Chapter 7

Simple Lie algebras

In this Chapter we present the general structural theory concerning complex semisimple Lie algebras and their corresponding real forms. The underlying idea is to generalise the procedure considered for the three dimensional Lie algebras and $\mathfrak{su}(3)$, specifically by the use of simultaneously diagonalisable linear operators that span a (maximal) Abelian subalgebra and the (quadratic) Casimir operators. This will lead to a structure (of crystallographic type) called root system that contains the essential structural information of the Lie algebra, allowing us in particular to determine the isomorphism classes of simple complex Lie algebras.

7.1 Some preliminaries

In this section, we review some important properties of linear operators that are required for our analysis, such as its diagonalisation and nilpotence properties, and the so-called Dunford decomposition of a linear operator (matrix) into a direct sum of semisimple and nilpotent operators.

7.1.1 *Basic properties of linear operators*

For clarity and completeness we recall some definitions and properties already mentioned in Chapter 2. An n-dimensional Lie algebra $\mathfrak{g}$ is generated by n elements $T_a, a = 1, \cdots, n$ obeying the commutation relations

$$\big[T_a, T_b\big] = i f_{ab}{}^c T_c \ , \tag{7.1}$$

and satisfying the Jacobi identity

$$\big[T_a, \big[T_b, T_c\big]\big] + \big[T_b, \big[T_c, T_a\big]\big] + \big[T_c, \big[T_a, T_b\big]\big] = 0 \ .$$

The latter identity can also be given in terms of the structure constants $f_{ab}{}^c$. Indeed, using (7.1) we obviously obtain

$$f_{bc}{}^d f_{ad}{}^e + f_{ca}{}^d f_{bd}{}^e + f_{ab}{}^d f_{cd}{}^e = 0 \ . \tag{7.2}$$

To the Lie algebra $\mathfrak{g}$ one can associate the Killing form defined by

$$g_{ab} = \kappa(T_a, T_b) = \mathrm{Tr}\Big(\mathrm{ad}(T_a)\mathrm{ad}(T_b)\Big),$$

where the adjoint representation is defined by

$$\mathrm{ad}(T_a)\cdot T_b = \big[T_b, T_a\big] \ , \quad \mathrm{ad}^k(T_a)\cdot T_b = \underbrace{\big[\cdots\big[T_b, T_a\big], T_a\big]\cdots, T_a\big]}_{k-\mathrm{times}} \quad (7.3)$$

(see Eq. (2.88)) and $\mathrm{ad}(T_a)_b{}^c = -if_{ab}{}^c$. Thus, we have

$$g_{ab} = -f_{ac}{}^d f_{bd}{}^c \ .$$

There are various types of Lie algebras, but in this Chapter only (semi)simple complex and real Lie algebras will be considered. Following the Cartan-Killing Theorem 2.15, a given Lie algebra is called semisimple *iff* its Killing form is non-degenerate. In this case, since the metric is non-degenerate, an inverse metric g^{ab} satisfying

$$g_{ac}g^{cb} = \delta_a{}^b \ ,$$

can be defined. The metric tensor and its inverse enable us to raise or lower indices. In particular, we can consider

$$f_{abc} = g_{cd}f_{ab}{}^d \ ,$$

which is fully antisymmetric

$$f_{abc} = -f_{bac} = -f_{acb} \ .$$

The first property is obvious and the second comes from the Jacoby identity. Indeed, we have

$$\begin{aligned} f_{abc} &= -f_{ab}{}^d \underbrace{f_{df}{}^e f_{ce}{}^f}_{g_{dc}} \underbrace{f_{ab}{}^d f_{fd}{}^e}_{\mathrm{Jacobi}} f_{ce}{}^f = -\Big(f_{bf}{}^d f_{ad}{}^e + f_{fa}{}^d f_{bd}{}^e\Big) f_{ce}{}^f \\ &= i\mathrm{Tr}\Big(\mathrm{ad}(T_b)\mathrm{ad}(T_a)\mathrm{ad}(T_c) - \mathrm{ad}(T_a)\mathrm{ad}(T_b)\mathrm{ad}(T_c)\Big) \ , \end{aligned}$$

which implies that f_{abc} is fully antisymmetric.

In addition, since for a simple complex Lie algebra we can always find a basis where the structure constants are real, the matrix elements in the adjoint representation are purely imaginary.

7.1.2 *Semisimple and nilpotent elements*

In the characterisation of the complex Lie algebras $\mathfrak{su}(2)$ and $\mathfrak{su}(3)$, we encountered two types of elements. For instance, for the Lie algebra $\mathfrak{su}(3)$ the elements H^1 and H^2 are diagonal and the elements $E_{\pm\alpha(i)}, i = 1, 2, 3$ (see Eq. (6.4)) are eigenvectors of the H's. We have also shown that the generators $E_{\pm\alpha(i)}$ are nilpotent, that is, they vanish at some power. The former elements are called semisimple and the latter nilpotent. It is however important to observe that the semisimple elements are given by a diagonal matrix only when the Lie algebra $\mathfrak{su}(3)$ is considered as a complex Lie algebra. Of course a similar classification for the elements of $\mathfrak{su}(2)$ occurs.

Having recalled these elementary properties which can be directly checked on (a complexification of) $\mathfrak{su}(3)$ (or of $\mathfrak{su}(2)$), one may naturally ask the question whether or not this decomposition (between nilpotent and semisimple elements) holds for any (semi)simple complex Lie algebra. In fact this decomposition remains true for any (semi)simple complex Lie algebra, and is due to the following

Proposition 7.1 (Dunford decomposition). *Let M be an $n \times n$ complex matrix. Then there exists a unique decomposition*

$$M = \Delta + N \ ,$$

such that

(1) Δ is diagonalisable;
(2) N is nilpotent;
(3) Δ and N commute as matrices: $\Delta M = M\Delta$.

