Lefschetz index of arbitrary planar homeomorphisms at isolated invariant continua

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October 30, 2014

Abstract

Let \( f : U \subset \mathbb{R}^2 \to f(U) \subset \mathbb{R}^2 \) be an arbitrary homeomorphism, with \( U \) an open set, and let \( K \subset U \) be an isolated invariant continuum. If \( K \) does not decompose the plane, the fixed point indices of the iterates of \( f \) at \( K \), \( i_{\mathbb{R}^2}(f^n, K) \), follow the same behavior than the indices at a locally maximal fixed point. This problem was solved by P. Le Calvez and J.C. Yoccoz in the orientation preserving case. In this paper we compute the indices for arbitrary, not only orientation preserving, homeomorphisms and arbitrary isolated continua and show some dynamical implications. In particular we show that proper invariant continua of the 2-sphere containing the set of periodic orbits of a homeomorphism are not isolated.

1. Introduction

The existence of a minimal homeomorphism of \( \mathbb{R}^m \) is an open problem suggested by Ulam in the Scottish Book [17]. In dimension 2, one can consider the problem of the existence or not of minimal homeomorphisms \( f : \mathbb{R}^2 \setminus K \to \mathbb{R}^2 \setminus K \) with \( K \) a compact set. If \( K = \emptyset \), the result follows from the Brouwer’s translation arcs theorem. Handel, in [10], proved that, if \( K \) has at least two points, there are not minimal homeomorphisms of \( \mathbb{R}^2 \setminus K \). Le Calvez and Yoccoz [15] solved completely the problem in the multi-punctured plane. Later Franks [6], gave an alternative proof using Conley index methods. The general result for \( K \) a compact set is given in [23]. Recently has been proved that there are not minimal orientation reversing homeomorphisms in \( \mathbb{R}^3 \) ([11]).

The key of the proof in [15] is the study of the fixed point indices of the iterations of an orientation preserving local homeomorphism \( f : U \subset \mathbb{R}^2 \to \mathbb{R}^2 \).
\( \mathbb{R}^2 \to \mathbb{R}^2 \) in a neighborhood of a point \( p \) which is not an attractor nor a repeller isolated invariant set. This result was extended in [21] for arbitrary homeomorphisms with a shorter proof, using the following ideas:

If \( f : U \subset \mathbb{R}^2 \to \mathbb{R}^2 \) is a local homeomorphism and \( p \in U \) is a non-repeller fixed point of \( f \) such that \( \{p\} \) is an isolated invariant set, then there are an AR (absolute retract), \( D \), containing a neighborhood \( V \subset \mathbb{R}^2 \) of \( p \), a finite subset \( \{q_1, \ldots, q_m\} \subset D \) and a map \( \overline{f} : D \to D \) such that \( \overline{f}|V = f|_V \) and for every \( k \in \mathbb{N} \), \( \text{Fix}(\overline{f})^k \subset \{p, q_1, \ldots, q_m\} \). Moreover,

a) (Le Calvez-Yoccoz) If \( f \) preserves the orientation, then

\[ i_{\mathbb{R}^2}(f^k, p) = \begin{cases} 1 - rq & \text{if } k \in r\mathbb{N} \\ 1 & \text{if } k \not\in r\mathbb{N} \end{cases} \]

where \( k \in \mathbb{N} \), \( q \) is the number of periodic orbits of \( \overline{f} \) (excluding \( p \)) and \( r \) is their period.

b) Assume that \( f \) reverses the orientation (see [21]). Then

\[ i_{\mathbb{R}^2}(f^k, p) = \begin{cases} 1 - \delta & \text{if } k \text{ odd} \\ 1 - (2q + \delta) & \text{if } k \text{ even} \end{cases} \]

where \( q \) and \( \delta \) are the number of orbits of period 2 and 1 of \( \overline{f} \) in \( \{q_1, \ldots, q_m\} \).

More results about the computation of the fixed point index for a fixed point \( p \) can be obtained in [24], [19], [14], [9], [20], [22], [11], [8], [9] and [16].

The aim of this paper is to extend the above result by computing the fixed point index \( i_{\mathbb{R}^2}(f^k, K) \) for \( K \) an isolated invariant continuum. The techniques we employ are based on Conley index ideas. From the results of [16] it follows that this sequence is periodic.

We will prove, in a constructive way, the existence of certain special isolating blocks and index pairs, which we call strong filtration pairs, associated to an isolated invariant continuum \( K \) of a homeomorphism \( f : U \subset \mathbb{R}^2 \to f(U) \subset \mathbb{R}^2 \), with \( U \) an open set. The isolating block, \( N \), of \( K \) is a connected manifold. \( N \) and \( K \) will decompose the plane in the same number of components. This construction permit us to prove the Main Theorems which compute the fixed point indices of the iterations of a homeomorphism \( f \) at \( K \).

**Main Theorem 1.** Let \( f : U \subset \mathbb{R}^2 \to f(U) \subset \mathbb{R}^2 \) be a homeomorphism and let \( K \) be an isolated invariant continuum. Then,

\[ i_{\mathbb{R}^2}(f^k, K) = \begin{cases} 2 - C(k) - P(k) & \text{if } f^k \text{ is orientation preserving} \\ -C'(k) - P(k) & \text{if } f^k \text{ is orientation reversing} \end{cases} \]

where \( C(k) \geq 0 \) is the number of components \( C \) of \( \mathbb{R}^2 \setminus K \) such that \( f^k(U \cap C) \subset C \). The integers \( C'(k) \) (\( k \) odd) are also defined only in terms of the above components as the number of them which are exit regions for \( f^k \) in a
neighborhood of $K$ minus the cardinal of the rest. The integers $P(k) \geq 0$ depend on the behavior of $f^k$ in the exit set of $N$.

Let us observe that, if $k$ is even or $f$ is orientation preserving, the index is $i_{\mathbb{R}^2}(f^k, K) = 2 - C(k) - P(k) \leq 2$. On the other hand, if $k$ is odd and $f$ is orientation reversing, $i_{\mathbb{R}^2}(f^k, K) = -C'(k) - P(k)$.

