \frac{k_1}{2^8} + \frac{k_2}{2^5} \mid \leq \frac{2}{3}$; this implies that $\frac{k_1}{2^8} + \frac{k_2}{2^5} \in [-\frac{1}{4m}, \frac{1}{4m}]$.
Equivalently, $\frac{k_1}{2^3} + k_2 \in [-\frac{2^5}{4m}, \frac{2^5}{4m}]$. Since $\frac{k_1}{8} \in [-\frac{1}{4m}, \frac{1}{4m}]$, we obtain that if $\frac{k_2}{31} + \mathbb{Z} \in \mathbb{T}_m$ (which is equivalent to $k_2 \in [-\frac{31}{4m}, \frac{31}{4m}]$), then $\frac{k_1}{2^3} + k_2 \in [-\frac{2^5}{4m}, \frac{2^5}{4m}]$; or equivalently, $\frac{k_1}{2^8} + \frac{k_2}{2^5} + \mathbb{Z} \in \mathbb{T}_m$.
We consider now $n = 4$.
$\frac{k}{2^{16}} + \mathbb{Z} = k_1 \frac{2}{2^{16}} + k_2 \frac{2^4}{2^{16}} + k_3 \frac{2^9}{2^{16}} + \mathbb{Z} = \frac{1}{2^7}(\frac{k_1}{2^8} + \frac{k_2}{2^5} + k_3) + \mathbb{Z}$. This number must be in $\mathbb{T}_m$. Once again by 4.3.1 $\frac{1}{2^7}(\frac{k_1}{2^8} + \frac{k_2}{2^5} + k_3) + \mathbb{Z} \in \mathbb{T}_m$ if and only if $\frac{1}{2^7}(\frac{k_1}{2^8} + \frac{k_2}{2^5} + k_3) \in [-\frac{1}{4m}, \frac{1}{4m}]$; or, equivalently $(\frac{k_1}{2^8} + \frac{k_2}{2^5} + k_3) \in [-\frac{128}{4m}, \frac{128}{4m}]$. We choose $k_3 \in [-\frac{127}{4m}, \frac{127}{4m}]$. Since $\frac{k_1}{2^8} + \frac{k_2}{2^5} \in [-\frac{1}{4m}, \frac{1}{4m}]$, $\frac{k}{2^{16}} + \mathbb{Z} \in \mathbb{T}_m$.
We can consider the following:

Proposition 4.3.3 *Let k be an integer and let $S = \{\frac{1}{2^{n^2}} + \mathbb{Z} \mid n \in \mathbb{N}\}$. If $k_0 = 0$, $\frac{k_1}{8} + \mathbb{Z} \in \mathbb{T}_m$ and $\frac{k_n}{\frac{2^{(n+1)^2}}{2^{n^2}} - 1} + \mathbb{Z} = \frac{k_n}{2^{2n+1}-1} + \mathbb{Z} \in \mathbb{T}_m$ for $2 \leq n \leq N$, then $\frac{k}{2^{(N+1)^2}} + \mathbb{Z} \in \mathbb{T}_m$. In particular, if $N \geq N(k)$ as defined in 4.1.1, then $k \in V_{S,m}$*

*P*roof: We will prove this result by induction on N.

We have already seen the proof if $N = 2, 3$.

We will now suppose the result true for N and prove it for $N + 1$.

We want $\frac{k}{2^{(N+1)^2}} + \mathbb{Z} = k_1 \frac{2}{2^{(N+1)^2}} + k_2 \frac{2^4}{2^{(N+1)^2}} + \cdots + k_N \frac{2^{N^2}}{2^{(N+1)^2}} + \mathbb{Z} = \frac{k_1}{2^{(N+1)^2-1}} + \frac{k_2}{2^{(N+1)^2-4}} + \cdots + \frac{k_N}{2^{(N+1)^2-N^2}} + \mathbb{Z} = \frac{1}{2^{(N+1)^2-N^2}}(\frac{k_1}{2^{N^2-1}} + \frac{k_2}{2^{N^2-4}} + \cdots + k_N) + \mathbb{Z} \in \mathbb{T}_m$.

By 4.3.1, $\frac{k}{2^{(N+1)^2}} + \mathbb{Z} \in \mathbb{T}_m$ if and only if $\frac{1}{2^{(N+1)^2-N^2}}(\frac{k_1}{2^{N^2-1}} + \frac{k_2}{2^{N^2-4}} + \cdots + k_N) + \mathbb{Z} \in \mathbb{T}_m$.

That is, we must see that $(\frac{k_1}{2^{N^2-1}} + \frac{k_2}{2^{N^2-4}} + \cdots + k_N) \in [-\frac{2^{(N+1)^2-N^2}}{4m}, \frac{2^{(N+1)^2-N^2}}{4m}]$.

Since the proposition is true for N, we know that $\frac{k_1}{2^{N^2-1}} + \cdots + \frac{k_{N-1}}{2^{N^2-(N-1)^2}} \in [-\frac{1}{4m}, \frac{1}{4m}]$.

If furthermore, $\frac{k_N}{2^{(N+1)^2-N^2}-1} + \mathbb{Z} \in \mathbb{T}_m$, or equivalently, $k_N \in [-\frac{2^{(N+1)^2-N^2}-1}{4m}, \frac{2^{(N+1)^2-N^2}-1}{4m}]$, then $(\frac{k_1}{2^{N^2-1}}) + \frac{k_2}{2^{N^2-4}} + \cdots + k_N \in [-\frac{2^{(N+1)^2-N^2}}{4m}, \frac{2^{(N+1)^2-N^2}}{4m}]$.

This proves the first statement. The second is a direct consequence of the decomposition found at 4.1.1.

QED

Remark 4.3.4 We have already seen that these conditions are sufficient for an integer k to be in $V_{S,m}$. The next example shows that they are not necessary.

Take $m = 1$, $k = 128$. We will see that k does not verify these conditions but $k \in V_{S,m}$.

$128 = 0 \cdot 1 + 0 \cdot 2 + 8 \cdot 16$.

$k_0 + \mathbb{Z} = 0 + \mathbb{Z} \in \mathbb{T}_+$.

$\frac{k_1}{8} + \mathbb{Z} = 0 + \mathbb{Z} \in \mathbb{T}_+$.

$\frac{k_2}{31} + \mathbb{Z} \approx 0'2580 + \mathbb{Z} \notin \mathbb{T}_+$.

But

$\frac{128}{2} + \mathbb{Z} = 0 + \mathbb{Z} \in \mathbb{T}_+$.

$\frac{128}{16} + \mathbb{Z} = 0 + \mathbb{Z} \in \mathbb{T}_+$.

$\frac{128}{512} + \mathbb{Z} = \frac{1}{4} + \mathbb{Z} \in \mathbb{T}_+$.

Trivially $\frac{128}{2^{k^2}} + \mathbb{Z} \in \mathbb{T}_+$ if $k > 3$.