In addition the matrices Δ and N are polynomial in M.

This means that considering a given semi(simple) complex Lie algebra $\mathfrak{g}$ with basis $T_a, a = 1, \cdots, n$, and considering the adjoint representation, the elements $\text{ad}(T_a)$ can be decomposed into a set of semisimple elements and a set of nilpotent elements. This is precisely this decomposition which allows to deduce a complete classification of simple complex Lie algebras.

In the theorem above, the matrix M is considered as a complex matrix. This is simply due to the fact that if a matrix can be diagonalised over the field of complex numbers, it may happen that it cannot be diagonalised over the field of real numbers. This simple observation applied in the context of simple Lie algebras means that the classification of real Lie algebras is more involved than the classification of complex Lie algebra. Historically, the complete classification of simple complex Lie algebras was first obtained by Killing, being corrected and completed by E. Cartan in his thesis, and subsequently extended to the classification of real simple Lie algebras by Cartan and Gantmacher. In this chapter we shall consider the complete classification of simple complex Lie algebras, and only some specific simple real Lie algebras will be considered, as the compact or the split real forms.

7.2 Some properties of simple complex Lie algebras

To obtain a classification of simple complex Lie algebras, following Proposition 7.1, we have to identify appropriate semisimple and nilpotent elements. A detailed analysis can be found *e.g.* in [Cornwell (1984b)]. For this chapter one can see the following references [Jacobson (1962); Gilmore (1974); Helgason (1978); Cahn (1984); Cornwell (1984b); Fuchs and Schweigert (1997); Georgi (1999); Frappat et al. (2000); Ramond (2010)].

7.2.1 *The Cartan subalgebra and the roots*

We extract from a complex Lie algebra $\mathfrak{g}$ the maximal set of generators $H^1, \cdots, H^r$ such that

(1) $\text{ad}(H^i)$, $i = 1, \cdots, r$ are diagonalisable;
(2) $[H^i, H^j] = 0$ for all $i, j = 1, \cdots, r$.

Such a maximal Abelian subalgebra of semisimple elements $\mathfrak{h} = \text{Span}(H^1, \cdots, H^r) \subset \mathfrak{g}$ is called a Cartan subalgebra and $r = \dim \mathfrak{h}$ is called the rank of the Lie algebra $\mathfrak{g}$. Obviously, the choice of the Cartan subalgebra is not unique, but it can be shown that all Cartan subalgebras are conjugate,[1] hence that all choices are equivalent [Helgason (1978)]. As we have previously seen, $\mathfrak{su}(2)$ is a rank one Lie algebra with Cartan subalgebra $\mathfrak{h} = \text{Span}(T_3)$ (see Eq. (5.7)) and $\mathfrak{su}(3)$ is a rank two Lie algebra with Cartan subalgebra $\mathfrak{h} = \text{Span}(T_3, T_8)$ (see Eq. (6.4)).

Since all the elements $\text{ad}(H^i)$ are diagonalisable and are pairwise commuting, they can be simultaneously diagonalised. Thus, one can find a basis of $\mathfrak{g} = \text{Span}(H^1, \cdots, H^r, E_{\alpha_{(1)}}, \cdots, E_{\alpha_{(n-r)}})$, called the Cartan-Weyl basis, such that the $(n-r)$ elements $E_{\alpha_{(1)}}, \cdots, E_{\alpha_{(n-r)}}$ are simultaneously eigenvectors of the H's:

$$\left[H^i, E_\alpha\right] = \alpha^i E_\alpha \ . \tag{7.4}$$

[1] For the case of real form Cartan subalgebras are no more conjugate.

For each element E_α, the commutation relations with the Cartan subalgebra are perfectly specified by the r numbers $\alpha^1, \cdots, \alpha^r$. All these numbers define an r-dimensional vector α called a root. We introduce the set of roots of the Lie algebra

$$\Sigma = \left\{\alpha_{(1)}, \cdots, \alpha_{(n-r)}\right\} ,$$

meaning that for any root $\alpha \in \Sigma$ there exists a nilpotent element in $\mathfrak{g}$ such that $[H^i, E_\alpha] = \alpha^i E_\alpha$.

The relation (7.4) is valid for the simple complex Lie algebra $\mathfrak{g}$ or for any of its real forms (if we allow however to have linear combinations with complex coefficients — see (5.6) and (6.4)). In particular for the real form corresponding to the compact Lie algebra all the generators of $\mathfrak{g}$, and in particular, the H's are Hermitean (see Eq. (7.34) below).[2] Thus their eigenvalues are real and consequently all the roots are real r-dimensional vectors. We thus consider now the real form of $\mathfrak{g}$ corresponding to its compact Lie algebra since the proofs are more easy.

Next, the fact that the Lie algebra $\mathfrak{g}$ is semisimple has strong consequences upon the roots

(1) if α is a root then $-\alpha$ is a root;
(2) for any root their is only one eigenvector;
(3) the only multiple of a root α which are a root are $\pm\alpha$.

The point 1 above is easy to prove, indeed if

$$[H^i, E_\alpha] = \alpha^i E_\alpha ,$$

by Hermitean conjugation, and using $(H^i)^\dagger = H^i$ we obtain

$$[H^i, E_\alpha^\dagger] = -\alpha^i E_\alpha^\dagger ,$$

and we can assume

$$E_\alpha^\dagger = E_{-\alpha} .$$

In particular, the difference $n - r$ is always an even number.