**Corollary 1.** Under the above conditions, if $f$ is orientation preserving, $i_{\mathbb{R}^2}(f^k, K) \leq 2$. Moreover, there exists $k_0 \in \mathbb{N}$ such that $i_{\mathbb{R}^2}(f^{nk_0}, K) \leq 1$ for every $n \in \mathbb{N}$.

**Corollary 2.** Given any homeomorphism $f$, if $K$ decomposes the plain into three or more components, there exists $k_0 \in \mathbb{N}$ such that $i_{\mathbb{R}^2}(f^{k_0}, K) < 0$. Consequently $f$ has a periodic orbit in $K$ for the orientation preserving and for the orientation reversing cases.

Note that a stronger version of the above result of existence of periodic orbits is well known to be true in the orientation reversing case (see [13] for example). However, we find periodic points of non-zero index.

The knowledge of the structure of the set of periodic orbits of homeomorphisms of the sphere is an interesting problem that was stated explicitly by Le Calvez in his lecture in the ICM 2006. The question of whether $cl(\text{Per}(f))$ is isolated has been studied in [23] with restrictions on the homeomorphism (area-preserving). The next corollary is valid for arbitrary homeomorphisms but we consider just invariant continua containing $\text{Per}(f)$.

**Corollary 3.** Given a homeomorphism $f : S^2 \to S^2$, if $K \subsetneq S^2$ is an invariant continuum which contains $\text{Per}(f)$, $K$ is not isolated. On the other hand, if $K$ has a finite amount of connected components, $K = \bigcup_{i=1}^n K_i$, and for each $K_i$, $S^2 \setminus K_i$ has no invariant components $U_{i,j}$ for some $f^{m_{i,j}}$ which are locally attracted to $K$ or repelled from $K$ for $f^{m_{i,j}}$ ($f^{m_{i,j}}(U_{i,j} \cap U_K) \subseteq U_{i,j} \cap U_K$ and $f^{m_{i,j}}(U_{i,j} \cap U_K) \supseteq U_{i,j} \cap U_K$ for every neighborhood $U_K$ of $K$), then $K$ is not isolated.

When $K$ is not locally maximal but $\mathbb{R}^2 \setminus K$ has a finite number of components, the computation of the index could be made by using Carathéodory’s prime ends techniques but we’re not treating this situation here.

The structure of the article is the following. In section 2 we introduce the Conley index techniques we will need for our particular setting. Section 3 and 4 are devoted to prove the Main Theorem. In section 5 we see a theorem which relates the fixed point indices with the local dynamics of $f$ in a neighborhood of $K$.

The reader is referred to the text of [3], [4], [12] and [18] for information about the fixed point index theory.
2. Preliminaries. Conley index and construction of adequate filtration pairs

Let $U \subset X$ be an open set. By a \textit{local semidynamical system} we mean a locally defined continuous map $f : U \to X$. We say that a function $\sigma : \mathbb{Z} \to X$ is a \textit{solution to $f$ through $x$ in $N \subset X$} if $f(\sigma(i)) = \sigma(i + 1)$ for all $i \in \mathbb{Z}$, $\sigma(0) = x$ and $\sigma(i) \in N$ for all $i \in \mathbb{Z}$. The \textit{invariant part of $N$, $\text{Inv}(N, f)$}, is defined as the set of all $x \in N$ that admit a solution to $f$ through $x$ in $N$, i.e. the set of all $x \in N$ such that there is a full orbit $\gamma$ such that $x \in \gamma \subset N$. $\text{Inv}^+ (N, f)$ (respectively $\text{Inv}^- (N, f)$) denotes the set of all $x \in N$ such that $f^n(x) \in N$ for every $n \in \mathbb{N}$ (respectively $f^{-n}(x)$ is well defined and belongs to $N$ for every $n \in \mathbb{N}$).

Given $A \subset B \subset N$, $\text{cl}(A), \text{cl}_B(A), \text{int}(A), \text{int}_B(A), \partial(A)$ and $\partial_B(A)$ will denote the closure of $A$, the closure of $A$ in $B$, the interior of $A$, the interior of $A$ in $B$, the boundary of $A$ and the boundary of $A$ in $B$.

A compact set $K \subset X$ is \textit{invariant} if $f(K) = K$. An invariant compact set $K$ is \textit{isolated with respect to $f$} if there exists a compact neighborhood $N$ of $K$ such that $\text{Inv}(N, f) = K$. The neighborhood $N$ is called an \textit{isolating neighborhood of $K$}.

A compact set $N$ is called \textit{isolating block} if $f(N) \cap N \cap f^{-1}(N) \subset \text{int}(N)$. If $K = \text{Inv}(N, f)$ then $N$ is an isolating neighborhood of $K$ and we will say that $N$ is an isolating block of $K$.

We consider the \textit{exit set of $N$} to be defined as

$$N^- = \{ x \in N : f(x) \notin \text{int}(N) \}.$$  

Let $f : U \subset \mathbb{R}^2 \to f(U) \subset \mathbb{R}^2$ be a homeomorphism, with $U$ open, and let $K \subset U$ be an isolated invariant continuum. Given $k \in \mathbb{N}$ we want to compute the fixed point index of $f^k$ in $K$, i.e. the fixed point index of each $f^k$ in a small enough neighborhood of $K$.

We begin this study with the choice of an adequate isolating block $N$ of $K$ with topological properties which give us information about the dynamical behavior of $f$ in a neighborhood of $K$.

Among all compact connected smooth manifolds (discs with a finite amount of holes), isolating blocks for $K$, we take $N$ to be one of them which decomposes the plane in the minimum possible number of bounded components. Let $\{D_1, \ldots, D_p\}$ be the components of $\text{int}(N)$ and let $D(N) = N \cup (\bigcup_{i=1}^p D_i)$ be the disc obtained by filling the holes of $N$.

If we consider $K \subset N \subset S^2$, the set $I(K) = S^2 \setminus K$ is an open subset of $S^2$ and $S^2 \setminus \text{int}(N)$ has $p + 1$ components $\{D_0, D_1, \ldots, D_p\}$ with $D_0$ unbounded. Let us denote $\{I(K)_i\}$ the connected components of $I(K)$ with $I(K)_0$ the component which contains $D_0$. 

