INVERTIBILITY OF LINEAR
NON-COOPERATIVE ELLIPTIC SYSTEMS

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Abstract. In this work we give a sufficient condition for the invertibility of a linear weakly coupled non-cooperative elliptic system under general boundary conditions. Then, we use this result to show the uniqueness of the coexistence state in a large class of Lotka-Volterra predator-prey models with diffusion and transport effects.

1. Introduction

In this work we analyze the eigenvalue problem

\[
\begin{pmatrix}
\mathcal{L}_1 & 0 \\
0 & \mathcal{L}_2
\end{pmatrix}
\begin{pmatrix}
u \\
v
\end{pmatrix} = \begin{pmatrix}
0 & -\alpha(x) \\
\beta(x) & 0
\end{pmatrix}
\begin{pmatrix}
u \\
v
\end{pmatrix} + \tau
\begin{pmatrix}
u \\
v
\end{pmatrix}
\text{ in } (a, b),
\]

\[
\mathcal{B}_{p_1, a} u = \mathcal{B}_{q_1, b} u = \mathcal{B}_{p_2, a} v = \mathcal{B}_{q_2, b} v = 0,
\]

where \(a, b \in \mathbb{R}, a < b\), and \(\mathcal{L}_j\) are second order differential operators of the form

\[
\mathcal{L}_j = -a_j(x) \frac{d^2}{dx^2} + b_j(x) \frac{d}{dx} + c_j(x), \quad j = 1, 2,
\]

with \(a_j, b_j, c_j \in C[a, b], a_j(x) > 0 \text{ for all } x \in [a, b]\), and \(p_j, q_j \in [0, \infty], j = 1, 2\). Given \(p, q \in [0, \infty]\) we have denoted

\[
\mathcal{B}_{p, a} w = \frac{w(a) - p w'(a)}{1 + p}, \quad \mathcal{B}_{q, b} w = \frac{w(b) + q w'(b)}{1 + q},
\]

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If \( p = 0 \), then \( B_{0,a} \) is the Dirichlet operator. In this case we set \( B_{0,a} = D_a \). Similarly, \( B_{0,b} = D_b \). We also consider the Neumann operators

\[
B_{\infty,a} w := N_a w := -w'(a), \quad B_{\infty,b} w := N_b w := w'(b).
\]

The main result of this paper (Theorem 3.1 of Section 3) shows that \( \tau = 0 \) is never an eigenvalue of (1.1) (in \((C[a,b])^2\)) if the following two conditions are satisfied:

(A1) The operators \((L_1; B_{p_1,a}, B_{q_1,b})\) and \((L_2; B_{p_2,a}, B_{q_2,b})\) satisfy the strong maximum principle;

(A2) \( \alpha \geq 0, \beta \geq 0, \) and \( \{ x \in (a, b) : \alpha(x) = 0 \}, \{ x \in (a, b) : \beta(x) = 0 \} \) have empty interior.

This result is known for Dirichlet boundary conditions, [9], [3]. To generalize it to the case of general boundary conditions we first characterize the maximum principle in terms of the positivity of the principal eigenvalue. This characterization is known in the case of Dirichlet boundary conditions [9], [2], [7], and in the context of weakly coupled cooperative elliptic systems, [8], but it is new in our general setting. In Section 2 we characterize the strong maximum principle by means of the existence of a strict positive supersolution and use it to get some basic comparison results between principal eigenvalues. In Section 3 we show the invertibility result and in Section 4 we use it to show the uniqueness of the coexistence state for a class of Lotka-Volterra predator-prey models with one-dimensional diffusion and convection subject to general boundary conditions. The existence of coexistence states can be characterized by using the theory of [4] and [6]. Our results are far from known because of the lack of a priori bounds for the positive solutions of such systems. We conclude the paper discussing the stability of the coexistence states. If some coexistence state loses stability when some of the coefficients of the model varies, then a couple of complex conjugate imaginary eigenvalues of the linearization across the imaginary axis in the complex plain. Our techniques provide us with the uniqueness for a wide class of one-dimensional reaction diffusion systems of predator-prey type.

2. The maximum principle and some properties of principal eigenvalues

Let \( a, b \in \mathbb{R}, a < b \), and set \( I := (a, b) \). In this section we consider a second order differential operator \( \mathcal{L} \) of the form

\[
\mathcal{L} = -a(x) \frac{d^2}{dx^2} + b(x) \frac{d}{dx} + c(x),
\]

where \( a, b, c \in C[a,b], a(x) > 0, x \in [a,b], \) and \( p, q \in [0, \infty] \), and we deal with the eigenvalue problem

\[
\mathcal{L} \varphi = \tau \varphi \quad \text{in} \quad I, \quad B_{p,a} \varphi = B_{q,b} \varphi = 0.
\]  

(2.1)

Problem (2.1) possesses a unique principal eigenvalue. We shall denote it by

\[
\sigma_1^\tau[\mathcal{L}; B_{p,a}, B_{q,b}].
\]
By a principal eigenvalue we mean an eigenvalue to a positive eigenfunction. Its existence and uniqueness can be obtained applying the classical maximum principle and the Krein-Rutman theorem to the operator $\mathcal{L}+C$, under homogeneous boundary conditions $B_p,a = 0$, $B_q,b = 0$, where $C$ is any constant satisfying $C + c > 0$, [1, Theorem 4.3]. Then,

$$
\sigma'_1[\mathcal{L};B_p,a, B_q,b] = \sigma'_1[\mathcal{L} + C;B_p,a, B_q,b] - C.
$$

Moreover, the principal eigenfunction is unique up to multiplicative constants. The main result of this section is the following characterization of the maximum principle by means of the sign of the principal eigenvalue. This result is well known for Dirichlet boundary conditions, [9], [7], [2], but it is new for general boundary conditions.

**Theorem 2.1.** The following conditions are equivalent:

(i) $\sigma'_1[\mathcal{L};B_p,a, B_q,b] > 0$.