4.4 Characterization of convergent sequences in τ_S. General case

Let $(l_j)_{j\in\mathbb{N}}$ be a sequence in $\mathbb{Z}$ converging to 0 in τ_S. By the proposition in the previous section, we can write $l_j = \sum_q k_{j,q}2^{a_q}$, where $(a_n)_{n\in\mathbb{N}}$ a strictly increasing sequence and $a_1 = 1$.
Our objective is to find necessary and/or sufficient conditions for the coefficients $k_{j,q}$, in order that k belongs to $V_{S,m}$.
In the previous chapter, we saw that $l_j \to 0 \iff \forall m \in \mathbb{N},\ \exists j_0$ such that $\forall n \in \mathbb{N},\ \frac{l_j}{2^{a_n}} + \mathbb{Z} \in \mathbb{T}_m$ for all $j \geq j_m$.
Since $l_j = \sum_q k_{j,q}2^{a_q}$, we can write $\frac{l_j}{2^{a_n}} = \sum_q k_{j,q}\frac{2^{a_q}}{2^{a_n}}$.
We observe that $\sum_q k_{j,q}\frac{2^{a_q}}{2^{a_n}} + \mathbb{Z} = \sum_{q=0}^{n-1} k_{j,q}\frac{2^{a_q}}{2^{a_n}} + \mathbb{Z},\ \forall n \in \mathbb{N}$.
What do these conditions mean?
For $n = 1$, $k_{j,0}\frac{1}{2} + \mathbb{Z} \in \mathbb{T}_m$.
For $n = 2$, $k_{j,0}\frac{1}{2^{a_2}} + k_{j,1}\frac{2}{2^{a_2}} + \mathbb{Z} \in \mathbb{T}_m$
For $n = 3$, $k_{j,0}\frac{1}{2^{a_3}} + k_{j,1}\frac{2}{2^{a_3}} + k_{j,2}\frac{2^{a_2}}{2^{a_3}} + \mathbb{Z} \in \mathbb{T}_m$
For $n = 4$, $k_{j,0}\frac{1}{2^{a_4}} + k_{j,1}\frac{2}{2^{a_4}} + k_{j,2}\frac{2^{a_2}}{2^{a_4}} + k_{j,3}\frac{2^{a_3}}{2^{a_4}} + \mathbb{Z} \in \mathbb{T}_m$
We study the first condition, $k_{j,0} \in \{0, 1\}$, then $\frac{k_{j,0}}{2} + \mathbb{Z} \in \{0 + \mathbb{Z}, \frac{1}{2} + \mathbb{Z}\}$. Since $\frac{k_{j,0}}{2} + \mathbb{Z} \in \mathbb{T}_m$, $k_{j,0} = 0$ for all $j > j_1$.
We turn now to the second one, we choose $j > j_1$, then the condition turns into $\frac{k_{j,1}}{2^{a_2-a_1}} + \mathbb{Z} \in \mathbb{T}_m$. For $m \geq a_2$, and $j \geq j_{a_2}$ $k_{j,1} = 0$.
Repeating the same arguments for each q, we choose $m = a_{q+1}$; for all $j \geq j_{a_{q+1}}$, we will get that $k_{j,q} = 0$.

4.5 Characterization of $V_{S,m}$

In this case, we will have that $k = \sum_{i=0}^{\infty} 2^{a_i}k_i$ and $|\ k_i\ |\leq \frac{2^{a_{i+1}}}{2\cdot 2^{a_i}}$.
We will try to reproduce 4.3.1. This time we will need that $a_{n+1} - a_n \geq 2$ forall n.

Lemma 4.5.1 $|\ \frac{k_1}{2^{a_{N+1}-1}} + \frac{k_2}{2^{a_{N+1}-a_2}} + \cdots + \frac{k_N}{2^{a_{N+1}-a_N}}\ |\leq \frac{2}{3}$*, for every N.*

*P*roof: $|\ \frac{k_1}{2^{a_{N+1}}-1} + \frac{k_2}{2^{a_{N+1}-a_2}} + \cdots + \frac{k_N}{2^{a_{N+1}-a_N}}\ |\leq \frac{|k_1|}{2^{a_{N+1}-1}} + \frac{|k_2|}{2^{a_{N+1}-a_2}} + \cdots + \frac{k_N}{2^{a_{N+1}-a_N}} = \frac{|k_1|+|k_2|2^{a_2-1}+\cdots|k_N|2^{a_N-1}}{2^{a_{N+1}-1}} \leq \frac{\sum_{j=1}^{\infty} 2^{a_{N+1}-2j}}{2^{a_{N+1}-1}} = \frac{\frac{2^{a_{N+1}-2}}{\frac{3}{4}}}{2^{a_{N+1}-1}} = \frac{2}{3}$

QED

Remark 4.5.2 This lemma has an important consequence, whenever we have $\frac{k_1}{2^{a_{N+1}-1}} + \frac{k_2}{2^{a_{N+1}-a_2}} + \cdots + \frac{k_N}{2^{a_{N+1}-a_N}} + \mathbb{Z} \in \mathbb{T}_m$ equivalently $\frac{k_1}{2^{a_{N+1}-1}} + \frac{k_2}{2^{a_{N+1}-a_2}} + \cdots + \frac{k_N}{2^{a_{N+1}-a_N}} \in [-\frac{1}{4m}, \frac{1}{4m}]$.