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Take $f^* : S^2 \to S^2$ to be a homeomorphism which extends $f|_N : N \to f(N)$.

**Lemma 1.** If $I(K)_j \subset I(K)$, then $I(K)_j$ contains a component $D_j \in \{D_0, \ldots, D_p\}$. As a consequence, $I(K)$ decomposes into a finite number of components on which $f^*$ acts as a permutation.

**Proof.** Let us suppose that there are two components $I(K)_j \subset I(K)_j$. Then, each image $f^{\ast}(I(K)_j) \subset I(K)$ is contained in $N$. 

In fact, if there exists $D_j \subset I(K)_j = f^*(I(K)_j)$, since $N$ is an isolating block, $f(\partial(D_j)) \subset \text{int}(D_j) \subset I(K)_j = f^*(I(K)_j)$. On the other hand, it is immediate to see that $f^{-1}(\partial(D_j)) \subset N$ and then $f(\partial(D_j)) \subset \text{int}(D_j) \subset I(K)_j = f^*(I(K)_j)$. Therefore all the images by $f^*$ of $I(K)_j$ have discs of the finite family $\{D_0, \ldots, D_p\}$. Then, there exists $n$ such that $(f^*)^n(I(K)_j) = I(K)_j$, and it must contain a component of $S^2 \setminus N$, but this is a contradiction.

Then, the set $K \cup d(\bigcup_{n \in \mathbb{Z}}(f^*(I(K)_j))) \neq K$ is an invariant continuum contained in $N$. This contradicts the fact that $\text{Inv}(N, f) = K$. \hfill $\Box$

**Lemma 2.** Let $D_j \subset I(K)_j$. Then

$$f^*(D_j) \not\subset N \text{ and } (f^*)^{-1}(D_j) \not\subset N.$$

**Proof.** Let us prove that $f^*(D_j) \not\subset N$ (the other statement has an analogous proof). If $f^*(D_j) \subset N$, since $f$ and $f^{-1}$ are defined in the isolating block $N$, then $D_j \subset U$. On the other hand, $(f^*)^{-1}(D_j) \subset \text{int}(D_i) \subset I(K)_i = (f^*)^{-1}(I(K)_j)$ with $D_i \neq D_j$. The set $N' = N \cup D_j \subset U$ is an isolating block but $N'$ has one hole less than $N$ which is a contradiction. \hfill $\Box$

**Lemma 3.** $I(K)_j$ contains a unique component $D_j$.

**Proof.** Let us suppose that there are two components $D_{j,1} \neq D_{j,2}$ in $I(K)_j$ and let $D^0 = \bigcup_{i=0}^p D_i \subset S^2$. Let $B_j$ be a topological disc with $D_{j,1} \cup D_{j,2} \subset$
$B_j \subset I(K)_j$. We consider a finite covering of $B_j$, in $I(K)_j$, formed by closed balls $\{B(x)\}_{x \in \mathcal{F}}$, with $\mathcal{F}$ a finite subset of $B_j$, and such that for all $B(x)$ there exists some $n_x \in \mathbb{Z}$ such that $f^{n_x}(B(x)) \subset \text{int}(D^0)$. Let $n_0$ be a natural number such that $|n_x| \leq n_0$ for all $x \in \mathcal{F}$.

Our goal is to construct an isolating block of $K$, $N_{n_0}$, with similar properties than $N$ and such that $S^2 \setminus N_{n_0}$ has less components than $S^2 \setminus N$ and we will arrive to a contradiction.

Define $E^0 = D^0 \cup V_1 \cup V_{-1} \subset S^2$, with $V_i$ compact manifolds homeomorphic to $(f^*)^i(D^0)$, $V_i \subset \text{int}((f^*)^i(D^0))$ for $i \in \{1, -1\}$. Furthermore, $V_i$ must be transversal to $D^0$, close enough to $(f^*)^i(D^0)$ and such that if $x \in \mathcal{F}$ with $(f^*)^i(B(x)) \subset \text{int}(D^0)$, then $B(x) \subset V_{-i}$.

Let $D^1$ be the set obtained by filling all the components of $S^2 \setminus E^0$ which do not contain $K$. Using the above lemma, one can choose $V_1$ and $V_{-1}$ such that $D^1$ has at most $p + 1$ components.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{DIAGRAM2.png}
\caption{Figure 2}
\end{figure}

Let us define

$$N_1 = N \setminus \text{int}(D^1)$$

Note that $K \subset \text{int}(N_1) \subset \text{int}(N)$ and $D^0 \subset D^1$. It is easy to see that $N_1$ is an isolating block of $K$ and a compact, connected manifold such that $S^2 \setminus N_1$ has at most $p + 1$ components. The set $N_1$ and the components $D^1$ are in the conditions of Lemma 2.

If we repeat the last construction $n_0$ times, we get the pairs of manifolds and holes $\{(N_1, D^1), \ldots, (N_{n_0}, D^{n_0})\}$ with $D^0 \subset D^1 \subset \cdots \subset D^{n_0}$ and $N_{n_0} \subset \cdots \subset N_1 \subset N$. Each $D^n$ is a finite union of discs and if $D^m \subset D^n$, each component of $D^m$ contains, at least, a component of $D^n$. Since $D^0$ has $p + 1$
components and $D_{j,1} \cup D_{j,2} \subset B_j \subset \bigcup_{x \in F} B(x) \subset D^n$, then $D^n$ has less than $p + 1$ components.

Finally, $N_{n_0}$ is a compact connected manifold, an isolating block of $K$ and $S^2 \setminus N_{n_0}$ has less than $p + 1$ components.

\[\square\]

**Definition 1.** [7] Let $K$ be a compact isolated invariant set and suppose $L \subset N$ is a compact pair contained in the interior of the domain of $f$. The pair $(N, L)$ is called a *filtration pair* for $K$ provided $N$ and $L$ are each the closure of their interiors and

1. $\text{cl}(N \setminus L)$ is an isolating neighborhood of $K$.
2. $f(\text{cl}(N \setminus L)) \subset \text{int}(N)$ and
3. $f(L) \cap \text{cl}(N \setminus L) = \emptyset$.

In the next proposition we prove the existence of strong filtration pairs.