(ii) There exists $\psi \in C^1[a,b] \cap C^2(a,b)$ such that $\psi(x) > 0$ for all $x \in I$ and

$$
\mathcal{L} \psi \geq 0 \quad \text{in} \quad I, \quad B_{p,a} \psi \geq 0, \quad B_{q,b} \psi \geq 0, \quad (2.2)
$$

with some of these inequalities strict, i.e. the problem

$$
\mathcal{L} w = 0 \quad \text{in} \quad I, \quad B_{p,a} w = B_{q,b} w = 0, \quad (2.3)
$$

has a strict positive supersolution. We will simply say that $(\mathcal{L};B_p,a, B_q,b)$ possesses a strict positive supersolution.

(iii) The operator $(\mathcal{L};B_p,a, B_q,b)$ satisfies the strong maximum principle, i.e. if $w \in C^1[a,b] \cap C^2(a,b)$ satisfies

$$
\mathcal{L} w \geq 0 \quad \text{in} \quad I, \quad B_{p,a} w \geq 0, \quad B_{q,b} w \geq 0, \quad (2.4)
$$

then $w \geq 0$ in $I$. Moreover, if some of these inequalities is strict, then $w \gg 0$ in the sense that $w(x) > 0$ for all $x \in I$, $w'(a) > 0$ if $w(a) = 0$, and $w'(b) < 0$ if $w(b) = 0$.

**Proof.** (i) $\Rightarrow$ (ii). Assume that $\sigma'_1[\mathcal{L};B_p,a, B_q,b] > 0$. Let $\varphi \gg 0$ denote its principal eigenfunction. Then, $\mathcal{L} \varphi > 0$ and so $\varphi$ is a strict positive supersolution of (2.3).

(ii) $\Rightarrow$ (iii). Let $\psi$ be a positive strict supersolution of (2.3). Consider $w \in C^1[a,b] \cap C^2(a,b)$ satisfying (2.4). To prove that $w \geq 0$ we argue by contradiction. Suppose that there exists $x_1 \in (a,b)$ such that $w(x_1) < 0$. Then, the minimum $t \geq 0$ for which $v_t := w + t \psi \geq 0$ in $I$ is positive. Denote it by $\mu > 0$. We claim that $v_\mu(x) > 0$ for all $x \in (a,b)$. On the contrary assume that $v_\mu(x_2) = 0$ for some $x_2 \in (a,b)$. Then, since $(\mathcal{L} + c^+ - c) v_\mu \geq \mathcal{L} v_\mu \geq 0$ in $I$, it follows from [10, Chap. 1, Theorem 3] that $v_\mu \equiv 0$. Thus, $w = -\mu \psi$ and hence, using (2.2) and (2.4), we find that $0 \leq \mathcal{L} w = -\mu \mathcal{L} \psi \leq 0$ in $I$, $0 \leq B_{p,a} w = -\mu B_{p,a} \psi \leq 0$ and $0 \leq B_{q,b} w = -\mu B_{q,b} \psi \leq 0$. Therefore, $\mathcal{L} \psi = 0$ in $I$, $B_{p,a} \psi = 0$ and $B_{q,b} \psi = 0$, which is impossible, because we are assuming that some of the inequalities of (2.2)
is strict. This contradiction shows that \( v_\mu(x) > 0 \) for all \( x \in I \). By the definition of \( \mu \) we find that either \( v_\mu(a) = 0 \), or \( v_\mu(b) = 0 \). Moreover, due to [10, Chap. 1, Theorem 4], \( v'_\mu(a) > 0 \) if \( v_\mu(a) = 0 \), and \( v'_\mu(b) < 0 \) if \( v_\mu(b) = 0 \). To complete the proof we distinguish several different cases according to the boundary conditions.

(a) Dirichlet boundary conditions: \( p = q = 0 \). In this case, it follows from [11, Theorem 2] that some of the following options occurs. Either there exists \( \beta < 0 \) such that \( w = \beta \psi \) in \( I \), or \( w \equiv 0 \) in \( I \), or \( w(x) > 0 \) for all \( x \in I \). As we are assuming that \( w(x_1) < 0 \), necessarily \( w = \beta \psi \) in \( I \) for some \( \beta < 0 \). Thus,

\[
0 \leq \mathcal{L} w = \beta \mathcal{L} \psi \leq 0, \quad 0 \leq \mathcal{D}_a w = \beta \mathcal{D}_a \psi \leq 0, \quad 0 \leq \mathcal{D}_b w = \beta \mathcal{D}_b \psi \leq 0 .
\]

Therefore, \( \mathcal{L} \psi = 0 \) and \( \mathcal{D}_a \psi = \mathcal{D}_b \psi = 0 \). As \( \psi \) is a strict positive supersolution, this is a contradiction. This contradiction shows that \( w \geq 0 \).

(b) Robin boundary conditions: \((p, q) \in (0, \infty)^2\). We already know that either \( v_\mu(a) = 0 \) and \( v'_\mu(a) > 0 \), or \( v_\mu(b) = 0 \) and \( v'_\mu(b) < 0 \). Thus, either \( \mathcal{B}_{p,a} v_\mu < 0 \) or \( \mathcal{B}_{q,b} v_\mu < 0 \). On the other hand, from (2.2), (2.4) and the definition of \( v_\mu \) we get

\[
\mathcal{B}_{p,a} v_\mu = \mathcal{B}_{p,a} w + \mu \mathcal{B}_{p,a} \psi \geq 0, \quad \mathcal{B}_{q,b} v_\mu = \mathcal{B}_{q,b} w + \mu \mathcal{B}_{q,b} \psi \geq 0 . \tag{2.5}
\]

This contradiction shows that \( w \geq 0 \).

(c) Neumann boundary conditions: \( p = \infty, q = \infty \). The same argument used in case (b) shows that either \( \mathcal{N}_a v_\mu = -v'_\mu(a) < 0 \), or \( \mathcal{N}_b v_\mu = v'_\mu(b) < 0 \). This contradicts (2.5) and shows that \( w \geq 0 \).