In the previous chapter we observed that $V_{S,m} = \{k \in \mathbb{Z} \mid \frac{k}{2^{a_n}} + \mathbb{Z} \in \mathbb{T}_m \ \forall n \in \mathbb{N}\}$.
Obviously, the necessary and sufficient conditions for $k \in \mathbb{Z}$, where $k = \sum_{j=0}^{N} \frac{k_j}{2^{a_j}}$ to be in $V_{S,m}$ would be $\frac{\sum_{i=0}^{n-1} k_i 2^{a_i}}{2^{a_n}} + \mathbb{Z} \in \mathbb{T}_m \ \forall n \in \mathbb{N}$.
As we can observe, the admissible values k_l depend on $k_0, \ldots, k_{l-1}$. Hence, it is too difficult to find necessary and sufficient conditions for each coefficient.
We put $k = k_0 + k_1 2^{a_1} + k_2 2^{a_2} + \cdots$, where $-\frac{2^{a_{n+1}}}{2\cdot 2^{a_n}} \leq k_n \leq \frac{2^{a_{n+1}}}{2\cdot 2^{a_n}}$; or equivalently, $\mid k_n \mid \leq 2^{a_{n+1}-a_n-1}$.
We start studying the case $n = 1$: $\frac{k}{2^{a_1}} + \mathbb{Z} = \frac{k_0}{2} + \mathbb{Z} \in \mathbb{T}_m$ if and only if $k_0 = 0$.
We now study what happens when $n = 2$: $\frac{k}{2^{a_2}} + \mathbb{Z} = k_0 \frac{1}{2^{a_2}} + k_1 \frac{2^{a_1}}{2^{a_2}} + \mathbb{Z} = \frac{k_1}{a_3} + \mathbb{Z} \in \mathbb{T}_m$. We know that $\mid k_1 \mid \leq 2^2$. Hence, the condition we were searching is that $\frac{k_1}{a_3} + \mathbb{Z} \in \mathbb{T}_m$.
We move directly to $n = 3$. We will obtain a condition for k_2.
$\frac{k}{2^{a_3}} + \mathbb{Z} = k_1 \frac{2}{2^{a_3}} + k_2 \frac{2^{a_2}}{2^{a_3}} + \mathbb{Z} \in \mathbb{T}_m$. We must find k_2 such that the previous holds. $\frac{k_1}{2^{a_3-1}} + \frac{k_2}{2^{a_3-a_2}} + \mathbb{Z} = \frac{1}{2^{a_3-a_2}}(\frac{k_1}{2^{a_2-a_1}} + k_2) + \mathbb{Z}$. At this point we remember that by 4.5.1 we have $\mid \frac{k_1}{2^{a_3-1}} + \frac{k_2}{2^{a_3-a_2}} \mid \leq \frac{2}{3}$; this implies that $\frac{k_1}{2^{a_3-1}} + \frac{k_2}{2^{a_3-a_2}} \in [-\frac{1}{4m}, \frac{1}{4m}]$. Equivalently, $\frac{k_1}{2^{a_2-1}} + k_2 \in [-\frac{2^{a_3-a_2}}{4m}, \frac{2^{a_3-a_2}}{4m}]$. Since $\frac{k_1}{2^{a_2-1}} \in [-\frac{1}{4m}, \frac{1}{4m}]$, we obtain that if $\frac{k_2}{2^{a_3-a_2}-1} + \mathbb{Z} \in \mathbb{T}_m$ (which is equivalent to $k_2 \in [-\frac{2^{a_3-a_2}-1}{4m}, \frac{2^{a_3-a_2}-1}{4m}]$), then $\frac{k_1}{2^{a_2-a_1}} + k_2 \in [-\frac{2^{a_3-a_2}}{4m}, \frac{2^{a_3-a_2}}{4m}]$; or equivalently, $\frac{k_1}{2^{a_2-a_1}} + \frac{k_2}{2^{a_3-a_2}} + \mathbb{Z} \in \mathbb{T}_m$.
We consider now $n = 4$.
$\frac{k}{2^{a_4}} + \mathbb{Z} = k_1 \frac{2}{2^{a_4}} + k_2 \frac{2^{a_2}}{2^{a_4}} + k_3 \frac{2^{a_3}}{2^{a_4}} + \mathbb{Z} = \frac{1}{2^{a_4-a_3}}(\frac{k_1}{2^{a_3-1}} + \frac{k_2}{2^{a_3-a_2}} + k_3) + \mathbb{Z}$. This number must be in $\mathbb{T}_m$. Once again by 4.5.1 $\frac{1}{2^{a_4-a_3}}(\frac{k_1}{2^{a_3-1}} + \frac{k_2}{2^{a_3-a_2}} + k_3) + \mathbb{Z} \in \mathbb{T}_m$ if and only if $\frac{1}{2^{a_4-a_3}}(\frac{k_1}{2^{a_3-a_1}} + \frac{k_2}{2^{a_3-a_2}} + k_3) \in [-\frac{1}{4m}, \frac{1}{4m}]$; or, equivalently $(\frac{k_1}{2^{a_3-1}} + \frac{k_2}{2^{a_3-a_2}} + k_3) \in [-\frac{2^{a_4-a_3}}{4m}, \frac{2^{a_4-a_3}}{4m}]$. We choose $k_3 \in [-\frac{2^{a_4-a_3-1}}{4m}, \frac{2^{a_4-a_3-1}}{4m}]$. Since $\frac{k_1}{2^{a_3-1}} + \frac{k_2}{2^{a_3-a_2}} \in [-\frac{1}{4m}, \frac{1}{4m}]$, $\frac{k}{2^{a_4}} + \mathbb{Z} \in \mathbb{T}_m$.
We can consider the following:

Proposition 4.5.3 *If* $k_0 = 0$, $\frac{k_1}{2^{a_2-1}} + \mathbb{Z} \in \mathbb{T}_m$ *and* $\frac{k_n}{\frac{2^{a_{n+1}}}{2^{a_n}}-1} + \mathbb{Z} = \frac{k_n}{2^{a_{n+1}-a_n}-1} + \mathbb{Z} \in \mathbb{T}_m$ *for* $2 \leq n \leq N$, *then* $\frac{k}{2^{a_{N+1}}} + \mathbb{Z} \in \mathbb{T}_m$. *In particular, if* $N \geq N(k)$ *defined in 4.1.1, then* $k \in V_{S,m}$

*P*roof: We will prove this result by induction on N.
We have already seen the proof if $N = 2, 3$.
We will now suppose the result true for N and prove it for $N + 1$.
We want $\frac{k}{2^{a_{N+1}}} + \mathbb{Z} = k_1 \frac{2}{2^{a_{N+1}}} + k_2 \frac{2^{a_2}}{2^{a_{N+1}}} + \cdots + k_N \frac{2^{a_N}}{2^{a_{N+1}}} + \mathbb{Z} = \frac{k_1}{2^{a_{N+1}-1}} + \frac{k_2}{2^{a_{N+1}-a_2}} + \cdots + \frac{k_N}{2^{a_{N+1}-a_N}} + \mathbb{Z} = \frac{1}{2^{a_{N+1}-a_N}}(\frac{k_1}{2^{a_N-1}} + \frac{k_2}{2^{a_N-a_2}} + \cdots + k_N) + \mathbb{Z} \in \mathbb{T}_m$.
By 4.5.1, $\frac{k}{2^{a_{N+1}}} + \mathbb{Z} \in \mathbb{T}_m$ if and only if $\frac{1}{2^{a_{N+1}-a_N}}(\frac{k_1}{2^{a_N-1}} + \frac{k_2}{2^{a_N-a_2}} + \cdots + k_N) + \mathbb{Z} \in \mathbb{T}_m$.

That is, we must see that $(\frac{k_1}{2^{a_N-1}} + \frac{k_2}{2^{a_N^2-a_2}} + \cdots + k_N) \in [-\frac{2^{a_{N+1}-a_N}}{4m}, \frac{2^{a_{N+1}-a_N}}{4m}]$.
Since the proposition is true for N, we know that $\frac{k_1}{2^{a_N-1}} + \cdots + \frac{k_{N-1}}{2^{a_N-a_{N-1}}} \in [-\frac{1}{4m}, \frac{1}{4m}]$.
If furthermore, $\frac{k_N}{2^{a_{N+1}-a_N-1}} \in \mathbb{T}_m$, or equivalently, $k_N \in [-\frac{2^{a_{N+1}-a_N-1}}{4m}, \frac{2^{a_{N+1}-a_N-1}}{4m}]$, then $(\frac{k_1}{2^{a_N-1}}) + \frac{k_2}{2^{a_N-a_2}} + \cdots + k_N \in [-\frac{2^{a_{N+1}-a_N}}{4m}, \frac{2^{a_{N+1}-a_N}}{4m}]$.
This proves the first statement. The second is a direct consequence of the decomposition found at 4.1.1.