**Proposition 1.** Let $f : U \subset \mathbb{R}^2 \rightarrow f(U) \subset \mathbb{R}^2$ be a homeomorphism, and let $K$ be an isolated invariant continuum. Then there exists a compact pair $(N, L)$, which we call strong filtration pair, where $N$ is a compact connected two-manifold, isolating block of $K$, with a minimum set of holes $\{D_1, \ldots, D_p\}$, and $L = L_1 \cup \cdots \cup L_m \subset N$ is a finite union of compact connected two-manifolds with, at most, one hole. The pair $(N, L)$ is such that:

1. $\partial_N(L_i)$ is an arc with $\partial_N(L_i) \cap \partial(N) = \{a_i, b_i\}$ two points, or $\partial_N(L_i)$ is a Jordan curve.
2. $\text{cl}(N \setminus L)$ is an isolating neighborhood of $K$.
3. $f(\text{cl}(N \setminus L)) \subset \text{int}(N)$.
4. For all $L_i$ there exists a set $B_{i_j}$ which is a disc or an annulus, such that $\partial_N(L_i) \subset B_{i_j} \subset L_i$, $B_{i_j} \cap N^- \neq \emptyset$ and $f(B_{i_j}) \cap \text{cl}(N \setminus L) = \emptyset$.

**Proof.** Let $N$ be a compact connected manifold, isolating block of $K$, such that $S^2 \setminus N$ has the minimum possible number of components $\{D_1, \ldots, D_p\}$. Following the steps of the proof of Theorema 3.7 in [7], we obtain a pair $(N, J)$, $J = J_1 \cup \cdots \cup J_n$ is a finite union of two-dimensional compact connected manifolds such that the pair $(N, J)$ is a filtration pair for $K$, with $J$ a submanifold of $N$, and $J_i \cap N^- \neq \emptyset$ for all $i \in \{1, \ldots, n\}$.

If we add to $J$ the holes of $J$ which are in $N$ and not intersect $K$, we obtain a compact manifold denoted by $D(J)$. Let us consider the pair $(N, D(J))$. Each $D(J_i)$ (defined as $D(J)$) has at most one hole. There are only three possible situations:

a) $J_i \subset I(K)_j$, $j \neq 0$, a connected component of $I(K)$. In this case $D(J_i)$ has, at most, one hole. If this hole exists, it contains $D_j$, the hole of $N$ in $I(K)_j$.

b) $J_i \subset I(K)_0$. Then $D(J_i)$ has only one hole which contains $K$.

c) In the remaining cases $D(J_i)$ has no holes.
We can suppose that $D(J) = D(J_1) \cup \cdots \cup D(J_n)$ are disjoints and left to the reader to verify that the pair $(N, D(J))$ is a filtration pair for $K$.

Let us define $L_i = N \setminus c.c.(N \setminus D(J_i), K)$, where $c.c.(N \setminus D(J_i), K)$ is the connected component of $N \setminus D(J_i)$ which contains $K$. Let $L = \bigcup_{i=1}^n L_i = \bigcup_{i=1}^m L_i$ with $m \leq n$, such that $\bigcup_{i=1}^m L_i$ is a disjoint union of discs with, at most, one hole.

The pair $(N, L)$ is the pair of the proposition and the proof of its properties is easy to check.

Let $f : U \subset \mathbb{R}^2 \rightarrow f(U) \subset \mathbb{R}^2$ be a homeomorphism, and let $K$ be a compact connected isolated invariant set. Let us consider a strong filtration pair $(N, L)$ as in the last proposition.

**Remark 1.** If $\partial N(L_j)$ is homeomorphic to $S^1$, then $L_j$ has a hole. Furthermore, $L_j \subset I(K)_0$ contains $K$ in its hole and $\partial(L_j) = \partial N(L_j) \cup \partial(D(N))$, or $L_j \subset I(K)_i$ for some $i \neq 0$ and the hole of $L_j$ is $D_i$. On the other hand, if $\partial N(L_j)$ is an arc, then $L_j$ is a disc without holes. Therefore, $cl(N \setminus L)$ is a disc with the same amount of holes than $N$.

Let $cl(N \setminus L)/\sim = N_L$ be the quotient space obtained by identifying each $\partial N(L_i) \subset cl(N \setminus L)$ to a point $q_i$. Take the projection $\pi : cl(N \setminus L) \rightarrow N_L$ and a retraction $r : N \rightarrow cl(N \setminus L)$ with $r(x) = x$ if $x \in cl(N \setminus L)$ and $r$ retracts each $L_i$ to $\partial N(L_i)$.

Now define $f' = \pi \circ r \circ f \circ \pi^{-1}, f' : N_L \setminus \{q_1, \ldots, q_m\} \rightarrow N_L$. The map $f'$ is continuous and, in a neighborhood of $K$, $f' \equiv f$. Since $f(\partial N(L_i)) \subset int(L_j)$, $f'$ admits a unique continuous extension.

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such that $\mathcal{F}(U'(q_i)) = q_j$ for a neighborhood $U'(q_i)$ of $q_i$.

We have that $\mathcal{F}((q_1, \ldots, q_m)) \subset \{q_1, \ldots, q_m\}$ if and only if $f(\partial N(L_i)) \subset int(L_j)$. Obviously $Fix(\mathcal{F}) \subset K \cup \{q_1, \ldots, q_m\}$ and, since $cl(N \setminus L)$ is an isolating neighborhood of $K$, $Fix((\mathcal{F})^k) \subset K \cup \{q_1, \ldots, q_m\}$.

By applying $f^*$ to the family $\{I(K)_0, \ldots, I(K)_p\}$ we obtain a finite union of cycles. We have $p + 1 = t_1 + \cdots + t_i$, with $\{t_1, \ldots, t_i\}$ the lengths of the cycles. Let us change the notation of the open sets $\{I(K)_i\}_{i=0}^p$ and the holes $\{D_i\}_{i=0}^p$ by $\{I(K)_{i,j}\}_{j=1}^{t_i}$ and $\{D_{i,j}\}_{j=1}^{t_i}$ where $D_{i,j} \subset I(K)_{i,j}$ and each family $\{I(K)_{i,j}\}_{j=1}^{t_i}$, with $i \in \{1, \ldots, l\}$, is a cycle of length $t_i$. Let us observe that $(f^*)^{t_i} : I(K)_{i,j} \to I(K)_{i,j}$ with $i \in \{1, \ldots, l\}, j \in \{1, \ldots, t_i\}$.