(d) Dirichlet-Robin boundary conditions: \( p = 0, q \in (0, \infty) \). If \( v_\mu(b) = 0 \) and \( v'_\mu(b) < 0 \), then \( \mathcal{B}_{q,b} v_\mu < 0 \) which contradicts (2.5). Hence, \( v_\mu(b) > 0 \), \( v_\mu(a) = 0 \) and \( v'_\mu(a) > 0 \). Due to the minimality of \( \mu \), there exists a sequence \((x_k)\), \( k \geq 1 \), in \((a, b)\) such that

\[
w(x_k) + (\mu - \frac{1}{k}) \psi(x_k) = v_\mu(x_k) - \frac{1}{k} \psi(x_k) < 0, \quad k \geq 1 . \tag{2.6}
\]

Let \( x^* \) be a limit point of some convergent subsequence of \((x_k)\), again denoted by \((x_k)\). We already know that \( v_\mu(x) > 0 \) for all \( x \in (a, b) \). Thus, if we assume that \( x^* > a \), then passing to the limit as \( k \to \infty \) in (2.6) we get \( 0 < v_\mu(x^*) = w(x^*) + \mu \psi(x^*) \leq 0 \), which is impossible. Thus, \( x^* = a \) and (2.6) yields

\[
(\mu - \frac{1}{k}) \psi(x_k) < -w(x_k) = w(a) - w(x_k) \leq M (x_k - a) = M d(x_k) , \tag{2.7}
\]

for some constant \( M > 0 \), where \( d(x) = \min\{|x-a|, |x-b|\} \). To obtain (2.7) we have used \( w(a) = 0 \). This follows from \( 0 = v_\mu(a) = w(a) + \mu \psi(a) \geq 0 \), taking into account that \( \mathcal{D}_a w = w(a) \geq 0 \) and that \( \mathcal{D}_a \psi = \psi(a) \geq 0 \). Now, it follows from [9, Lemma 1] that \( v_\mu(x) > \delta d(x), x \simeq a, \) for some \( \delta > 0 \). Therefore, we find from (2.7) that

\[
v_\mu(x_k) - \frac{1}{k} \psi(x_k) > \delta (1 - \frac{M}{\mu k - 1}) d(x_k) > 0
\]
for $k$ sufficiently large. This relation contradicts to (2.6) and completes the proof of $w \geq 0$. The above argument can be easily adapted to cover the case Robin at $a$ and Dirichlet at $b$. This technique was also used to prove Theorem 2 of [11].

(e) **Dirichlet-Neumann boundary conditions:** $p = 0$, $q = \infty$. The same argument as in case (d) shows that $w \geq 0$.

(f) **Neumann-Robin boundary conditions:** $p = \infty$, $q \in (0, \infty)$. From (2.2), (2.4) and the definition of $v_{\mu}$ we find that

$$
N_a v_{\mu} = -v_{\mu}'(a) \geq 0, \quad B_{q,b} v_{\mu} = \frac{v_{\mu}(b) + q v_{\mu}'(b)}{1 + q} \geq 0. \tag{2.8}
$$

In particular, $v_{\mu}'(a) \leq 0$. Therefore, $v_{\mu}(b) = 0$ and $v_{\mu}'(b) < 0$. This contradicts the second relation of (2.8) and shows that $w \geq 0$. A similar argument shows that $w \geq 0$ in case Robin-Neumann.

We have already seen that $w \geq 0$ in $[a, b]$. If we assume that some of the inequalities of (2.4) is strict, then $w > 0$ in $I$. Moreover, $(\mathcal{L} + c^+ - c) w \geq \mathcal{L} w \geq 0$ in $(a, b)$. Thus, it follows from [10, Chap.1, Th. 3] that $w(x) > 0$ for all $x \in (a, b)$. Finally, [10, Chap.1, Th. 4] completes the proof of (ii) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (i). We argue by contradiction assuming that $\sigma^\sharp_1[\mathcal{L}; B_{p,a}, B_{q,b}] \leq 0$. Let $\varphi \gg 0$ be its associated principal eigenfunction. Then, $\mathcal{L}(-\varphi) = \sigma^\sharp_1[\mathcal{L}; B_{p,a}, B_{q,b}](\varphi) \geq 0$ in $I$, $B_{p,a}(-\varphi) = 0$ and $B_{q,b}(-\varphi) = 0$. Therefore, (iii) implies $\varphi \leq 0$. This contradiction shows $\sigma^\sharp_1[\mathcal{L}; B_{p,a}, B_{q,b}] > 0$ and completes the proof of the theorem. $\Box$

We now use this characterization to get some comparison results between principal eigenvalues. First, we give the following of min-max characterization of $\sigma^\sharp_1[\mathcal{L}; B_{p,a}, B_{q,b}]$.

**Theorem 2.2.** Let $\mathcal{P}_{a,b}$ denote the set of functions $\psi \in C^1[a, b] \cap C^2(a, b)$ such that $\psi(x) > 0$ for all $x \in [a, b]$, $B_{p,a}\psi > 0$ and $B_{q,b}\psi > 0$. Then,

$$
\sigma^\sharp_1[\mathcal{L}; B_{p,a}, B_{q,b}] = \sup_{\psi \in \mathcal{P}_{a,b}} \inf_{x \in I} \frac{\mathcal{L}\psi(x)}{\psi(x)}. \tag{2.9}
$$

**Proof.** Let $\lambda < \sigma^\sharp_1[\mathcal{L}; B_{p,a}, B_{q,b}]$ be. Then $\sigma^\sharp_1[\mathcal{L} - \lambda; B_{p,a}, B_{q,b}] > 0$ and it follows from Theorem 2.1 that the unique solution of

$$
(\mathcal{L} - \lambda) \phi = 1 \quad \text{in} \quad I, \quad B_{p,a} \phi = B_{q,b} \phi = 1,
$$

satisfies $\phi(x) > 0$ for all $x \in [a, b]$. In particular,

$$
\lambda \leq \inf_{x \in I} \frac{\mathcal{L}\phi(x)}{\phi(x)} \leq \sup_{\psi \in \mathcal{P}_{a,b}} \inf_{x \in I} \frac{\mathcal{L}\psi(x)}{\psi(x)}.
$$

Since this inequality holds for any $\lambda < \sigma^\sharp_1[\mathcal{L}; B_{p,a}, B_{q,b}]$ we find that

$$
\sigma^\sharp_1[\mathcal{L}; B_{p,a}, B_{q,b}] \leq \sup_{\psi \in \mathcal{P}_{a,b}} \inf_{x \in I} \frac{\mathcal{L}\psi(x)}{\psi(x)}.
$$
To complete the proof we argue by contradiction. Suppose that

$$\sigma_1^I[\mathcal{L}; B_{p,a}, B_{q,b}] \leq \sup_{\psi \in \mathcal{P}_{a,b}} \inf_{x \in I} \frac{\mathcal{L}\psi(x)}{\psi(x)}.$$ 