QED

4.6 General results

We start the section proving the following algebraic result.

Proposition 4.6.1 *Let $(a_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence and let $S = \{2^{-a_n} + \mathbb{Z} \mid n \in \mathbb{N}\}$. Then $\langle S \rangle = \mathbb{Z}(2^\infty)$.*

*P*roof: Let $S_0 = \{2^{-n} + \mathbb{Z} \mid n \in \mathbb{N}\}$. We know that $\mathbb{Z}(2^\infty) = \langle S_0 \rangle$.
Since $S \subset S_0$, $\langle S \rangle \subset \langle S_0 \rangle = \mathbb{Z}(2^\infty)$.
Conversely, let $x + \mathbb{Z} \in \mathbb{Z}(2^\infty)$. $x = \frac{k}{2^n}$ for some $k \in \mathbb{Z}$ and some $n \in \mathbb{N}$.
Since (a_n) is strictly increasing $a_n \geq n$.
Hence, $x = \frac{k}{2^n} = \frac{k2^{a_n-n}}{2^{a_n}}$. Thus $x \in \langle S \rangle$.

QED

We now recover our interest in convergent sequences in τ_S.

Theorem 4.6.2 *Let $S = \{2^{-n^2} + \mathbb{Z} \mid n \in \mathbb{N}\}$. Then $\tau_2 \neq \tau_S$.*

*P*roof: It suffices to prove that there exists a sequence $l_j \to 0$ in τ_2 such that $l_j \nrightarrow 0$ in τ_S.
We use the criteria found in the previous chapter.
Fix $l_j = 2^j$.
Obviously, $l_j \to 0$ in τ_2.
We observed that, $l_j \to 0$ in $\tau_S \iff \forall m \in \mathbb{N}\, \exists j_m$ such that $\forall n\ \frac{l_j}{2^{n^2}} + \mathbb{Z} \in \mathbb{T}_m$ for all $j \geq j_m$.
We now want to see that $l_j \nrightarrow 0$ in τ_S
Fix m. It suffices to show that $A = \{j_n \in \mathbb{N} \mid \exists n$ such that $\frac{l_j}{2^{n^2}} + \mathbb{Z} \notin \mathbb{T}_m$ if $j \geq j_m\}$ is cofinal in $\mathbb{N}$.
Fix n. Find j_n such that $\frac{l_{j_n}}{2^{n^2}} = \frac{1}{2} \iff l_{j_n} = 2^{n^2-1} \iff j_n = n^2 - 1$.
Thus, $A = \{n^2 - 1 \mid n > 1\}$ and it is obviously cofinal.

Hence $l_j \nrightarrow 0$ in τ_S and $\tau_2 \neq \tau_S$.

QED

Can we generalize this result? The answer is affirmative

Theorem 4.6.3 *Let (a_n) be a strictly increasing sequence and let $S = \{\frac{1}{2^{a_n}} + \mathbb{Z} \mid n \in \mathbb{N}\}$. Then $\tau_2 \neq \tau_S$.*

*P*roof: As above it suffices to show the existence of a sequence $l_j \to 0$ in τ_2 such that $l_j \nrightarrow 0$ in τ_S.
Fix $l_j = 2^j$.
As before, $l_j \to 0$ in τ_2.
We observed that, $l_j \to 0$ in $\tau_S \iff \forall m \in \mathbb{N}\, \exists j_m$ such that $\forall n\ \frac{l_j}{2^{a_n}} + \mathbb{Z} \in \mathbb{T}_m$ if $j \geq j_m$.
Fix $m = 1$. It suffices to show that $B = \{j_n \in \mathbb{N} \mid \exists n$ such that $\frac{l_j}{2^{a_n}} + \mathbb{Z} \notin \mathbb{T}_+$ if $j \geq j_1\}$ is cofinal in $\mathbb{N}$.
Fix n. Find j_n such that $\frac{l_{j_n}}{2^{a_n}} = \frac{1}{2} \iff l_{j_n} = 2^{a_n - 1} \iff j_n = a_n - 1$.
Thus, $B = \{a_n - 1 \mid n > 1\}$. Since (a_n) is strictly increasing, B is cofinal.
Hence $l_j \nrightarrow 0$ in τ_S and $\tau_2 \neq \tau_S$.

QED

Next question would be:

Question 4.6.4 Fix $(a_n) \neq (b_n)$ strictly increasing sequences. $S_1 = \{2^{-a_n} + \mathbb{Z} \mid n \in n\}$, $S_2 = \{2^{-b_n} + \mathbb{Z} \mid n \in \mathbb{N}\}$. Is $\tau_{S_1} \neq \tau_{S_2}$?

We now compare two topologies on $\mathbb{Z}$. On the one hand, τ_S, the topology of uniform convergence on $S = \{\frac{1}{2^{a_n}} + \mathbb{Z} \mid n \in \mathbb{N}\} \subset \mathbb{T}$, and on the other hand the linear group topology generated by the sequence $\{2^{a_n} \mid n \in \mathbb{N}\}$, whose neighbourhood basis of zero is given by $\{2^{a_n}\mathbb{Z} \mid n \in \mathbb{N}\}$

Question 4.6.5 [Question 1] Let $(a_n)_{n\in\mathbb{N}}$ be an increasing sequence. Then $\{2^{a_n}\mathbb{Z} \mid n \in \mathbb{N}\}$ forms a neighbourhood basis for a Haussdorff group topology on $\mathbb{Z}$.

Question 4.6.6 [Question 2] Let $(a_n)_{n\in\mathbb{N}}$ be an increasing sequence. Let $S = \{\frac{1}{2^{a_n}} + \mathbb{Z} \mid n \in \mathbb{N}\}$. Then $\{2^{a_n}\mathbb{Z} \mid n \in \mathbb{N}\}$ is a neighbourhood basis for $(\mathbb{Z}, \tau_S)$.

We shall see in this section the answer to both questions. The first one is true, but the resulting topology is the $2-$*adic* topology. Unfortunately, the second one is false.