Remark 2. a) Given a cycle $\{I(K)_{i,j}\}$, if $I(K)_{i,1}$ contains a component of $L$, $L_{h_1}$, with $D_{i,1}$ the hole of $L_{h_1}$, then there exists $L_{h_j} \subset I(K)_{i,j}$ with $D_{i,j}$ the hole of $L_{h_j}$. Moreover, there exists only one component of $L$, $L_{h_j}$, in each $I(K)_{i,j}$.

b) Given a cycle $\{I(K)_{i,j}\}$, if $I(K)_{i,1} \cap L = \emptyset$, then $I(K)_{i,j} \cap L = \emptyset$.

Let us observe that $\partial(N \setminus L)/\sim$ is a finite and disjoint union of Jordan curves and points.

Definition 2. Let $\theta = \{p_1, \ldots, p_s\} \subset \{q_1, \ldots, q_m\}$ be such that $\mathcal{F}(\theta) = \theta$. We say that $p_i, p_j \in \theta$ are adjacent in $\theta$ if there is an arc $\gamma \subset \partial(N \setminus L)/\sim$ such that $\gamma \cap \theta = \{p_i, p_j\}$.

Lemma 4. (21) Let $\theta = \{p_1, \ldots, p_s\} \subset \{q_1, \ldots, q_m\}$ such that $\mathcal{F}(\theta) = \theta$. If $p_i, p_j$ are adjacent in $\theta$, then their images by $\mathcal{F}$, $p_{i+1}$ and $p_{j+1}$ are adjacent in $\theta$.

For a proof of this result see Proposition 1 of [21].


In this section we consider an orientation preserving homeomorphism $f : U \subset \mathbb{R}^2 \to f(U) \subset \mathbb{R}^2$.

Proposition 2. If $f$ is orientation preserving and $\{I(K)_{i,j}\}_{j=1}^{t_i}$ is a cycle, the set of periodic orbits of $\mathcal{F}|_{(\pi\text{or})'(\cup_{i=1}^{t_i} \partial(D_{i,j}))}$ is such that all its orbits have the same period $r_i = n_i t_i$ for some $n_i \in \mathbb{N}$.

Proof. Let us fix an orientation in $I(K)_{i,1} \cap \partial(N \setminus L) \simeq S^1$. The Jordan curve $f(I(K)_{i,1} \cap \partial(N \setminus L)) \subset I(K)_{i,2}$ bounds $D_{i,2} \subset I(K)_{i,2}$ preserving orientation.

In case a) of Remark 2 we only have a periodic orbit of period $t_i$. 

In case b) of Remark 2 the result is obvious because \((\pi \circ r)(\bigcup_{j=1}^{t_1} \partial(D_{i,j})) \cap \{q_1, \ldots, q_m\} = \emptyset\).

In any other case, given two periodic orbits \(\theta_1 = \{p_{r_1}, \ldots, p_{r_1}\}\) and \(\theta_2 = \{p'_{r_1}, \ldots, p'_{r_1}\}\), by Lemma 4 it easy to see that \(r_1 = r_2 = n_i t_i\) for some \(n_i \in \mathbb{N}\).

\[\square\]

**Corollary 4.** In the conditions of the last proposition, and given \(k \in \mathbb{N}\), \((\mathcal{F})^k\) has fixed points in \((\pi \circ r)(\bigcup_{j=1}^{t_1} \partial(D_{i,j})) \subset \partial(N \setminus L)/ \sim\) if and only if \(k = r_i N\). Then, the number of fixed points is \(r_i q^i\), with \(q^i\) the number of periodic orbits of \(\mathcal{F}\) in \((\pi \circ r)(\bigcup_{j=1}^{t_1} \partial(D_{i,j}))\).

**Definition 3.** Let us decompose the set of lengths of cycles of \(I(K) \subset S^2\), \(\{t_1, \ldots, t_l\}\), into three disjoint sets \(t_A, t_R, t_S \subset \{t_1, \ldots, t_l\}\) such that \(t_A \cup t_R \cup t_S = \{t_1, \ldots, t_l\}\), with \(t_A\) the set of lengths of cycles corresponding to case b) of Remark 2 (attracting), \(t_R\) the set of lengths of cycles corresponding to case a) of Remark 2 (repelling), and \(t_S\) the rest of lengths of \(\{t_1, \ldots, t_l\}\) (hyperbolicity). It is obvious that \(r_j = q^i = 0\) for all \(t_j \in t_A\), \(r_j = t_j\) with \(q^i = 1\) for all \(t_j \in t_R\) and \(r_j = n_j t_j\) for all \(t_j \in t_S\).

The family of periods \(\{r_1, \ldots, r_l\}\) and the number of periodic orbits of each period \(\{q^1, \ldots, q^l\}\) permit us to compute the number of fixed points of \((\mathcal{F})^k\) in \(\{q_1, \ldots, q_m\}\) for all \(k \in \mathbb{N}\).

**Proposition 3.** If \(f\) is orientation preserving, then

\[i_N_L((\mathcal{F})^k, N_L) = 2 - \sum_{\substack{t_i \in t_A \cup t_S \\ k \in t_1 N}} t_i\]

**Proof.** If \(K\) is not a repeller, \(N_L\) is homeomorphic to a disc with a finite amount of holes. On the other hand, \(H_0(N_L) = \mathbb{Q}\), \(H_1(N_L) = \mathbb{Q} \oplus \cdots \oplus \mathbb{Q}\) with \(\sum_{t_i \in t_A \cup t_S} t_i = 1\) generators and \(H_2(N_L) = 0\). Since \(i_N_L((\mathcal{F})^k, N_L) = \Lambda((\mathcal{F})^k)\), from the study of the trace of \((\mathcal{F})^k : H_1(N_L) \to H_1(N_L)\) it is easy to obtain the value of the fixed point index (see Figure 5).