Properties 2 and 3 are more involved and come from the fact that the Killing form is non-degenerate, see *e.g.* [Cornwell (1984b)] [Appendix E, Sec. 7]. Note also that there is a nice physical demonstration in [Georgi (1999)] of these two latter points.

[2] If instead of considering the real form corresponding to the compact Lie algebra, we are considering the complex Lie algebra itself, the various proofs are more complicated. One can see *e.g.* [Cornwell (1984b)].

To any root $\alpha \in \Sigma$ the space

$$\mathfrak{g}_\alpha = \mathrm{Span}(E_\alpha) \ ,$$

is called the root-space associated to the root α and because of the point (2) above $\dim \mathfrak{g}_\alpha = 1$. We also introduce for latter convenience

$$\mathfrak{h} = \mathfrak{g}_0 \ ,$$

i.e., we identify the operators associated to the vector $\alpha = 0$ with the Cartan subalgebra. Of course now $\dim \mathfrak{g}_0 = r$.

Let now give some useful properties, of the roots and of their associated operator E_α. Now, considering two different roots $\alpha, \beta \in \Sigma$, the Jacobi identity applied to E_α, E_β, H^i implies

$$\big[H^i, \big[E_\alpha, E_\beta\big]\big] = (\alpha^i + \beta^i)\big[E_\alpha, E_\beta\big] \ .$$

As a consequence, the following situations can appear:

(1) if $\alpha + \beta = 0$, since H^i commutes with $\big[E_\alpha, E_{-\alpha}\big]$, $\big[E_\alpha, E_{-\alpha}\big]$ automatically belongs to the Cartan subalgebra;
(2) if $\gamma = \alpha + \beta \in \Sigma$, since there is only one eigenvector associated to γ, we have $\big[H_\alpha, H_\beta\big] \sim E_\gamma$;
(3) if $\alpha + \beta \neq 0$ and $\alpha + \beta$ is not a root, the commutator vanishes.

All this can be summarised in

$$\big[E_\alpha, E_\beta\big] = \begin{cases} \lambda_i H^i & \text{if } \alpha + \beta = 0 \ , \\ \mathcal{N}_{\alpha,\beta} E_\gamma & \text{if } \alpha + \beta \in \Sigma \ , \\ 0 & \text{if } \alpha + \beta \notin \Sigma \ . \end{cases} \tag{7.5}$$

Now for any $\alpha, \beta \in \Sigma$ there exists an integer n such that

$$\mathrm{ad}^n(E_\alpha) \cdot E_\beta = 0 \ .$$

Otherwise this would imply that $\beta + k\alpha \in \Sigma$ for any k in $\mathbb{N}$. But since Σ is finite, this is a contradiction. So $\mathrm{ad}(E_\alpha)$ is nilpotent. The Lie algebra decomposes then into

$$\mathfrak{g} = \underbrace{\mathfrak{h}}_{\text{semisimple elements}} \oplus \underbrace{\mathrm{Span}(E_\alpha, \alpha \in \Sigma)}_{\text{nilpotent elements}} = \mathfrak{h} \oplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha \ . \tag{7.6}$$

The basis $\{H^1, \cdots, H^r, E_\alpha, \alpha \in \Sigma\}$ is called the Cartan-Weyl basis of $\mathfrak{g}$.

Now for every pair of (different) roots $\alpha, \beta \in \Sigma$ there exist two integer numbers p and q such that

$$\begin{array}{l} \mathrm{ad}^q(E_\beta) \cdot E_\alpha \neq 0 \quad \text{and } \mathrm{ad}^{q+1}(E_\beta) \cdot E_\alpha = 0 \\ \mathrm{ad}^p(E_{-\beta}) \cdot E_\alpha \neq 0 \text{ and } \mathrm{ad}^{p+1}(E_{-\beta}) \cdot E_\alpha = 0 \ . \end{array} \tag{7.7}$$

Thus, the set of roots $\alpha-p\beta,\cdots,\alpha-\beta,\alpha,\alpha+\beta,\cdots,\alpha+q\beta$ called a β-chain through α belongs to Σ and we have

$$\alpha - p\beta \ \ \cdots\cdots \ \ \xleftarrow{E_{-\beta}} \alpha-\beta \xleftarrow{E_{-\beta}} \ \alpha \xrightarrow{E_\beta} \alpha+\beta \xrightarrow{E_\beta} \ \ \cdots\cdots \ \ \alpha+q\beta$$

We introduce the subspace

$$\mathfrak{g}_{\alpha,\beta} = \oplus_{k=-p}^{q} \mathfrak{g}_{\alpha+k\beta} \ .$$

Having defined the roots of simple Lie algebras, we can put an order relation, which for instance is taken to be the lexicographic order. Of course this order relation is not unique since *e.g.* it depends on the ordering $H^1,\cdots,H^r$ of the Cartan subalgebra. However, the properties of a given simple Lie algebra $\mathfrak{g}$ do not depend of this ordering. A root α is said to be positive if its first non-vanishing component is positive. The set of roots of $\mathfrak{g}$ decomposes then into positive and negative roots

$$\Sigma = \Sigma_+ \oplus \Sigma_- \ .$$

The decomposition of roots can even be refined, introducing the so-called simple roots. If a Lie algebra has dimension n and rank r, the $1/2(n-r)$ positive roots cannot be independent. For instance, for $\mathfrak{su}(3)$ we have seen that $\alpha_{(2)} = \alpha_{(1)} + \alpha_{(3)}$. A positive root is said to be simple if it is positive and cannot be obtained by a positive sum of simple roots. It can be shown that for a rank r simple Lie algebra one can identify r linearly independent simple roots denoted $\beta_{(1)},\cdots,\beta_{(r)}$. Let us emphasise again that any positive root $\alpha\in\Sigma_+$ decomposes uniquely into a sum of simple roots with positive integer coefficients