Proposition 4.6.7 *If $\{2^{a_n}\mathbb{Z} \mid n \in \mathbb{N}\}$ is a neighbourhood basis for τ_S, then $(\mathbb{Z}, \tau_S)^\wedge = \langle S \rangle$. By proposition 4.6.1, this would mean that $(\mathbb{Z}, \tau_S)^\wedge = \mathbb{Z}(2^\infty)$.*

*P*roof: In 3.5.2 we have already seen that $\mathbb{Z}(2^\infty) \subset (\mathbb{Z}, \tau_S)^\wedge$.
Let $\chi \in (\mathbb{Z}, \tau_S)^\wedge$.Since χ is continuous, there exists a neighbourhood $U = 2^{a_n}\mathbb{Z}$ such that $\chi(U) = \chi(2^{a_n}\mathbb{Z}) \subset \mathbb{T}_+$. Now $2^{a_n}\mathbb{Z}$ is a subgroup of $\mathbb{Z}$, hence, its image by a homomorphism will be again a subgroup in $\mathbb{T}$. The only subgroup of $\mathbb{T}$ contained in $\mathbb{T}_+$ is $\{0\}$. Hence, $\chi(2^{a_n}\mathbb{Z}) = \{0 + \mathbb{Z}\}$. Let $x + \mathbb{Z} = \chi(1)$. Hence, $\chi(2^{a_n}) = 2^{a_n}x + \mathbb{Z} = 0 + \mathbb{Z}$. This implies that $2^{a_n}x \in \mathbb{Z}$.
Hence: $(\mathbb{Z}, \tau_S)^\wedge \subset \{\frac{k}{2^{a_n}} + \mathbb{Z} \mid k \in \mathbb{Z}, n \in \mathbb{N}\} = \langle S \rangle = \mathbb{Z}(2^\infty)$.

QED

*P*roof:[Question 2 is false if $(a_{n+1} - a_n)$ bounded] Suppose $S = \{2^{-n} + \mathbb{Z}\}$ and that question 2 is true. We have seen in 3.3.1 that in this case, $\tau_S = \tau_{dis}$.
Hence $(\mathbb{Z}, \tau_S) = (\mathbb{Z}, \tau_{dis})$. We dualize.
By 4.6.7 and 4.6.1 $(\mathbb{Z}, \tau_S)^\wedge = \langle S \rangle = \mathbb{Z}(2^\infty)$. On the other hand, $(\mathbb{Z}, \tau_{dis})^\wedge = \mathbb{T}$. This leads to the contradiction $\mathbb{Z}(2^\infty) = \mathbb{T}$.

QED

Proposition 4.6.8 *Let (a_n) as in 4.1.1, $\mathcal{U} = (2^{a_n}\mathbb{Z})$ and $\mathcal{V} = (2^n\mathbb{Z})$. The $\tau_{\mathcal{U}} = \tau_{\mathcal{V}}$*

*P*roof:[proposition + question 1] $\mathcal{U} \subseteq \mathcal{V}$, hence $\tau_{\mathcal{U}} \leq \tau_{\mathcal{V}}$.
Fix now $U \in \mathcal{U}$, $U = 2^{a_{n_0}}\mathbb{Z}$. Since (a_n) is increasing, $V = 2^{a_{n_0}} \subseteq U$. Hence $\mathcal{U} \supseteq \mathcal{V}$.
Since $\mathcal{U}$ is a neighbourhood basis for the $2 - adic$ topology (which is Hausdorff), we have, also, proven question 1. QED

*P*roof: [Question 2 is false] Suppose it is true, then τ_S has $(2^{b_n}\mathbb{Z})$ as neighbourhood basis. By previous proposition $\tau_S = \tau_2$, which we know is false by 4.6.3

QED

Chapter 5

Generalization for S-topologies

In chapter 3 we focused our interest on the 2-adic topology and on S-topologies generated by $S = \{2^{-a_n} + \mathbb{Z} \mid n \in \mathbb{N}\}$, which refine the 2-adic topology.
In this chapter we shall generalize these topologies into a wider frame.
We shall define new topologies and obtain some general results.

5.1 Linear group topologies on the integers

Proposition 5.1.1 *Let* $(b_n)_{n\in\mathbb{N}}$ *be a sequence as in* 4.1.1*, then* $\mathcal{U} = \{b_n\mathbb{Z} \mid n \in \mathbb{N}\}$ *is a neighbourhood basis at* 0 *for a Hausdorff topology.*

*P*roof: We should check conditions (i)-(iv) in 1.1.2.
(i) It is trivial that $0 \in b_n\mathbb{Z}$ for all $n \in \mathbb{N}$.
(ii) We shall prove that $b_n\mathbb{Z}$ is symmetric. Let $k \in b_n\mathbb{Z}$. By definition $k = b_n z$ for some $z \in \mathbb{Z}$. Since $\mathbb{Z}$ is symmetric, $-z \in \mathbb{Z}$ and $-k = b_n(-z) \in b_n\mathbb{Z}$.
(iii) We shall prove that $b_n\mathbb{Z} + b_n\mathbb{Z} = b_n\mathbb{Z}$. Obviously, $b_n\mathbb{Z} + b_n\mathbb{Z} \supseteq b_n\mathbb{Z}$.
Let $a, b \in b_n\mathbb{Z}$, we must check $a + b \in b_n\mathbb{Z}$. By definition, $a = b_n k_1$; $b = b_n k_2$. Hence, $a + b = b_n(k_1 + k_2)$. Since $k_1 + k_2 \in \mathbb{Z}$, we get $a + b \in b_n\mathbb{Z}$.
(iv) Let $b_n\mathbb{Z}, b_m\mathbb{Z} \in \mathcal{U}$, and suppose $n \leq m$. We shall prove $b_n\mathbb{Z} \cap b_m\mathbb{Z} \supseteq b_m\mathbb{Z}$. It clearly suffices to show that $b_m\mathbb{Z} \subseteq b_n\mathbb{Z}$. Let $x \in b_m\mathbb{Z}$, then $x = b_m z = \frac{b_m}{b_n} b_n z$. By definition of (b_n) and, since $m \geq n$, $\frac{b_m}{b_n} \in \mathbb{Z}$. Hence $x \in b_n\mathbb{Z} = b_n\mathbb{Z}$.
In order to prove that the resulting topology is Hausdorff, we must see that $\cap_{n\in\mathbb{N}} b_n\mathbb{Z} = \{0\}$.
Obviously $\cap_{n\in\mathbb{N}} b_n\mathbb{Z} \supseteq \{0\}$.
$x \in b_n\mathbb{Z} \iff x$ is multiple of b_n. Hence $x \in \cap b_n\mathbb{Z} \iff b_n \mid x$ for all $n \Rightarrow x = 0$

QED

Notation 5.1.2 *Let $(\mathbb{Z}, \tau_{(b_n\mathbb{Z})})$ be the group of the integers endowed with the topology defined by the neighbourhood basis $\mathcal{U} = \{b_n\mathbb{Z} \mid n \in \mathbb{N}\}$.*

Proposition 5.1.3 $(\mathbb{Z}, \tau_{(b_n\mathbb{Z})})^\wedge = \{\frac{k}{b_n} + \mathbb{Z} \mid n \in \mathbb{N}, k \in \mathbb{Z}\}$.