\[
\begin{align*}
\Lambda((\mathcal{F})^k) &= \begin{cases}
a \rightarrow b \\
- a - b - d - e
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\Lambda((\mathcal{F})^k) &= \begin{cases}
a \rightarrow b \\
- a - b - d - e
\end{cases}
\end{align*}
\]

Figure 5
If \( K \) is a repeller, \( N_L \simeq S^2 \) and \( H_0(N_L) = \mathbb{Q}, \ H_1(N_L) = 0, \ H_2(N_L) = \mathbb{Q} \). We obtain \( i_{N_L}(\mathcal{F})^k, N_L) = 2 \).

\[ \square \]

**Main Theorem 2. (Orientation preserving case)** Let \( f : U \subset \mathbb{R}^2 \rightarrow f(U) \subset \mathbb{R}^2 \) be an orientation preserving homeomorphism and let \( K \) be an isolated invariant continuum. Then,

\[ i_{\mathbb{R}^2}(f^k, K) = 2 - \sum_{t_i \in \mathcal{A} \cup R} t_i - \sum_{k \in \mathcal{N}} t_i n_i q_i^i \leq 2 \]

**Proof.** By the additivity property of the fixed point index,

\[ i_{N_L}(\mathcal{F})^k, N_L) = i_{\mathbb{R}^2}(f^k, K) + \sum_{q_i \in \{q_1, ..., q_m\}} i_{N_L}(\mathcal{F})^k, U(q_i)) \]

Since the value of each index in the last summation is 1 (\( \mathcal{F}^k \) is constant in a neighborhood of each \( q_i \)), we have

\[ i_{\mathbb{R}^2}(f^k, K) = 2 - \sum_{t_i \in \mathcal{A} \cup R} t_i - \sum_{r_i \in \{r_1, ..., r_l\}} r_i q_i^i \]

obtaining the result we are looking for.

\[ \square \]

**Remark 3.** (Proofs of Corollaries 1, 2 and 3) Let us observe that the sequence \( \{i_{\mathbb{R}^2}(f^k, K)\}_k \) is periodic and, if \( p \) is the number of holes of \( K \),

\[ 1 - p - \sum_{t_i \in \mathcal{S}} t_i n_i q_i^i \leq i_{\mathbb{R}^2}(f^k, K) \leq 2 \]

for all \( k \). If \( k \in \left( \prod_{t_i \in \mathcal{A} \cup R} t_i \right) \left( \prod_{t_i \in \mathcal{S}} n_i t_i \right) \mathbb{N} \), then \( i_{\mathbb{R}^2}(f^k, K) = 1 - p - \sum_{t_i \in \mathcal{S}} t_i n_i q_i^i \). This proves Corollary 1. Consequently, if \( p \geq 2 \), \( K \) has periodic orbits of period which divides \( \left( \prod_{t_i \in \mathcal{A} \cup R} t_i \right) \left( \prod_{t_i \in \mathcal{S}} n_i t_i \right) \) which proves Corollary 2. The proof of Corollary 3 is also easy. If \( f : S^2 \rightarrow S^2 \) is a homeomorphism with \( K \) an invariant continuum which contains \( \text{Per}(f) \), then \( K \) is not isolated because, if \( K \) were isolated, there should be \( k_0 \in \mathbb{N} \) with \( i_{\mathbb{R}^2}(f^{nk_0}, K) \leq 1 \). Then,

\[ 2 = i_{S^2}(f^k, S^2) = i_{\mathbb{R}^2}(f^{nk_0}, K) \leq 1 \]

and we obtain a contradiction. A similar argument permit us to prove the same result if \( K \) has a finite amount of connected components \( K = \bigcup_{i=1}^n K_i \) with each \( K_i \) in the conditions of the Corollary 3. If \( K \) were isolated, there should be \( k_0 \in 2\mathbb{N} \) such that each \( K_i \) is invariant and isolated for \( f^{k_0} \) and \( i_{\mathbb{R}^2}(f^{k_0}, K_i) \leq 0 \) (\( t_S \neq \emptyset \) for each \( K_i \)). Then
Let us see an example of index 2 with $U$ an open set which is not an open ball. Let $p = (p_1, p_2) \in U = \mathbb{R}^2 \setminus \{0\}$. We define the homeomorphisms:

- $S_1 : U \to U$ given by $S_1(p) = \frac{1}{||p||} p$.
- $S_2 : \mathbb{R}^2 \to \mathbb{R}^2$ given by $S_2(p_1, p_2) = (-p_1, p_2)$.
- $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(p) = ||p||p$.

The homeomorphisms $S_1$ and $S_2$ are orientation reversing and $f$ is orientation preserving. The map $g = f \circ S_2 \circ S_1 : U \to U$ is an orientation preserving homeomorphism. The unit circle $K = \{p : ||p|| = 1\}$ is an isolated invariant continuum for $g$ and it is easy to see if we consider the extension $\bar{g} : S^2 \to S^2$ that $i_{\mathbb{R}^2}(g, K) = 2$.

4. **Main theorem. Orientation reversing case**

**Proposition 4.** Let $f : U \subset \mathbb{R}^2 \to f(U) \subset \mathbb{R}^2$ be an orientation reversing homeomorphism and let $K$ be an isolated invariant continuum. Given a cycle $\{I(K)_{i,j}\}_{j=1}^{t_i}$, there are two possibilities:

a) $t_i$ is even. Then $\overline{f} |_{(\partial(D_{i,j}))}$ has $q^i$ periodic orbits of the same period $r_{i,1} = t_i$.

b) $t_i$ is odd. Then $\overline{f} |_{(\partial(D_{i,j}))}$ has $q_{i,1}^{1,1} \leq 2$ periodic orbits of period $r_{i,1} = t_i$ and $q_{i,2}^{1,2}$ periodic orbits of period $r_{i,2} = 2t_i$.