$$\alpha = n^i\beta_{(i)}, \ n^1,\cdots,n^r \in \mathbb{N} \ . \tag{7.8}$$

This is a direct and simple consequence of Eq. (7.5) which simply means that any operator associated to a positive non-simple root can be obtained through a multiple commutator of operators associated to simple roots. Note also that the simple roots depend on the choice of the Cartan subalgebra, but fortunately the properties of semisimple Lie algebras $\mathfrak{g}$ are independent of this choice. We shall show later on that the simple roots are linearly independent.

Finally, considering only the generator associated positive roots we define the Borel subalgebra

$$\mathfrak{b} = \mathfrak{h} \ \oplus_{\alpha\in\Sigma_+} \mathfrak{g}_\alpha \subset \mathfrak{g} \ ,$$

and its derived algebra

$$\mathfrak{b}' = \big[\mathfrak{b},\mathfrak{b}\big] = \oplus_{\alpha\in\Sigma_+}\mathfrak{g}_\alpha \subset \mathfrak{b} \subset \mathfrak{g} \ .$$

The Borel $\mathfrak{b}$ subalgebra is clearly solvable, since the derived series

$$\mathcal{D}^0\mathfrak{b} = \mathfrak{b} \subset \mathcal{D}^1\mathfrak{b} = \left[\mathcal{D}^0\mathfrak{b}, \mathcal{D}^0\mathfrak{b}\right] \subset \cdots \subset \mathcal{D}^k\mathfrak{b} = \left[\mathcal{D}^{k-1}\mathfrak{b}, \mathcal{D}^{k-1}\mathfrak{b}\right]$$

stops for some k, and $\mathfrak{b}'$ nilpotent because the central descending series

$$\mathcal{C}^0\mathfrak{b}' = \mathfrak{b}' \subset \mathcal{C}^1\mathfrak{b}' = \left[\mathfrak{b}', \mathfrak{b}'\right] \subset \cdots \subset \mathcal{C}^k\mathfrak{b}' = \left[\mathcal{C}^{k-1}\mathfrak{b}', \mathfrak{b}'\right] ,$$

stops for some k (see Sec. 2.11.2). The former corresponds (in any representation) to the set of upper triangular matrices and the latter to the set of strictly upper triangular matrices. These two properties (which are related since the derived algebra of a solvable algebra is always nilpotent) directly follows from the definition of simple roots, as any nilpotent operator associated with a non-simple root can always be given in terms of a multiple commutator of nilpotent operators associated with simple roots. This property has an interesting consequence on the decomposition of $\mathfrak{b}'$

$$\mathfrak{b}' = \oplus_k \mathfrak{g}_k^+ ,$$

where

$$\mathfrak{g}_k^+ = \left\{E_\alpha \ , \quad \alpha \in \Sigma_+ \ , \text{s.t.}, \ \alpha = n^i \beta_{(i)} \ , \text{with} \ \sum_{i=1}^r n^i = k\right\} .$$

A positive root which satisfies the property above is called a level k-root, and the subspace $\mathfrak{g}_k^+$ is spanned by the nilpotent generators associated to level k-roots. The level 1-roots just correspond to the set of simple roots.

7.2.2 *Block structure of the Killing form*

In order to obtain more concise information concerning the commutation relations of $\mathfrak{g}$, we now briefly focus on some relevant properties of the Killing form. We recall that

$$g_{ab} = \text{Tr}\Big(\text{ad}(T_a)\text{ad}(T_b)\Big) .$$

Due to the decomposition (7.6), this implies several simplifications of the Killing form:

(1) $\text{Tr}\Big(\text{ad}(E_\alpha)\text{ad}(E_\beta)\Big) = 0$ if $\alpha + \beta \neq 0$ and $\text{Tr}\Big(\text{ad}(E_\alpha)\text{ad}(H^i)\Big) = 0$. Consider the first identity. It is enough to show that $\text{ad}(E_\alpha)\text{ad}(E_\beta)$ has no diagonal element to prove that the trace vanishes. We thus consider the action of $\text{ad}(E_\alpha)\text{ad}(E_\beta)$ on an arbitrary element of $\mathfrak{g}$. Consider γ in Σ and denote $E_{\tilde{\gamma}} = E_\gamma, H^i$, $\tilde{\gamma} = \gamma, 0$. Compute

$$\Big[\big[E_{\tilde{\gamma}}, E_\beta\big], E_\alpha\Big] = \text{ad}(E_\alpha)\text{ad}(E_\beta) \cdot E_{\tilde{\gamma}} .$$

Using (7.5) we have

$$\begin{array}{ll} \mathrm{ad}(E_\alpha)\mathrm{ad}(E_\beta)\cdot E_{\tilde\gamma} \in \mathfrak{h} & \text{if } \alpha+\beta+\tilde\gamma = 0 \\ \mathrm{ad}(E_\alpha)\mathrm{ad}(E_\beta)\cdot E_{\tilde\gamma} \sim E_{\alpha+\beta+\tilde\gamma} & if\ \alpha+\beta+\tilde\gamma \in \Sigma \\ \mathrm{ad}(E_\alpha)\mathrm{ad}(E_\beta)\cdot E_{\tilde\gamma} = 0 & \text{if } \alpha+\tilde\beta+\gamma \notin \Sigma \end{array}$$

In the second case, since $\alpha+\beta \neq 0$, we conclude that $\mathfrak{g}_{\alpha+\beta+\tilde\gamma} \cap \mathfrak{g}_{\tilde\gamma} = \{0\}$. Thus there is no contribution to the trace. The other cases are analysed in a similar manner. This means that $\mathrm{Tr}\Big(\mathrm{ad}(E_\alpha)\mathrm{ad}(E_\beta)\Big) = 0$. The second equality is proved along the same lines.