Then, there exist $\varepsilon > 0$ and $\psi \in \mathcal{P}_{a,b}$ such that

$$\sigma_1^I[\mathcal{L}; B_{p,a}, B_{q,b}] + \varepsilon < \inf_{x \in I} \frac{\mathcal{L}\psi(x)}{\psi(x)}.$$ 

Hence,

$$(\mathcal{L} - \sigma_1^I[\mathcal{L}; B_{p,a}, B_{q,b}]) - \varepsilon)\psi > 0$$

and so, $\psi$ is a s.p. supersolution of $(\mathcal{L} - \sigma_1^I[\mathcal{L}; B_{p,a}, B_{q,b}] - \varepsilon; B_{p,a}, B_{q,b})$. Thus, due to Theorem 2.1 we obtain that

$$\sigma_1^I[\mathcal{L} - \sigma_1^I[\mathcal{L}; B_{p,a}, B_{q,b}] - \varepsilon; B_{p,a}, B_{q,b}] = -\varepsilon > 0.$$ 

This contradiction completes the proof. \(\Box\)

Most of the assertions of next theorem are special cases of Lemma 15.5 and Proposition 17.5 of [5], because Hölder continuity of the coefficients are not needed in the proofs. By the sake of completeness we include an alternative proof of all these properties with none additional regularity requirement.

**Theorem 2.3.** The following properties of the principal eigenvalue hold:

(a) Let $V_j \in C[a,b], j = 1, 2, \text{ with } V_1 < V_2$. Then,

$$\sigma_1^I[\mathcal{L} + V_1; B_{p,a}, B_{q,b}] < \sigma_1^I[\mathcal{L} + V_2; B_{p,a}, B_{q,b}].$$

(b) The following estimates are satisfied

$$\sigma_1^I[\mathcal{L}; B_{p,a}, N_0] < \sigma_1^I[\mathcal{L}; B_{p,a}, B_{q,b}] < \sigma_1^I[\mathcal{L}; B_{p,a}, D_b], \quad (p, q) \in [0, \infty) \times (0, \infty),$$

$$\sigma_1^I[\mathcal{L}; N_a, B_{q,b}] < \sigma_1^I[\mathcal{L}; B_{p,a}, B_{q,b}] < \sigma_1^I[\mathcal{L}; D_a, B_{q,b}], \quad (p, q) \in (0, \infty) \times [0, \infty].$$

(c) If $a < b < c$, then

$$\sigma_1^{(a,c)}[\mathcal{L}; B_{p,a}, B_{q,c}] < \sigma_1^{(a,b)}[\mathcal{L}; B_{p,a}, D_b],$$

$$\sigma_1^{(a,c)}[\mathcal{L}; B_{p,a}, B_{q,c}] < \sigma_1^{(b,c)}[\mathcal{L}; D_b, B_{q,c}].$$

**Proof.** (a) Let $\varphi$ be the pr. eig. associated with $\sigma_1^I[\mathcal{L} + V_1; B_{p,a}, B_{q,b}]$. Then,

$$(\mathcal{L} + V_2 - \sigma_1^I[\mathcal{L} + V_1; B_{p,a}, B_{q,b}]) \varphi = (V_2 - V_1) \varphi > 0 \quad \text{in} \quad I,$$

$$B_{p,a} \varphi = B_{q,b} \varphi = 0,$$
and so $\varphi$ is a s.p. supersolution of $(\mathcal{L} + V_2 - \sigma_1^I[\mathcal{L} + V_1; \mathcal{B}_{p,a}, \mathcal{B}_{q,b}]; \mathcal{B}_{p,a}, \mathcal{B}_{q,b})$. Thus, it follows from Theorem 2.1 that

$$\sigma_1^I[\mathcal{L} + V_2 - \sigma_1^I[\mathcal{L} + V_1; \mathcal{B}_{p,a}, \mathcal{B}_{q,b}]; \mathcal{B}_{p,a}, \mathcal{B}_{q,b}] > 0.$$  

This completes the proof of Part (a).

(b) Pick up $(p, q) \in [0, \infty] \times (0, \infty)$ and denote by $\psi$ to the principal eigenfunction associated with $\sigma_1^I[\mathcal{L}; \mathcal{B}_{p,a}, \mathcal{N}_b]$. It is clear that $\psi(b) \geq 0$. Since $\psi'(b) = 0$, if $\psi(b) = 0$, then the uniqueness of the Cauchy problem associated with $\mathcal{L} - \sigma_1^I[\mathcal{L}; \mathcal{B}_{p,a}, \mathcal{N}_b]$ implies that $\psi \equiv 0$. Thus, $\psi(b) > 0$ and hence $\mathcal{B}_{q,b}\psi = \psi(b) > 0$. So, $\psi$ is a strict positive supersolution of $(\mathcal{L} - \sigma_1^I[\mathcal{L}; \mathcal{B}_{p,a}, \mathcal{N}_b]; \mathcal{B}_{p,a}, \mathcal{B}_{q,b})$. Therefore, we find from Theorem 2.1 that

$$\sigma_1^I[\mathcal{L} - \sigma_1^I[\mathcal{L}; \mathcal{B}_{p,a}, \mathcal{N}_b]; \mathcal{B}_{p,a}, \mathcal{B}_{q,b}] > 0.$$  

This completes the proof of the first inequality in the statement of Part (b). To prove the second one we denote by $\varphi$ the principal eigenfunction associated with $\sigma_1^I[\mathcal{L}; \mathcal{B}_{p,a}, \mathcal{B}_{q,b}]$. Then, $\varphi(b) + q\varphi'(b) = 0$. Moreover, $\varphi(b) \geq 0$. Thus, if we assume that $\varphi(b) = 0$, then $\varphi'(b) = 0$ and the uniqueness of the Cauchy problem implies $\varphi \equiv 0$, which is not possible. Thus, $\varphi(b) = \mathcal{D}_b\varphi > 0$. Hence, $\varphi$ is a strict positive supersolution of $(\mathcal{L} - \sigma_1^I[\mathcal{L}; \mathcal{B}_{p,a}, \mathcal{B}_{q,b}]; \mathcal{B}_{p,a}, \mathcal{D}_b)$ and therefore Theorem 2.1 implies

$$\sigma_1^I[\mathcal{L} - \sigma_1^I[\mathcal{L}; \mathcal{B}_{p,a}, \mathcal{B}_{q,b}]; \mathcal{B}_{p,a}, \mathcal{D}_b] > 0.$$  

This completes the proof of the second inequality of Part (b). Some similar arguments complete the proof of Part (b).