*P*roof: First we show that $(\mathbb{Z}, \tau_{(b_n\mathbb{Z})})^\wedge \subseteq \{\frac{k}{b_n} + Z \mid n \in \mathbb{N}, k \in \mathbb{Z}\}$ holds. Let $\chi \in (\mathbb{Z}, \tau_{(b_n\mathbb{Z})})^\wedge$. Since χ is continuous, there exists a neighbourhood $b_n\mathbb{Z}$ such that $\chi(b_n\mathbb{Z}) \subset \mathbb{T}_+$. Hence, $\chi(b_n\mathbb{Z})$ is a subgroup contained in $\mathbb{T}_+$; therefore $\chi(b_n\mathbb{Z}) = \{0 + \mathbb{Z}\}$.
Let $x+\mathbb{Z} = \chi(1) \in \mathbb{T}, \chi(b_n) = b_n x+\mathbb{Z} = 0+\mathbb{Z}$. Hence $b_n x \in \mathbb{Z} \iff$ there exists $k \in \mathbb{Z}$ such that $x = \frac{k}{b_n}$.
In conclussion: $(\mathbb{Z}, \tau_{(b_n\mathbb{Z})})^\wedge \subseteq \{\frac{k}{b_n} + \mathbb{Z} \mid n \in \mathbb{N}, k \in \mathbb{Z}\}$
In order to prove that $(\mathbb{Z}, \tau_{(b_n\mathbb{Z})})^\wedge \supseteq \{\frac{k}{b_n} + \mathbb{Z} \mid n \in \mathbb{N}, k \in \mathbb{Z}\}$; let $k \in \mathbb{Z}, n \in \mathbb{N}$.
Observe that $\ker(\frac{k}{b_n} + \mathbb{Z}) = \{j \in \mathbb{Z} \mid j\frac{k}{b_n} + \mathbb{Z} = 0 + \mathbb{Z}\} \supseteq b_n\mathbb{Z}$.
By 1.1.3, $\ker(\frac{k}{b_n} + \mathbb{Z})$ is open.
Since $(\frac{k}{b_n} + \mathbb{Z})^{-1}(\mathbb{T}_n) \supset \ker(\frac{k}{b_n} + \mathbb{Z})$ our homomorphism is continuous.

QED

Let $(l_j) \subset \mathbb{Z}$ be a sequence. We want to find a criterion for $l_j \to 0$ in $\tau_{(b_n\mathbb{Z})}$.

Proposition 5.1.4 $l_j \to 0$ *in* $\tau_{(b_n\mathbb{Z})} \iff$ *For all $n \in \mathbb{N}$ there exists j_n such that $b_n \mid l_j$ if $j \geq j_n$*

*P*roof: Let $l_j \to 0$ in $\tau_{(b_n\mathbb{Z})}$; by definition of convergence, for every $n \in \mathbb{N}$, there exists j_n such that $l_j \in b_n\mathbb{Z}$ if $j \geq j_n$; or equivalently: $l_j \to 0$ in $\tau_{(b_n\mathbb{Z})} \iff$ for all $n \in \mathbb{N}$ there exists j_n such that $b_n \mid l_j$ if $j \geq j_n$.

QED

5.2 Generalization of S-topologies

Notation 5.2.1 *Let $(b_n)_{n\in\mathbb{N}_0}$ as in 4.1.1. Define $S = \{\frac{1}{b_n} + \mathbb{Z} \mid n \in \mathbb{N}\}$.*
Then $V_{S,m} = \{k \in \mathbb{Z} \mid$ for all $z + \mathbb{Z} \in S$ $zk + \mathbb{Z} \in \mathbb{T}_m\} = \{k \in \mathbb{Z} \mid \frac{k}{b_n} + \mathbb{Z} \in \mathbb{T}_m$ for all $n \in \mathbb{N}\}$

We look for a characterization of sequences converging to 0 in τ_S.

Proposition 5.2.2 $l_j \to 0$ *in* $\tau_S \iff$ *for all $m \in \mathbb{N}$ there exists j_m such that $\frac{l_j}{b_n} + \mathbb{Z} \in \mathbb{T}_m$ for all $n \in \mathbb{N}$ and $j \geq j_m$*

*P*roof: Fix $l_j \to 0$ in τ_S.
By definition (of convergence) for all $m \in \mathbb{N}$, there exists j_m such that $l_j \in V_{S,m}$ if $j \geq j_m$.
$l_j \in V_{S,m} \iff \frac{l_j}{b_n} \in \mathbb{T}_m$ for all $n \in \mathbb{N}$.
Combining both statements we get: $l_j \to 0$ in $\tau_S \iff$ for all $m \in \mathbb{N}$ there exists j_m such that $\frac{l_j}{b_n}+\mathbb{Z} \in \mathbb{T}_m$ for all $n \in \mathbb{N}$ if $j \geq j_m$.

QED

Proposition 5.2.3 *Let $(b_n)_{n\in\mathbb{N}_0}$ as in 4.1.1. Put $S = \{\frac{1}{b_n} + \mathbb{Z} \mid n \in \mathbb{N}\}$ and $(l_j) \subset \mathbb{Z}$. Then $l_j \to 0$ in $\tau_S \Rightarrow l_j \to 0$ in $\tau_{(b_n\mathbb{Z})}$.*

*P*roof: Let $n_0 \in \mathbb{N}$. Fix $m = b_{n_0}$.
By hypothesis, there exists j_m such that $\frac{l_j}{b_n} + \mathbb{Z} \in \mathbb{T}_m$ for all n and $j \geq j_m$.
This means that $\frac{l_j}{b_n} + \mathbb{Z}, \ldots, \frac{ml_j}{b_n} + \mathbb{Z} \in \mathbb{T}_+ \iff \frac{l_j}{b_n} + \mathbb{Z}, \ldots, \frac{b_{n_0} l_j}{b_n} + \mathbb{Z} \in \mathbb{T}_+$ for all n.
In particular this is true when $n = n_0$; then $\frac{l_j}{b_{n_0}} + \mathbb{Z}, \ldots, \frac{b_{n_0} l_j}{b_{n_0}} + \mathbb{Z} \in \mathbb{T}_+$.
But $\frac{l_j}{b_{n_0}} + \mathbb{Z}, \ldots, \frac{b_{n_0} l_j}{b_{n_0}} + \mathbb{Z} = \langle \frac{l_j}{b_{n_0}} + \mathbb{Z} \rangle$, which is a subgroup contained in $\mathbb{T}_+$.
Hence $\frac{l_j}{b_{n_0}} + \mathbb{Z} = 0 + \mathbb{Z}$. Or, equivalently, $b_{n_0} \mid l_j$ if $j \geq j_m$. That is, $l_j \to 0$ in $\tau_{(b_n\mathbb{Z})}$.