(2) $\mathrm{Tr}\Big(\mathrm{ad}(H^i)\mathrm{ad}(H^j)\Big) = g^{ij}$ with g^{ij} a non-degenerate $r\times r$ matrix and $\mathrm{Tr}\Big(\mathrm{ad}(E_\alpha)\mathrm{ad}(E_{-\alpha})\Big) = 2p_\alpha \neq 0$. Indeed, if g^{ij} is singular or one of the $p_\alpha = 0$, the Killing form is degenerate, contradicting the simplicity of $\mathfrak{g}$.

Since $E_\alpha^\dagger = E_{-\alpha}$ we have $p_\alpha > 0$. Furthermore, the eigenvectors $E_\alpha \in \mathfrak{g}_\alpha$ are defined up to a non-vanishing scale factor. This means that the nilpotent element E_α can be chosen such that all the $p_\alpha = 1$. In the basis $\{H^1,\cdots,H^r, E_{\alpha_{(1)}}, E_{-\alpha_{(1)}},\cdots, E_{\alpha_{(\frac{n-r}{2})}}, E_{-\alpha_{(\frac{n-r}{2})}}\}$ when the $E_{\pm\alpha}$ are correctly normalised the Killing form becomes

$$g = \begin{pmatrix} g^{ij} & & & \\ & \begin{array}{|cc|} \hline 0 & 1 \\ 1 & 0 \\ \hline \end{array} & & \\ & & \ddots & \\ & & & \begin{array}{|cc|} \hline 0 & 1 \\ 1 & 0 \\ \hline \end{array} \end{pmatrix} . \tag{7.9}$$

This special form enables us to endow the root space with the scalar product

$$(\alpha,\beta) = \alpha^i\beta^j g_{ij} \ ,$$

with g_{ij} the inverse of g^{ij}

$$g^{ik}g_{kj} = \delta^i{}_k \ .$$

7.2.3 *Commutation relations in the Cartan-Weyl basis*

We now give the commutation relations of $\mathfrak{g}$ in the Cartan-Weyl basis.

(1) By definition in the Cartan subalgebra we obviously have

$$\left[H^i, H^j\right] = 0 \ . \tag{7.10}$$

(2) By definition of the roots we have

$$\left[H^i, E_\alpha\right] = \alpha^i E_\alpha \ . \tag{7.11}$$

(3) We have previously seen that

$$\left[E_\alpha, E_{-\alpha}\right] = \lambda_i H^i \ .$$

We now compute the coefficients λ_i. Applying H^j on both sides, going into the adjoint representation and taking the trace we obtain

$$\underbrace{\mathrm{Tr}\Big(\mathrm{ad}(H^j)\mathrm{ad}(E_\alpha)\mathrm{ad}(E_{-\alpha}) - \mathrm{ad}(H^j)\mathrm{ad}(E_{-\alpha})\mathrm{ad}(E_\alpha)\Big)}_{\substack{=\mathrm{Tr}\left(\left[\mathrm{ad}(H^j),\mathrm{ad}(E_\alpha)\right]\mathrm{ad}(E_{-\alpha})\right) \\ =\alpha^j\,\mathrm{Tr}\left(\mathrm{ad}(E_\alpha)\mathrm{ad}(E_{-\alpha})\right)=\alpha^j}}$$
$$= \lambda_i \underbrace{\mathrm{Tr}\Big(\mathrm{ad}(H^j)\mathrm{ad}(H^i)\Big)}_{=g^{ij}} \ .$$

We have used the expression of the Killing form (7.9), the cyclicity of the trace and (7.11). Thus,

$$\lambda_i = g_{ij}\alpha^i = \alpha_i \ ,$$

and

$$\left[E_\alpha, E_{-\alpha}\right] = \alpha_i H^i \ . \tag{7.12}$$

(4) If $\alpha + \beta \notin \Sigma$

$$\left[E_\alpha, E_\beta\right] = 0 \ . \tag{7.13}$$

(5) If $\alpha + \beta \in \Sigma$

$$\left[E_\alpha, E_\beta\right] = \mathcal{N}_{\alpha,\beta} E_{\alpha+\beta} \ . \tag{7.14}$$

The coefficients $\mathcal{N}$ have the following symmetry properties

$$\mathcal{N}_{\alpha,\beta} = -\mathcal{N}_{\beta,\alpha} = \mathcal{N}_{-\beta,-\alpha} \ . \tag{7.15}$$

and

$$\mathcal{N}_{\alpha,\beta} = \mathcal{N}_{\beta,-\alpha-\beta} = \mathcal{N}_{-\alpha-\beta,\alpha} \ . \tag{7.16}$$

The fact that the structure constants ${f_{ab}}^c$ are real allows to chose the coefficients $\mathcal{N}_{\alpha,\beta}$ to be real [Cornwell (1984b)]. This property, together with the fact that $E_\alpha^\dagger = E_{-\alpha}$ and the antisymmetry of the commutator, leads to (7.15). To prove (7.16) we consider three roots such that $\beta + \gamma + \alpha = 0$. Then

$$\big[E_\alpha, \big[E_\beta, E_\gamma\big]\big] = \alpha_i \mathcal{N}_{\beta,\gamma} H^i \ .$$