**Proof.** The case a) follows from Lemma 4. Let us see the case b). If $t_i$ is odd, the periodic orbits have period $r_{i,1} = t_i$ or $r_{i,2} = 2t_i$. In fact, take a periodic orbit $\theta = \{p_1, \ldots, p_r\}$ of period $r$. Let $p_1 < p_2$ be adjacent in $\theta$ (with the ordering given by the orientation of $\partial(N \setminus L) / \sim$). By Lemma 4, $(\overline{f})^{|i|}(p_1) > (\overline{f})^{|i|}(p_2)$ are adjacent in $\theta$. Let us observe that if $(\overline{f})^{|i|}(p_1) = p_2$, then $(\overline{f})^{|i|}(p_2) = p_1$ and $r = 2t_i$. Let us suppose that $r > 2t_i$, and let $p_3 \in \theta$, $p_3 \neq p_1$, such that $p_2 < p_3$ are adjacent in $\theta$. Then, by an induction argument we find a point of $\theta$ with period $\leq 2t_i$, which is a contradiction. Then $r \leq 2t_i$.

Let us prove that the number of periodic orbits of period $t_i$ is $\leq 2$. In fact, let $\theta_1, \theta_2$ be two periodic orbits of period $t_i$. If there is another periodic
orbit $\theta_3$ we obtain, by using Lemma 4 and the fact that $f$ is orientation reversing, that the period of $\theta_3$ is $> t_i$.

\[ \square \]

**Proposition 5.** If $f$ is orientation reversing, then

\[ i_{\mathbb{R}^2}(f^k, K) = 1 + (-1)^k - \sum_{t_i \in 1 \cup \mathbb{N}} (-1)^k t_i \]

The proof of this result is analogous to the proof of Proposition 3 except that now $f$ is orientation reversing. We leave it to the reader.

The family of lengths of cycles $\{t_1, \ldots, t_i\}$ decomposes into two disjoint sets: the set of even lengths $t_P$ and the set of odd lengths $t_I$.

**Main Theorem 3.** (Orientation reversing case) Let $f : U \subset \mathbb{R}^2 \to f(U) \subset \mathbb{R}^2$ be an orientation reversing homeomorphism and let $K$ be an isolated invariant continuum. Then

\[ i_{\mathbb{R}^2}(f^k, K) = 1 + (-1)^k - \sum_{t_i \in 1 \cup \mathbb{N}} (-1)^k t_i q_i + \sum_{t_i \in 2 \cup \mathbb{N}} 2t_i q_i \]

where each index of the last summation is 1. Then the result follows automatically.

\[ \square \]

**Remark 4.** If $f$ is orientation reversing, the sequence $\{i_{\mathbb{R}^2}(f^k, K)\}_k$ is periodic and, if $p$ is the number of holes of $K$,

\[ 1 - p - \sum_{t_i \in t_P \setminus t_S} n_i t_i q_i - \sum_{t_i \in t_I \setminus t_S} t_i q_i - \sum_{t_i \in t_I \setminus t_S} 2t_i q_i \leq i_{\mathbb{R}^2}(f^k, K) \leq 2 \]

for all $k$ even. If $k \in 2 \left( \prod_{t_i \in t_P \cup t_S} t_i \right) \left( \prod_{t_i \in t_S} n_i t_i \right) \mathbb{N}$, with $n_i = 1$ if $t_i \in t_I \setminus t_S$, then $i_{\mathbb{R}^2}(f^k, K) = 1 - p - \sum_{t_i \in t_P \cap t_S} n_i t_i q_i - \sum_{t_i \in t_I \cap t_S} t_i q_i - \sum_{t_i \in t_I \setminus t_S} 2t_i q_i$. Consequently, if $p \geq 2$, $K$ has periodic orbits of period which divides $2 \left( \prod_{t_i \in t_P \cup t_S} t_i \right) \left( \prod_{t_i \in t_S} n_i t_i \right)$ with $n_i = 1$ if $t_i \in t_I \cap t_S$.

**Remark 5.** Let $f : U \subset \mathbb{R}^2 \to f(U) \subset \mathbb{R}^2$ be a homeomorphism and let $K$ be a compact isolated invariant set with a finite amount of connected components $K = \bigcup_{i=1}^{m_1} K_{i,1} \cup \cdots \cup \bigcup_{i=1}^{m_n} K_{i,n}$, such that $f(K_{i,j}) = K_{i,j+1}$ and $f(K_{i,m_i}) = K_{i,1}$. Then by the additivity property of the fixed point index applied to $f^k$, $i_{\mathbb{R}^2}(f^k, K) = \sum_{k \in m_i} m_i i_{\mathbb{R}^2}(f^k, K_{i,1})$. 

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Corollary 6. Let \( f : S^2 \to S^2 \) be a homeomorphism and let \( K \) be a compact, invariant and proper set with a finite amount of connected components. Then \( f : S^2 \setminus K \to S^2 \setminus K \) is not minimal.

Proof. Our proof is analogous to the given in [15] for \( K \) a finite amount of points. If \( f|_{S^2 \setminus K} \) is minimal, \( K \) is an isolated invariant set which is not an attractor nor a repellor. For an adequate \( 2n \) we have that the components of \( K \) and \( S^2 \setminus K \) are invariant for the orientation preserving homeomorphism \( f^{2n} \), and, by Remarks 3 and 5 applied to \( f^{2n} \), \( i_{S^2}(f^{2n}, K) \leq 0 \), but it is not possible because \( 2 = \Lambda(f_n) = i_{S^2}(f^{2n}, S^2) = i_{S^2}(f^{2n}, K) \).

\( \square \)

5. Dynamical behavior and fixed point index.

In this section we study the relationship between the fixed point indices \( i_{S^2}(f^n, K) \) and the local dynamics of \( f \) in a neighborhood of \( K \). If \((N, L)\) is a strong filtration pair, define \( cl(N \setminus L)_K \) and \( R^2_K \) as the spaces obtained from \( cl(N \setminus L) \) and \( R^2 \) by the identification of \( K \) to a point \( [k] \) and define \( f : cl(N \setminus L)_K \to R^2_K \) as the map induced by \( f \) (with the same notation).

Let us observe that \( cl(N \setminus L)_K \) is the pointed union of a finite family of \( p + 1 \) discs, \( E_i \), where \( p + 1 \) is the number of components of \( R^2 \setminus K \). Denote \( cl(N \setminus L)_K = \bigvee_{i=1}^{p+1} E_i \) with \( E_i \cap E_j = [k] \) for all \( i \neq j \).