QED

Remark 5.2.4 The previous proposition implies that

$$\mathrm{id} : (\mathbb{Z}, \tau_S) \to (\mathbb{Z}, \tau_{(b_n\mathbb{Z})})$$

is continuous. Since it is also surjective, the mapping:

$$\mathrm{id}^\wedge : (\mathbb{Z}, \tau_{(b_n\mathbb{Z})})^\wedge \to (\mathbb{Z}, \tau_S)^\wedge$$

is injective, and $(\mathbb{Z}, \tau_{(b_n\mathbb{Z})})^\wedge \subseteq (\mathbb{Z}, \tau_S)^\wedge$

5.3 Characterization of sequences converging to 0 in τ_S

By 4.1.1, we can write $l_j = \sum_{s=0}^{N(l_j)} b_s k_{s,j}$. We want to find information about $k_{s,j}$ in order that (l_j) is convergent to 0 in τ_S.

Proposition 5.3.1 *Let (l_j) be a null sequence in τ_S. Let $k_{q,j}$ be the coefficients introduced above. For any s there exists j_q such that $k_{q,j} = 0$ if $j \geq j_q$.*

*P*roof: We prove this result by induction.

We begin by proving the result for $q = 0$. Since $l_j \to 0$ in $\tau_{(b_n\mathbb{Z})}$, there exists $j_0 \in \mathbb{N}$ such that $b_1 \mid l_j$ for all $j \geq j_0$.
$l_j = \sum_{s=0}^{N(l_j)} b_s k_{s,j}$. Then $\frac{l_j}{b_1} = \frac{b_0}{b_1}k_{0,j} + \sum_{s=1}^{N(l_j)} \frac{b_s}{b_1}k_{s,j}$. Since, $\frac{b_s}{b_1} \in \mathbb{Z}$ for $s \geq 1$, we get that $\sum_{s=1}^{N(l_j)} \frac{b_s}{b_1}k_{s,j} \in \mathbb{Z}$ and hence, $\frac{b_0}{b_1}k_{0,j} \in \mathbb{Z}$, $\forall j \geq j_0$. Since $\mid k_{0,j} \mid \leq \frac{b_1}{2b_0}$, we get $\mid \frac{b_0}{b_1}k_{0,j} \mid \leq \frac{1}{2}$. Hence $\frac{b_0}{b_1}k_{0,j} \in \mathbb{Z}$ if and only if $k_{0,j} = 0$.
As conclusion, $k_{0,j} = 0$ if $j \geq j_0$.
Let us prove the inductive step; that is, we suppose that there exist $j_0 \leq j_1 \leq \cdots \leq j_q$ such that $k_{q,j} = 0$ for all $j \geq j_q$.
By definition, there exists $j_{q+1} \geq j_q$ such that $b_{q+2} \mid l_j$ for all $j \geq j_{q+1}$ (indexes are different from the ones in previous section).
Now, $l_j = \sum_{s=0}^{N(l_j)} b_s k_{s,j} = \sum_{s=q+1}^{N(l_j)} b_s k_{s,j}$ for $j \geq j_{q+1}$; the last equality holds by the inductive hypothesis.
$\frac{l_j}{b_{q+2}} = \frac{b_{q+1}}{b_{q+2}}k_{q+1,j} + \sum_{s=q+2}^{N(l_j)} \frac{b_s k_{s,j}}{b_{q+2}}$.
Since $\frac{l_j}{b_{q+2}} \in \mathbb{Z}$ for $j \geq j_{q+1}$, $\sum_{s=q+2}^{N(l_j)} \frac{b_s k_{s,j}}{b_{q+2}} \in \mathbb{Z}$ and the equality holds, we deduce that $\frac{b_{q+1}}{b_{q+2}}k_{q+1,j} \in \mathbb{Z}$.
As before, $\mid \frac{b_{q+1}}{b_{q+2}}k_{q+1,j} \mid \leq \frac{1}{2}$. This implies that $\frac{b_{q+1}}{b_{q+2}}k_{q+1,j} \in \mathbb{Z}$ if and only if $k_{q+1,j} = 0$. And the result follows.

QED

5.4 Characterization of the elements of $V_{S,m}$

In this section, we consider $(b_n)_{n\in\mathbb{N}_0}$ as in 4.1.1; $S = \{\frac{1}{b_n} + \mathbb{Z} \mid n \in \mathbb{N}\} \subset \mathbb{T}$, and, consequently, $V_{S,m} = \{k \in \mathbb{Z} \mid \frac{k}{b_n} \in \mathbb{T}_m \text{ for all } n\}$.
First, we shall find a characterization for k to be in $V_{S,m}$. Later, we try to find weaker (but on the other hand easier) sufficient or necessary conditions for k to be in $V_{S,m}$.

Theorem 5.4.1 *Let $k = \sum_{s=0}^{N(k)} k_s b_s$ be an integer, and let $k_0, \ldots, k_{N(k)}$ be the coefficients obtained in 4.1.1. Then $k \in V_{S,m} \iff \sum_{s=0}^{n-1} \frac{k_s b_s}{b_n}\mathbb{Z} \in \mathbb{T}_m$ for all $n \in \mathbb{N}$.*

*P*roof: By 4.1.1 we write $k = \sum_{s=0}^{N(k)} k_s b_s$.
Then, $\frac{k}{b_n} = \sum_{s=0}^{N(k)} \frac{k_s b_s}{b_n}$, and $\frac{k}{b_n} + \mathbb{Z} = \sum_{s=0}^{N(k)} \frac{k_s b_s}{b_n} + \mathbb{Z}$.
Since $b_n \mid b_m$ if $m \geq n$ and $k_m \in \mathbb{Z}$, since $\frac{k}{b_n} + \mathbb{Z} \in \mathbb{T}_m$ the assertion follows.

QED

Lemma 5.4.2 $\mid \sum_{s=0}^{n-1} \frac{k_s b_s}{b_n} \mid \leq \frac{1}{2}$ *for all $n \in \mathbb{N}$*

*P*roof: $\mid \sum_{s=0}^{n-1} \frac{k_s b_s}{b_n} \mid = \frac{|\sum_{s=0}^{n-1} k_s b_s|}{b_n} \overset{4.1.1}{\leq} \frac{\frac{b_n}{2}}{b_n} = \frac{1}{2}$.

QED

The following corollary is the characterization we were seeking.