(c) Let $\psi$ denote the pr. eig. associated with $\sigma_1^{(a,c)}[\mathcal{L}; \mathcal{B}_{p,a}, \mathcal{B}_{q,b}]$. Then, $\mathcal{D}_b\psi = \psi(b) > 0$ and so the function $\psi$ is a strict positive supersolution of the operator $(\mathcal{L} - \sigma_1^{(a,c)}[\mathcal{L}; \mathcal{B}_{p,a}, \mathcal{B}_{q,b}]; \mathcal{B}_{p,a}, \mathcal{D}_b)$. Due to Theorem 2.1 we find that

$$\sigma_1^{(a,b)}[\mathcal{L} - \sigma_1^{(a,c)}[\mathcal{L}; \mathcal{B}_{p,a}, \mathcal{B}_{q,b}]; \mathcal{B}_{p,a}, \mathcal{D}_b] > 0.$$  

This completes the proof of the first inequality. This argument can be adapted to complete the proof of the theorem. □

3. Invertibility of linear non-cooperative elliptic systems

In this section we show that under conditions (A1) and (A2) $(u, v) = (0, 0)$ is the unique solution of

$$L_1 u = -\alpha(x) v \quad \text{in} \quad I := (a, b),  \tag{3.1a}$$  

$$L_2 v = \beta(x) u$$  

$$\mathcal{B}_{p_1,a} u = \mathcal{B}_{q_1,b} u = \mathcal{B}_{p_2,a} v = \mathcal{B}_{q_2,b} v = 0. \tag{3.1b}$$

To precise, the following result holds.
**Theorem 3.1.** Suppose (A1) and (A2) of Section 1. Then, \((u, v) = (0, 0)\) is the unique solution of (3.1) in \((C[a,b])^2\).

**Remark 3.2.** Any solution \((u, v) \in (C[a,b])^2\) of (3.1a) is a classical solution, i.e. \((u, v) \in (C^2[a,b])^2\).

To prove Theorem 3.1 we use two general properties of the solutions of (3.1a).

**Lemma 3.3.** Assume that (3.1a) has a solution \((u, v)\) with \(u \neq 0\) and \(v \neq 0\). Then the set of zeros of \(u\) and \(v\) is discrete in \([a,b]\). In particular, the sets \(\{x \in (a,b) : u(x) = 0\}\) and \(\{x \in (a,b) : v(x) = 0\}\) have empty interior.

**Proof.** Let \((u, v)\) be a solution of (3.1a) with \(u \neq 0\) and \(v \neq 0\). First, we show that the zeros of \(u(x)\) do not accumulate at some \(x_0 \in [a,b]\). We argue by contradiction. Assume that the zeros of \(u\) accumulate at some \(x_0 \in [a,b]\). Then \(u(x_0) = u'(x_0) = 0\) and necessarily either \(v(x_0) \neq 0\), or \(v'(x_0) \neq 0\), because of the uniqueness of the Cauchy problem at \(x_0\) for the first order system associated with (3.1a). In any case \(v(x)\) has constant sign in \((x_0 - \varepsilon, x_0) \cap [a,b]\) and \((x_0, x_0 + \varepsilon) \cap [a,b]\); not necessarily the same if \(v(x_0) = 0\). Moreover, there exists a sequence \((x_m)\) such that \(u(x_m) = 0\) and either \(x_m \downarrow x_0\) or \(x_m \uparrow x_0\). Assume for instance that \(x_m \downarrow x_0\) and choose \(m\) sufficiently large so that \(x_m < x_0 + \varepsilon\) and

\[
\sigma_1^{(x_0,x_m)}[\mathcal{L}_1; \mathcal{D}_{x_0}, \mathcal{D}_{x_m}] > 0.
\]

Note that in fact the principal eigenvalue goes to infinity as the length of the interval goes to zero. It is clear that \(\alpha(x) v(x)\) does not change of sign in \((x_0, x_m)\). Therefore, it follows from the first equation of (3.1a) and the maximum principle that \(u\) does not vanish on \((x_0, x_m)\). This is impossible. Similarly, we get also a contradiction if \(x_m \uparrow x_0\). The same argument shows that the zeros of \(v\) do not accumulate. \(\Box\)

**Lemma 3.4.** Let \((u, v)\) be a non-trivial solution of (3.1a) such that \(u(z_1) = u(z_2) = 0\), for some \(z_1, z_2 \in [a,b]\), \(z_1 < z_2\), \(u < 0\) in \((z_1, z_2)\) and \(v(z_1) \leq 0\). Then, \(v(z_2) > 0\).

**Remark 3.5.** Changing the signs of \(u\) and \(v\) it is clear that if \((u, v)\) is a non-trivial solution of (3.1a) such that \(u(z_1) = u(z_2) = 0\), \(u > 0\) in \((z_1, z_2)\) and \(v(z_1) \geq 0\), then \(v(z_2) < 0\).