Theorem 1. Let \( K \) be an isolated continuum for a homeomorphism \( f : U \subset R^2 \to f(U) \subset R^2 \) which decompose the plane in \( p + 1 \) components and let \((N, L)\) be a strong filtration pair. Then there exist two families of closed discs, \( \{A_1, \ldots, A_s\}, \{R_1, \ldots, R_s\} \), in \( cl(N \setminus L)_K = \bigvee_{i=1}^{p+1} E_i \) and two families of continua without holes in \( cl(N \setminus L)_K \), \( \{S_1, \ldots, S_s\}, \{U_1, \ldots, U_s\} \), with

\[
\begin{align*}
    s &= -i_{S^2}(f^d, K) + 1 - p, \\
    a &= -i_{S^2}(f^d, K) + 2 - \sum_{t_i \in R \cup S} t_i - s, \\
    r &= -i_{S^2}(f^d, K) + 2 - \sum_{t_i \in A \cup R} t_i - s.
\end{align*}
\]

for \( d = 2 \left( \prod_{i \in t \setminus S} t_i n_i \right) \left( \prod_{i \in t \setminus A \cup R} t_i \right), \) \( n_i = 1 \) if \( f \) is orientation reversing and \( t_i \in t \setminus S \). These sets satisfy the following properties:

1. \( \bigcup_{i=1}^{s} S_i \subset K^+ \) and \( \bigcup_{i=1}^{s} U_i \subset K^- \). The set \( K^+ \) is the connected component of \( \text{Inv}^+(cl(N \setminus L)_K, f) \) which contains \([k]\) and the set \( K^- \) is the connected component of \( \text{Inv}^-(cl(N \setminus L)_K, f) \) which contains \([k]\).

2. \( S_i \cap S_j = U_i \cap U_j = S_i \cap U_j = [k] \), every \( S_i \subset E_k \) for some \( k = 1, \ldots, p+1 \) and every \( U_j \subset E_{k'} \) for some \( k' = 1, \ldots, p+1 \).

3. \( f^d(S_i) \subset S_i \), \( f^{-d}(U_i) \subset U_i \), \( \bigcap_{n \in \mathbb{N}} f^{nd}(S_i) = \bigcap_{n \in \mathbb{N}} f^{-nd}(U_i) = [k] \).

4. The sets \( \{S_i\}_i \) and \( \{U_i\}_i \), alternate in the circles of \( \partial(cl(N \setminus L)_K) \subset cl(N \setminus L)_K \).
5. \( f^d(A_i) \subset \text{int}(A_i), \ f^{-d}(R_j) \subset \text{int}(R_j) \) and 
\( \bigcap_{n \in \mathbb{N}} f^{nd}(A_i) = \bigcap_{n \in \mathbb{N}} f^{-nd}(R_j) = [k]. \) for all \( i = 1, \ldots, a \) and \( j = 1, \ldots, r. \)
Let $K_i = \bigcap_{n \in \mathbb{N}} (\mathcal{F})^{nd}(cl(A(p_i))), i = 1, \ldots, s$. Since $(\mathcal{F})^d(cl(A(p_i)) \subset cl(A(p_i))), K_i$ is a continuum which contains $p_i$ and $[k]$ with $(\mathcal{F})^d(K_i) = K_i \subset C_j$. Define

$$U_i = (\pi^{-1}_K(K_i) \cap K^-)$$

Let us construct the sets $S_i$. If $p_{i-1}, p_i \in \theta$ are adjacent in $\pi_K(\partial(cl(N \setminus L)))$, let $\gamma \subset \pi_K(\partial(cl(N \setminus L)))$ be an arc joining $p_{i-1}$ and $p_i$ with $\gamma \cap \theta = \{p_{i-1}, p_i\}$ and $\gamma \subset C_j$ for some disc $C_j$ of the pointed union $N_{L,K}$. There is a component $K_{p_i}$ of $\partial(A(p_i))$ separating $p_i$ from $\theta \setminus p_i$, $[k] \in K_{p_i}$ with $\lim_{n \to \infty} (\mathcal{F})^{nd}(x) = [k]$ for all $x \in K_{p_i}$, and such that $K_{p_i} \cap \gamma \neq \emptyset$. Let $B_i$ be the connected component of $C_j \setminus (K_{i-1} \cup K_i)$ containing $K_{p_i} \cap \gamma$. Define

$$S_i = (\pi^{-1}_K(B_i) \cup [k]) \cap K^+$$

It is easy to prove that the sets $U_i$ and $S_i$ satisfy the properties 1 to 4 of the theorem. The details are left to the reader.

The equality relating the number $s$ of sets $\{U_i\}$ and $\{S_i\}$ with the fixed point index, $s = -i_{\mathbb{R}^2}(f^d, K) + 1 - p$ follows, in the orientation preserving case, from the Main Theorem 1:

$$s = \sum_{t_i \in t_S} n_i t_i q^1 = -i_{\mathbb{R}^2}(f^d, K) + 1 - p.$$  

For the orientation reversing case we have, from the Main Theorem 2:

$$s = \sum_{t_i \in t_{p \cap t_S}} n_i t_i q^1 + \sum_{t_i \in t_{p \cap d_S}} t_i q^1 + \sum_{t_i \in t_{p \cap d_S}} 2t_i q^2 = -i_{\mathbb{R}^2}(f^d, K) + 1 - p.$$  

For every disc $E_i$ of the pointed union $\bigcup_{i=1}^{p+1} E_i$ such that $E_i \cap L = \emptyset$, define $A_i = E_i$ and for every $E_j$ such that $E_j \cap L$ is a circumference, define $R_j = E_j$. The proof of the property 5 is immediate.

The equalities relating $a$ and $r$ with the fixed point index follow from

$$a = \sum_{t_i \in t_{t}} t_i = -i_{\mathbb{R}^2}(f^d, K) + 2 - \sum_{t_i \in t_{t \cap t_S}} t_i - s.$$  

$$r = \sum_{t_i \in t_{R}} t_i = -i_{\mathbb{R}^2}(f^d, K) + 2 - \sum_{t_i \in t_{R \cap d_S}} t_i - s.$$  

$\square$

References