Taking cyclic permutations, and using the Jacobi identity we get

$$\Big(\alpha_i \mathcal{N}_{\beta,\gamma} + \beta_i \mathcal{N}_{\gamma,\alpha} + \gamma_i \mathcal{N}_{\alpha,\beta}\Big) H^i = 0 \ .$$

Multiplying by H^j, taking the trace and using $\gamma = -\alpha - \beta$ we obtain

$$\alpha^i \big(\mathcal{N}_{\beta,-\alpha-\beta} - \mathcal{N}_{\alpha,\beta}\big) + \beta^i \big(\mathcal{N}_{-\alpha-\beta,\alpha} - \mathcal{N}_{\alpha,\beta}\big) = 0 \ .$$

Since the roots α and β are arbitrary, this gives (7.16).

Finally, one can obtain the coefficients $\mathcal{N}_{\alpha,\beta}$. Since this is central in the following classification, we give the computation of those coefficients in several steps. In fact, as it is usually the case for Lie algebras, the intensive use of the Jacobi identity puts constraints upon the algebraic structure. In particular, for simple Lie algebras it allows to get the fundamental relation

$$2\frac{(\alpha,\beta)}{(\beta,\beta)} = p - q \ , \tag{7.17}$$

with p, q defined in (7.7), together with

$$\mathcal{N}^2_{\alpha,\beta} = \frac{1}{2} q(p+1)(\beta,\beta) \ . \tag{7.18}$$

We observe that this relation does not fix completely the sign of $\mathcal{N}_{\alpha,\beta}$. In fact this sign depends on the one hand on the relative sign of E_α, E_β, and on the other hand on the sign of $E_{\alpha+\beta}$. See for instance the commutation relations of $\mathfrak{su}(3)$. Anticipating to the fact that (β,β) is always a rational number, this shows that the $\mathcal{N}'^2$s are rational numbers. This relation will have a natural interpretation latter on (see (7.37)).

Step one

For $-p \le k \le q$ we have

$$\beta_i(\alpha^i + k\beta^i) = \mathcal{F}(k) - \mathcal{F}(k-1) \ , \ \text{with} \ \ \mathcal{F}(k) = \mathcal{N}_{\beta,\alpha+k\beta}\mathcal{N}_{-\beta,-\alpha-k\beta} \ . \tag{7.19}$$

Considering the Jacobi identity for $E_\beta, E_{-\beta}, E_{\alpha+k\beta}, -p \leq k \leq q$ gives

$$\underbrace{\underbrace{\mathcal{N}_{-\beta,\alpha+k\beta}}_{\substack{=\ \mathcal{N}_{-\alpha-(k-1)\beta,-\beta} \\ =-\mathcal{N}_{-\beta,-\alpha-(k-1)\beta},}}\mathcal{N}_{\beta,\alpha+(k-1)\beta}}_{=-\mathcal{F}(k-1)} + \underbrace{\underbrace{\mathcal{N}_{\alpha+k\beta,\beta}}_{=-\mathcal{N}_{\beta,\alpha+k\beta}}\ \underbrace{\mathcal{N}_{-\beta,\alpha+(k+1)\beta}}_{\substack{=\ \mathcal{N}_{-\alpha-k\beta,-\beta} \\ =-\mathcal{N}_{-\beta,-\alpha-k\beta}}}}_{=\mathcal{F}(k)} = \beta_i(\alpha^i + k\beta^i)\ , \tag{7.20}$$

where we have used $\mathcal{N}_{A,B} = \mathcal{N}_{-A-B,A}$ coming from (7.16) and $\mathcal{N}_{A,B} = -\mathcal{N}_{B,A}$ from (7.15). This gives the desired property.

Step two

We have

$$\mathcal{F}(k) = (k-q)\beta_i\Big[\alpha^i + \frac{1}{2}(k+q+1)\beta^i\Big]\ .$$

To prove this identity we first observe that if we take $k = q$ in (7.19), using that $[E_{\alpha+q\beta}, E_\beta] = 0$, the identity (7.20) leads to

$$\mathcal{F}(q-1) = -\beta_i(\alpha^i + q\beta^i)\ .$$

Now, by an induction argument, this gives for any $k = -p, \cdots, q$

$$\mathcal{F}(k) = (k-q)\beta_i\Big[\alpha^i + \frac{1}{2}(k+q+1)\beta^i\Big]\ , \tag{7.21}$$

as we now show. It is obvious to check that $\mathcal{F}(q) = 0$ and $\mathcal{F}(q-1) = -\beta_i(\alpha^i + q\beta^i)$. Assuming that $\mathcal{F}(k)$ is given by (7.21), using (7.19) it is direct to check

$$\mathcal{F}(k-1) = (k-1-q)\beta_i\Big[\alpha^i + \frac{1}{2}(k+q)\beta^i\Big]\ ,$$

which ends the proof.

Step three

We have the relations

$$2\frac{(\beta,\alpha)}{(\beta,\beta)} = p - q\ .$$

To prove this very important relation, we proceed as in step one but with $k = -p$. Since $[E_{-\beta}, E_{\alpha-p\beta}] = 0$, we have that

$$\mathcal{F}(-p-1) = (q+p+1)\Big(\beta_i\alpha^i + \frac{1}{2}(q-p)\beta_i\beta^i\Big) = 0\ .$$

The latter easily reduces to (7.17).