**Proof of Lemma 3.4.** By (A1) and Theorem 2.1, \(\sigma_1^1[\mathcal{L}_j; \mathcal{B}_{p_j,a}, \mathcal{B}_{q_j,b}] > 0\). Thus, we find from Theorem 2.3(b) that

\[
\sigma_1^{(z_1,z_2)}[\mathcal{L}_1; \mathcal{D}_{z_1}, \mathcal{D}_{z_2}] \geq \sigma_1^{(a,b)}[\mathcal{L}_1; \mathcal{D}_a, \mathcal{D}_b] > 0.
\]

So, if we assume that \(v \leq 0\) in \((z_1, z_2)\), then

\[
\mathcal{L}_1 u = -\alpha v \geq 0 \text{ in } (z_1, z_2), \quad u(z_1) = u(z_2) = 0,
\]

and hence we find from Theorem 2.1 that \(u \geq 0\), which is a contradiction. Therefore, \(v\) changes of sign in \((z_1, z_2)\). Note that in particular \(v \neq 0\). On the other hand,

\[
\mathcal{L}_2 v = \beta u \leq 0 \text{ in } (z_1, z_2), \quad v(z_1) \leq 0,
\]
and since \( \sigma^{(z_1,z_2)}_1[\mathcal{L}_2;\mathcal{D}_{z_1},\mathcal{D}_{z_2}] > 0 \), if we assume that \( v(z_2) \leq 0 \), then we get from Theorem 2.1 that \( v \leq 0 \) in \( (z_1,z_2) \), which is not possible. Therefore, \( v(z_2) > 0 \). This completes the proof. \( \square \)

**Proof of Theorem 3.1.** We will argue by contradiction. Assume that (3.1) possesses a solution \((u,v) \neq (0,0)\). Then, \( u \neq 0 \) and \( v \neq 0 \). Indeed, if we assume that \( v = 0 \), then

\[
\mathcal{L}_1 u = 0 \quad \text{in} \quad (a,b), \quad \mathcal{D}_a u = \mathcal{D}_b u = 0,
\]

and since \( \sigma^{(a)}_1[\mathcal{L}_1;\mathcal{D}_a,\mathcal{D}_b] > 0 \), we obtain \( u = 0 \). Similarly, if \( u = 0 \) then we find from the second equation of (3.1a) that \( v = 0 \). Therefore, \( u \neq 0 \) and \( v \neq 0 \). We claim that \( u \) and \( v \) change of sign. On the contrary, assume that \( u > 0 \). Then, \( \mathcal{L}_2 v = \beta u \geq 0 \) in \((a,b)\) and so we find from the maximum principle that \( v \geq 0 \). Thus, \( \mathcal{L}_1 u = -\alpha v \leq 0 \) and hence \( u \leq 0 \), which contradicts \( u > 0 \). Therefore, \( u \) changes of sign. By symmetry, \( v \) changes of sign as well.

By Lemmas 3.3, 3.4 the set of zeros of \( u \) where it changes of sign is discrete in \([a,b]\). Let

\[
a = z_0 < z_1 < \cdots < z_N = b
\]

be such set. By changing the signs of \( u \) and \( v \), if necessary, we can assume that

\[
\begin{align*}
u > 0 & \quad \text{in} \quad (z_{2j}, z_{2j+1}) \quad 0 \leq j \leq N. \\
u < 0 & \quad \text{in} \quad (z_{2j-1}, z_{2j})
\end{align*}
\]

In particular,

\[
\begin{align*}
u > 0 & \quad \text{in} \quad (a,z_1) \\
u < 0 & \quad \text{in} \quad (a,z_1)
\end{align*}
\]

Due to Theorem 2.3(c) we have

\[
\sigma^{(a,z_1)}_1[\mathcal{L}_2;\mathcal{B}_{p_2,a},\mathcal{D}_{z_1}] > \sigma^{(a,b)}_1[\mathcal{L}_2;\mathcal{B}_{p_2,a},\mathcal{B}_{q_2,b}] > 0. \quad (3.2)
\]

Moreover, \( \mathcal{B}_{p_2,a} v = 0 \). We claim that \( v(z_1) < 0 \). On the contrary, assume that \( \mathcal{D}_{z_1} v \geq 0 \). Then, since

\[
\mathcal{L}_2 v = \beta u \geq 0 \quad \text{in} \quad (a,z_1), \quad \mathcal{B}_{p_2,a} v = 0, \quad \mathcal{D}_{z_1} v \geq 0,
\]

it follows from (3.2) and Theorem 2.1 that \( v > 0 \) in \((a,z_1)\). Thus,

\[
\mathcal{L}_1 u = -\alpha v < 0 \quad \text{in} \quad (a,z_1), \quad \mathcal{B}_{p_1,a} u = 0, \quad \mathcal{D}_{z_1} u = 0 \quad (3.3)
\]

On the other hand, it follows from Theorem 2.3(c) that

\[
\sigma^{(a,z_1)}_1[\mathcal{L}_1;\mathcal{B}_{p_1,a},\mathcal{D}_{z_1}] > \sigma^{(a,b)}_1[\mathcal{L}_1;\mathcal{B}_{p_1,a},\mathcal{B}_{q_1,b}] > 0,
\]

and so we can apply Theorem 2.1 to (3.3) to get \( u < 0 \) in \((a,z_1)\). This contradiction shows that \( v(z_1) < 0 \). Now, thanks to Lemma 3.3 and 3.4, some of the following situations occurs. Either

\[
v(z_N) < 0 \quad \text{and} \quad u < 0 \quad \text{in} \quad (z_N,b),
\]
In this section we consider the following nonlinear elliptic boundary value problem
\[ \mathcal{E}_1 u = \ell(x) u - A(x) u^2 - B(x) u v \quad \text{in} \quad I := (a, b), \]
\[ \mathcal{E}_2 v = m(x) v + C(x) u v - D(x) v^2 \]
and
\[ B_{p_j, a} u = B_{q_j, b} v = 0, \]
where \( \mathcal{E}_j, j = 1, 2, \) are two second order differential operators of the form
\[ \mathcal{E}_j = -\alpha_j(x) \frac{d^2}{dx^2} + \beta_j(x) \frac{d}{dx}, \quad j = 1, 2, \]
with \( \alpha_j, \beta_j \in C[a, b], \alpha_j(x) > 0 \) for all \( x \in [a, b], \) and we assume \( \ell, m, A, B, C, D \in C[a, b] \) to satisfy the following hypothesis:
\begin{enumerate}
  \item[(H1)] \( A > 0, D > 0, \) i.e. there exist \( x_1, x_2 \in (a, b) \) such that \( A(x_1) > 0, D(x_2) > 0. \)
  \item[(H2)] \( B > 0, C > 0, \) and \( \{ x \in I : B(x) = 0 \}, \{ x \in I : C(x) = 0 \} \) have empty interior.
\end{enumerate}
The boundary operators \( B_{p_j, a} \) and \( B_{q_j, b} \) are of the same type as those of (1.3). Note that (H1) does not exclude the possibility that \( A(x), \) or \( D(x), \) vanishes on some subinterval of \( I. \) Our main result shows that (4.1) possesses at most a coexistence state. By a coexistence state we mean a solution couple \( (u_0, v_0) \) with both component positive. Due to the strong maximum principle, if \( (u_0, v_0) \) is a coexistence state, then \( u_0(x) > 0 \) and \( v_0(x) > 0 \) for all \( x \in I. \) Moreover, \( u_0'(a) > 0 \) if \( p_1 = 0 \) and \( u_0'(b) < 0 \) if \( q_1 = 0. \) Similarly, \( v_0'(a) > 0 \) if \( p_2 = 0 \) and \( v_0'(b) < 0 \) if \( q_2 = 0. \)