Corollary 5.4.3 *$k \in V_{S,m} \iff \mid \sum_{s=0}^{n-1} \frac{k_s b_s}{b_n} \mid \leq \frac{1}{4m}$ for all $n \in \mathbb{N}_0$.*

*P*roof: The proof is an inmediate consequence of 5.4.1 and 5.4.2. QED

Proposition 5.4.4 *$k \in V_{S,m}$ implies $\mid \frac{k_n b_n}{b_{n+1}} \mid \leq \frac{3}{8m}$ for all $n \in \mathbb{N}_0$.*

*P*roof: For $n = 0$, we have $\mid \frac{k_0 b_0}{b_1} \mid \overset{5.4.3}{\leq} \frac{1}{4m} \leq \frac{3}{8m}$
Since $k \in V_{S,m}$, by 5.4.3, $\mid \sum_{s=0}^{n-1} \frac{k_s b_s}{b_n} \mid \leq \frac{1}{4m}$, for all $n \in \mathbb{N}$.
We suppose that for some $n \geq 1$, we have $\frac{b_n |k_n|}{b_{n+1}} > \frac{3}{8m}$.
$\mid \sum_{s=0}^{n-1} \frac{k_s b_s}{b_{n+1}} \mid = \mid \sum_{s=0}^{n} \frac{k_s b_s}{b_{n+1}} - \frac{k_n b_n}{b_{n+1}} \mid \overset{5.4.3}{>} \frac{3}{8m} - \frac{1}{4m} = \frac{1}{8m}$.
$\frac{|\sum_{s=0}^{n-1} k_s b_s|}{b_n} = \frac{|\sum_{s=0}^{n-1} k_s b_s|}{b_{n+1}} \frac{b_{n+1}}{b_n} > \frac{1}{8m} 2 = \frac{1}{4m}$; which contradicts the characterization in 5.4.3.

QED

Proposition 5.4.5 *If $\mid \frac{k_n b_n}{b_{n+1}} \mid \leq \frac{1}{8m}$ for $0 \leq n \leq N$ then $k = \sum_{n=0}^{N} b_n k_n \in V_{S,m}$*

*P*roof: Let $0 \leq n \leq N$, then $\mid \sum_{p=0}^{n-1} \frac{k_p b_p}{b_n} \mid \leq \sum_{p=0}^{n-1} \mid \frac{k_p b_p}{b_n} \mid = \sum_{p=0}^{n-1} \frac{b_{p+1}}{b_n} \frac{|k_p| b_p}{b_{p+1}} \leq \frac{1}{8m} \sum_{p=0}^{n-1} \frac{b_{p+1}}{b_n}$.
Since $\frac{b_{n-1}}{b_n} \leq \frac{1}{2}$, $\frac{b_{n-k}}{b_n} \leq \frac{1}{2^k}$.
In the first equality, we consider the change $k = n - 1 - p$. $\sum_{p=0}^{n-1} \frac{b_{p+1}}{b_n} = \sum_{k=0}^{n-1} \frac{b_{n-k}}{b_n} \leq \sum_{k=0}^{n-1} \frac{1}{2^k} \leq \sum_{k=0}^{\infty} 2^{-k} = 2$.
Hence, $\mid \sum_{p=0}^{n-1} \frac{k_p b_p}{b_n} \mid \leq \frac{1}{8m} \sum_{p=0}^{n-1} \frac{b_{p+1}}{b_n} \leq \frac{1}{4m}$.
Hence $k \in V_{S,m}$.

QED

5.5 Relation between linear topologies and S-topologies associated

In this section we will prove some differences between both types of topologies.
Let $\mathbf{b} = (b_n)_{n \in \mathbb{N}}$ be a sequence as in 4.1.1. We define the linear topology associated to $\mathbf{b}$ as the one having $\{b_n \mathbb{Z} \mid n \in \mathbb{N}\}$ as neighbourhood basis at 0. We shall denote it by $\tau_{(b_n \mathbb{Z})}$. We shall also deal with the S-topology where $S = \{\frac{1}{b_n} + \mathbb{Z} \mid n \in \mathbb{N}\}$.

We already know some facts of each topology; in this section we will prove that $\tau_S \neq \tau_{(b_n\mathbb{Z})}$ for any **b**.
In 3.3.1 we have already seen that $\tau_S \neq \tau_{(b_n\mathbb{Z})}$ if $(\frac{b_{n+1}}{b_n})$ is bounded ($\tau_{(b_n\mathbb{Z})}$ is never discrete).
As a consequence we must only prove the result if $(\frac{b_{n+1}}{b_n})$ is unbounded.

Proposition 5.5.1 *Let* $\mathbf{b} = (b_n)$ *be a sequence as in 4.1.1, and such that* $(\frac{b_{n+1}}{b_n})$ *is unbounded. Let* $\tau_{(b_n\mathbb{Z})}$ *and* τ_S *the linear and the S-topology associated to* **b**, *respectively. Then* $\tau_S \neq \tau_{\mathfrak{L}}$.

*P*roof: Since $(\frac{b_{n+1}}{b_n})$ is unbounded, there exists a sequence $(n_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ such that $\frac{b_{n_k}}{b_{n_k+1}} \to 0$.
It suffices to see that there exists a sequence (l_j) such that $l_j \to 0$ in $\tau_{b_n\mathbb{Z}}$, but $l_j \nrightarrow 0$ in τ_S.
Fix $l_j = b_j[\frac{b_{j+1}}{2b_j}]$, where $[x]$ is the biggest integer which is smaller or equal to x.
Since $b_j \mid l_j$ it is obvious that $l_j \to 0$ in $\tau_{\mathfrak{L}}$.
Let us show that $l_j \nrightarrow 0$ in τ_S. Since $(\frac{b_{n_k}}{b_{n_k+1}})_k \to 0$, there exists k_0 such that $\frac{b_{n_k}}{b_{n_k+1}} \leq \frac{1}{16}$ if $k \geq k_0$.
We prove that for $A := \{n_k \mid k \geq k_0\}$ and for every $j = n_{k_0}$, there exists $n \in \mathbb{N}$ (we prove that $n = n_{k_0} + 1$) such that $\frac{l_j}{b_n}\mathbb{Z} \notin \mathbb{T}_+$, which shows that $l_j \nrightarrow 0$ in τ_S.
$\frac{l_j}{b_{j+1}} + \mathbb{Z} = \frac{b_j}{b_{j+1}}[\frac{b_{j+1}}{2b_j}] + \mathbb{Z}$.
If $\frac{b_{j+1}}{b_j}$ is even, then $[\frac{b_{j+1}}{2b_j}] = \frac{b_{j+1}}{2b_j}$, and $\frac{l_j}{b_{j+1}} + \mathbb{Z} = \frac{b_j}{b_{j+1}}\frac{b_{j+1}}{2b_j} + \mathbb{Z} = \frac{1}{2} + \mathbb{Z} \notin \mathbb{T}_+$.
If $\frac{b_{j+1}}{b_j}$ is odd then $[\frac{b_{j+1}}{2b_j}] = \frac{b_{j+1}}{2b_j} - \frac{1}{2}$, and $\frac{l_j}{b_{j+1}} + \mathbb{Z} = \frac{b_j}{b_{j+1}}(\frac{b_{j+1}}{2b_j} - \frac{1}{2}) + \mathbb{Z} = \frac{1}{2}(1 - \frac{b_j}{b_{j+1}})$. By the choice of k_0, $\frac{l_j}{b_{j+1}} + \mathbb{Z} \notin \mathbb{T}_+$.
The result follows.

QED

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