Step four

We have

$$\mathcal{N}^2_{\alpha,\beta} = \frac{1}{2} q(p+1)(\beta,\beta) \ . \tag{7.22}$$

This comes from $\mathcal{F}(0) = \mathcal{N}_{\beta,\alpha}\mathcal{N}_{-\beta,-\alpha} = -q(\beta,\alpha) - \frac{1}{2}q(q+1)(\beta,\beta)$ using (7.15) and (7.17) which expresses (α,β) in terms of (β,β).

We now summarise the commutation relations in the Cartan-Weyl basis

$$\begin{aligned} \left[H^i, H^j\right] &= 0 \ , \\ \left[H^i, E_\alpha\right] &= \alpha^i E_\alpha \ , \\ \left[E_\alpha, E_\beta\right] &= \begin{cases} \alpha_i H^i & \text{if } \alpha+\beta = 0 \ , \\ \mathcal{N}_{\alpha,\beta} E_{\alpha+\beta} & \text{if } \alpha+\beta \in \Sigma \ , \\ 0 & \text{if } \alpha+\beta \notin \Sigma . \end{cases} \end{aligned} \tag{7.23}$$

7.2.4 *Fundamental properties of the roots*

In the previous subsection we have obtained the relation (7.17) letting $\text{ad}(E_{\pm\beta})$ act on E_α. Of course, we can proceed on the opposite way, *i.e.*, with $\text{ad}(E_{\pm\alpha})$ acting on E_β. This means that for any two different roots α and β, there exist (p,q) and (p',q') such that

$$\begin{aligned} &\left.\begin{array}{l} \text{ad}(E_\beta)^{q+1}\cdot E_\alpha = 0 \\ \text{ad}(E_{-\beta})^{p+1}\cdot E_\alpha = 0 \end{array}\right\} \quad \Rightarrow \quad 2\frac{(\alpha,\beta)}{(\beta,\beta)} = p - q = n \ , \\ &\left.\begin{array}{l} \text{ad}(E_\alpha)^{q'+1}\cdot E_\beta = 0 \\ \text{ad}(E_{-\alpha})^{p'+1}\cdot E_\beta = 0 \end{array}\right\} \quad \Rightarrow \quad 2\frac{(\alpha,\beta)}{(\alpha,\alpha)} = p' - q' = n' \ . \end{aligned} \tag{7.24}$$

Taking the ratio of the two equations gives

$$\frac{n}{n'} = \frac{(\alpha,\alpha)}{(\beta,\beta)} > 0 \ , \tag{7.25}$$

and multiplying the two equations implies

$$\frac{(\alpha,\beta)^2}{(\alpha,\alpha)(\beta,\beta)} = \cos^2\theta_{\alpha,\beta} = \frac{1}{4}nn' \leq 1 \ ,$$

with $\theta_{\alpha,\beta}$ the angle between the two roots α and β. In addition if $\cos^2\theta_{\alpha,\beta} = 1$ this means that α and β are proportional, which is not possible. This considerably restricts the possible angles between the roots α and β, and the only possible solutions are given by (we do not orientate the angles, and make no distinction between θ and $-\theta$)

(1) $(n, n') = (0, 0)$, then $(\alpha, \alpha)/(\beta, \beta)$ is unspecified and

$$\theta_{\alpha,\beta} = \frac{\pi}{2} \ .$$

(2) $(n, n') = (1, 1)$ corresponding to $\sqrt{(\beta, \beta)} = \sqrt{(\alpha, \alpha)}$ and

$$\theta_{\alpha,\beta} = \frac{\pi}{3} \ ,$$

(3) $(n, n') = (-1, -1)$ corresponding to $\sqrt{(\beta, \beta)} = \sqrt{(\alpha, \alpha)}$ and

$$\theta_{\alpha,\beta} = \frac{2\pi}{3} \ ,$$

(4) $(n, n') = (1, 2)$ corresponding to $\sqrt{(\beta, \beta)} = \sqrt{2}\sqrt{(\alpha, \alpha)}$ and

$$\theta_{\alpha,\beta} = \frac{\pi}{4} \ ,$$

(5) $(n, n') = (-1, -2)$ corresponding to $\sqrt{(\beta, \beta)} = \sqrt{2}\sqrt{(\alpha, \alpha)}$ and

$$\theta_{\alpha,\beta} = \frac{3\pi}{4} \ ,$$

(6) $(n, n') = (1, 3)$ corresponding to $\sqrt{(\beta, \beta)} = \sqrt{3}\sqrt{(\alpha, \alpha)}$ and

$$\theta_{\alpha,\beta} = \frac{\pi}{6} \ ,$$

(7) $(n, n') = (-1, -3)$ corresponding to $\sqrt{(\beta, \beta)} = \sqrt{3}\sqrt{(\alpha, \alpha)}$ and

$$\theta_{\alpha,\beta} = \frac{5\pi}{6} \ ,$$

We summarise the results in Table 7.1.

Table 7.1 Relative length and angle between two roots.

n	n'	$\theta_{\alpha,\beta}$	$\frac{\sqrt{(\beta,\beta)}}{\sqrt{(\alpha,\alpha)}}$
0	0	$\frac{\pi}{2}$	unspecified
1	1	$\frac{\pi}{3}$	1
-1	-1	$\frac{2\pi}{3}$	1
1	2	$\frac{\pi}{4}$	$\sqrt{2}$
-1	-2	$\frac{3\pi}{4}$	$\sqrt{2}$
1	3	$\frac{\pi}{6}$	$\sqrt{3}$
-1	-3	$\frac{5\pi}{6}$	$\sqrt{3}$

7.2.5 *The Chevalley-Serre basis and the Cartan matrix*

Since the components of the roots $\alpha \in \Sigma$ are not necessarily integer numbers, the commutation relations in the Cartan-Weyl basis involves in general irrational numbers (see for instance for $\mathfrak{su}(3)$). There exists however one basis where all the commutation relations involve integer number: the Chevalley-Serre basis.