**Theorem 4.1.** Suppose (H1) and (H2). Then, (4.1) admits at most a coexistence state.

**Proof.** Assume that (4.1) admits a coexistence state, say \( (u_1, v_1). \) Let \( (u_2, v_2) \) be any other coexistence state of (4.1). Set
\[ U := u_1 - u_2, \quad V := v_1 - v_2, \]
Theorem 4.2. Suppose that there exist \( a_h < b_h \) such that \([a_h, b_h] \subset (a, b)\) and that \( h(x) = 0 \) if and only if \( x \in [a_h, b_h] \). Then, (4.3) possesses a positive solution if, and only if,
\[
\sigma_1^{(a,b)}[\mathcal{E} - n; B_{p,a}, B_{q,b}] < 0 < \sigma_1^{(a_h, b_h)}[\mathcal{E} - n; D_{a_h}, D_{b_h}].
\]
Let \( \theta[\mathcal{E} - n; h;p;q] \) denote the maximal non-negative solution of (4.3). Due to Theorem 4.2, \( \theta[\mathcal{E} - n; h;p;q] = 0 \) if \( \sigma_1^{(a,b)}[\mathcal{E} - n; B_{p,a}, B_{q,b}] \geq 0 \) or \( \sigma_1^{(a_h, b_h)}[\mathcal{E} - n; D_{a_h}, D_{b_h}] \leq 0 \), and \( \theta[\mathcal{E} - n; h;p;q] \gg 0 \) in any other case. By \( \gg \) we mean that it lies in the interior of the cone of positive functions of \( C_0^1[a, b] \). If \( h(x) \) satisfies the assumptions of Theorem 4.2 and \( \theta[\mathcal{E} - n; h;p;q] \gg 0 \) then \( \theta[\mathcal{E} - n; h;p;q] \) is a global attractor for the positive solutions of the parabolic problem associated with (4.3), [4]. Moreover,
\[
\lim_{\sigma_1^{(a_h, b_h)}[\mathcal{E} - n; D_{a_h}, D_{b_h}] \downarrow 0} \|\theta[\mathcal{E} - n; h;p;q]\|_{\infty} = \infty,
\]
and therefore, the positive solutions of (4.1) do not admit a priori bounds. As all the results found for this type of systems dealt with the case where a priori bounds are available, the following characterization of the existence is new.
Theorem 4.3. Suppose that
\[
\sigma_1^{(a_A,b_A)}[\mathcal{E}_1 - \ell; \mathcal{D}_{a_A}, \mathcal{D}_{b_A}] > 0, \tag{4.4a}
\]
\[
\sigma_1^{(a_D,b_D)}[\mathcal{E}_2 - C \theta[\mathcal{E}_1 - \ell; \mathcal{D}_{p_1,q_1}] - m; \mathcal{D}_{a_D}, \mathcal{D}_{b_D}] > 0. \tag{4.4b}
\]
Then, (4.1) possesses a coexistence state if, and only if,
\[
\sigma_1^{(a,b)}[\mathcal{E}_1 + B \theta[\mathcal{E}_2 - m; \mathcal{D}_{p_2,q_2}] - \ell; \mathcal{B}_{p_1,a}, \mathcal{B}_{q_1,b}] < 0, \tag{4.5a}
\]
\[
\sigma_1^{(a,b)}[\mathcal{E}_2 - C \theta[\mathcal{E}_1 - \ell; \mathcal{D}_{p_1,q_1}] - m; \mathcal{B}_{p_2,a}, \mathcal{B}_{q_2,b}] < 0, \tag{4.5b}
\]
\[ \text{i.e. if any of the semi-trivial positive solutions is linearly unstable.}\]

Remark 4.4. Since \( \theta[\mathcal{E}_2 - m; \mathcal{D}_{p_2,q_2}] \geq 0 \), (4.5a) implies
\[
\sigma_1^{(a,b)}[\mathcal{E}_1 - \ell; \mathcal{B}_{p_1,a}, \mathcal{B}_{q_1,b}] < 0. \tag{4.6}
\]

If (4.6) holds, then (4.4a) and Theorem 4.2 give \( \theta[\mathcal{E}_1 - \ell; \mathcal{D}_{p_1,q_1}] \gg 0 \).

Remark 4.5. It can be easily seen that (4.4) holds provided \( b_A - a_A \) and \( a_D - b_D \) are sufficiently small. The fact that (4.5) is equivalent to the instability of any of the semi-trivial states (those with one component vanishing) follows by a direct calculation from the strong maximum principle. Note that the linearization of (4.1) at any semi-trivial state possesses a triangular structure and so estimating its spectrum is reduced to estimating the spectrum of a boundary value problem for a single equation.

Proof. Suppose (4.4). Let \((u_0, v_0)\) be a coexistence state of (4.1). Then,
\[
\mathcal{E}_1 u_0 = \ell(x) u_0 - A(x) u_0^2 - B(x) u_0 v_0 < \ell(x) u_0 - A(x) u_0^2,
\]
and hence, \(u_0 > 0\) is a strict subsolution of
\[
\mathcal{E}_1 w = \ell(x) w - A(x) w^2 \quad \text{in} \quad I, \quad \mathcal{B}_{p_1,a} w = \mathcal{B}_{q_1,b} w = 0.
\]
By (4.4a) we can use the method of sub and supersolutions to show that this implies
\[
0 < u_0 < \theta[\mathcal{E}_1 - \ell; \mathcal{D}_{p_1,q_1}]. \tag{4.7}
\]
Moreover, due to Theorem 4.2 we get (4.6). Thus, (4.6) is necessary for the existence of a solution with the first component positive. Now, going back to the second equation of (4.1) yields
\[
\mathcal{E}_2 v_0 = m(x) v_0 + C(x) u_0 v_0 - D(x) v_0^2 < m(x) v_0 + C(x) \theta[\mathcal{E}_1 - \ell; \mathcal{D}_{p_1,q_1}] v_0 - D(x) v_0^2,
\]
and hence, \(v_0 > 0\) is a strict subsolution of
\[
(\mathcal{E}_2 - C(x) \theta[\mathcal{E}_1 - \ell; \mathcal{D}_{p_1,q_1}]) w = m(x) w - D(x) w^2 \quad \text{in} \quad I, \quad \mathcal{B}_{p_2,a} w = \mathcal{B}_{q_2,b} w = 0.
\]
Therefore, the method of sub and supersolutions implies

\[ 0 < v_0 < \theta [\mathcal{E}_2 - C \theta [\mathcal{E}_1 - \ell; A; p_1; q_1] - m; D; p_2; q_2] \cdot \] (4.8)

In particular, it follows from Theorem 4.2 that

\[ \sigma_1^{(a,b)} [\mathcal{E}_2 - C \theta [\mathcal{E}_1 - \ell; A; p_1; q_1] - m; B_{p_2,a}, B_{q_2,b}] < 0 . \]

Similarly, since \( \sigma_1^{(a,D,b,D)} [\mathcal{E}_2 - m; D_{a,D}, D_{b,D}] > 0 \), the second equation of (4.1) gives

\[ v_0 > \theta [\mathcal{E}_2 - m; D; p_2; q_2] . \]

Going back to the first equation and substituting the previous estimate gives

\[ \mathcal{E}_1 u_0 = \ell(x) u_0 - A(x) u_0^2 - B(x) u_0 v_0 < \ell(x) u_0 - A(x) u_0^2 - B(x) \theta [\mathcal{E}_2 - m; D; p_2; q_2] u_0 . \]

So, \( u_0 > 0 \) is a strict subsolution of

\[ (\mathcal{E}_1 + B(x) \theta [\mathcal{E}_2 - m; D; p_2; q_2]) w = \ell(x) w - A(x) w^2 \text{ in } I , \quad B_{p_1,a} w = B_{q_1,b} w = 0 , \]

and therefore, it follows from the method of sub and supersolutions that

\[ 0 < u_0 < \theta [\mathcal{E}_1 + B \theta [\mathcal{E}_2 - m; D; p_2; q_2] - \ell; A; p_1; q_1] . \]

Due to Theorem 4.2, this relation gives

\[ \sigma_1^{(a,b)} [\mathcal{E}_1 + B \theta [\mathcal{E}_2 - m; D; p_2; q_2] - \ell; B_{p_1,a}, B_{q_1,b}] < 0 . \]

This shows that (4.5) is necessary for the existence of a coexistence state.

We now show that under condition (4.4), (4.5) is sufficient for the existence of a coexistence state. To show the sufficiency of this condition we can apply Theorem 4.1 of [6]. Thought this result was obtained for Dirichlet boundary conditions, it can be easily adapted to deal with general boundary conditions. We will omit the details here. To apply this theorem is useful regarding to (4.1) as the particularization at \((\lambda, \mu) = (0, 0)\) of

\[ (\mathcal{E}_1 - \ell) u = \lambda u - A(x) u^2 - B(x) u v \text{ in } I := (a,b) , \]

\[ (\mathcal{E}_2 - m) v = \mu v + C(x) u v - D(x) v^2 \]

\[ B_{p_1,a} u = B_{q_1,b} u = B_{p_2,a} v = B_{q_2,b} v = 0 , \]

where \((\lambda, \mu)\) is a two-dimensional continuation parameter. Observe that any coexistence state of (4.1) must satisfy (4.7) and (4.8). Moreover, due to (4.4), it follows from the theory developed in [4] that the functions on the right hand sides
of these inequalities are bounded above uniformly on compact subsets of the parameter space. Therefore, under condition (4.4) we have a priori bounds for the coexistence states. This completes the proof.

Finally, a remark about the stability of the coexistence states of (4.1). Let \((u_0, v_0)\) be a coexistence state of (4.1). Then, the stability of \((u_0, v_0)\) as a solution of the parabolic system associated with (4.1) is given by the real parts of the eigenvalues of
\[
\begin{align*}
(\mathcal{E}_1 + 2A u_0 + B v_0) u &= -B(x) u_0(x) v + \tau u & \text{in } I := (a,b), \\
(\mathcal{E}_2 + 2D v_0 - C u_0) v &= C(x) v_0(x) u + \tau v \\
& \mathcal{B}_{p_1,a} u = \mathcal{B}_{q_1,b} u = \mathcal{B}_{p_2,a} v = \mathcal{B}_{q_2,b} v = 0.
\end{align*}
\]
If \(\text{Re } \tau > 0\) for any eigenvalue \(\tau\), then \((u_0, v_0)\) is asymptotically stable. If it possesses some eigenvalue \(\tau\) with \(\text{Re } \tau < 0\), then \((u_0, v_0)\) is unstable. Since \((u_0, v_0)\) is a coexistence state of (4.1) we have
\[
\sigma_1^I[\mathcal{E}_1 + A u_0 + B v_0 - \ell; \mathcal{B}_{p_1,a}, \mathcal{B}_{q_1,b}] = 0, \quad \sigma_1^I[\mathcal{E}_2 + D v_0 - C u_0 - m; \mathcal{B}_{p_2,a}, \mathcal{B}_{q_2,b}] = 0.
\]
Thus, it follow from Theorem 2.3(a) that
\[
\sigma_1^I[\mathcal{E}_1 + 2A u_0 + B v_0 - \ell; \mathcal{B}_{p_1,a}, \mathcal{B}_{q_1,b}] > 0,
\]
\[
\sigma_1^I[\mathcal{E}_2 + 2D v_0 - C u_0 - m; \mathcal{B}_{p_2,a}, \mathcal{B}_{q_2,b}] > 0,
\]
and hence we find from Theorem 3.1 that \(\tau = 0\) can not be an eigenvalue of the linearization of (4.1) at \((u_0, v_0)\). Therefore, if \((u_0, v_0)\) loses its stability is because some sort of Hopf bifurcation occurs.

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References