If we consider now two simple roots $\beta_{(i)}$ and $\beta_{(j)}$, the difference $\beta_{(i)} - \beta_{(j)}$ is either a positive root, a negative root or not a root at all. Suppose that $\beta_{(i)} - \beta_{(j)}$ is a positive root. Because of (7.8) we can write $\beta_{(i)} - \beta_{(j)} = n^k \beta_{(k)}$, with n_k positive. This contradicts the fact that the root $\beta_{(i)}$ is simple. We conclude that the difference of two simple roots $\beta_{(i)}$ and $\beta_{(j)}$ can never be a root. This in particular means that in (7.24) $p, p' = 0$ and

$$2\frac{(\beta_{(i)}, \beta_{(j)})}{(\beta_{(i)}, \beta_{(i)})} = 0, -1, -2, -3 \ . \tag{7.26}$$

Thus, the angle between two simple roots can only be $\pi/2, 2\pi/3, 3\pi/4$ and $5\pi/6$.

We define now the Cartan matrix by

$$A_{ij} = 2\frac{(\beta_{(i)}, \beta_{(j)})}{(\beta_{(i)}, \beta_{(i)})} \ , \tag{7.27}$$

which has the following properties[3]

$$\begin{aligned} (i) &: A_{ii} = 2 \ , \\ (ii) &: A_{ij} = 0 \ , -1, -2, -3 \ , \quad i \neq j \ , \\ (iii) &: A_{ij} = 0 \quad \Leftrightarrow \quad A_{ji} = 0 \ , \\ (iv) &: \det(A) \neq 0 \ . \end{aligned} \tag{7.28}$$

All the properties of A above are obvious, but the last one. To prove it we firstly show that the simple roots are linearly independent. Consider $\alpha = x^i \beta_{(i)}$ and show that

$$x^i \beta_{(i)} = 0 \quad \Rightarrow \quad x^i = 0 \ .$$

A priori the coefficients x^i can be either of the same sign or of both signs. If we suppose that all the $x^i \geq 0$ (the case $x^i \leq 0$ being identical), since all

[3]If in the definition of the Cartan matrix we replace (ii) by $A_{ij} \leq 0$ for $i \neq j$, and we suppress condition (iv), the matrix A is called a generalised Cartan matrix which allows to define the so-called Kac-Moody algebras [Moody (1968); Macdonald (1986); Kac (1990)].

the roots are positive $\alpha = 0$ *iff* all the x^i vanish. If we now assume that the coefficients are of both signs, we can write

$$\alpha = \sum_{x^i \geq 0} x^i \beta_{(i)} - \sum_{-x^i \geq 0} (-x^i)\beta_{(i)} = \alpha_+ - \alpha_- \ .$$

Now

$$(\alpha, \alpha) = (\alpha_+, \alpha_+) + (\alpha_-, \alpha_-) - 2(\alpha_-, \alpha_+) \geq (\alpha_+, \alpha_+) + (\alpha_-, \alpha_-)$$

because the scalar product of two different simple roots is negative and thus $-2(\alpha_-, \alpha_+) \geq 0$. Assuming that $\alpha = 0$ leads to

$$(\alpha_+, \alpha_+) = (\alpha_-, \alpha_-) = 0 \ ,$$

but since the scalar product is positive definite, we deduce that $\alpha_+ = \alpha_- = 0$. As before, these two relations imply that all coefficients x^i vanish.

We prove now the non-vanishing of the determinant. Assume that the simple roots are not linearly independent. Without loss of generality we can suppose

$$\beta_{(1)} = \sum_{i=2}^{r} k^i \beta_{(i)} \ .$$

Thus we have for the first column of the Cartan matrix

$$\begin{aligned} A_{i1} &= 2\frac{(\beta_{(i)}, \beta_{(1)})}{(\beta_{(i)}, \beta_{(i)})} = 2\sum_{j=2}^{r} k^j \frac{(\beta_{(i)}, \beta_{(j)})}{(\beta_{(i)}\beta_{(i)})} = \sum_{j=2}^{r} k^j A_{ij} \ , \quad i > 1 \ , \\ A_{11} &= 2 = 2\frac{(\beta_1, \beta_1)}{(\beta_1, \beta_1)} = 2\sum_{j=2}^{r} k^j \frac{(\beta_{(1)}, \beta_{(j)})}{(\beta_{(1)}\beta_{(1)})} = \sum_{j=2}^{r} k^j A_{1j} \ . \end{aligned}$$

Since the first column is a linear combination of the others the determinant of A vanishes. Which ends the proof.

Considering $\beta_{(1)}, \cdots, \beta_{(r)}$ the simple roots of $\mathfrak{g}$, for each simple root we introduce three different operators

$$\begin{aligned} h_i &= \frac{2}{(\beta_{(i)}, \beta_{(i)})} \beta_{(i)j} H^j \ , \\ e_i^+ &= \sqrt{\frac{2}{(\beta_{(i)}, \beta_{(i)})}} E_{\beta_{(i)}} \ , \\ e_i^- &= \sqrt{\frac{2}{(\beta_{(i)}, \beta_{(i)})}} E_{-\beta_{(i)}} \ . \end{aligned} \tag{7.29}$$

It is direct to check from (7.23) that

$$\begin{aligned} \left[h_i, e_i^{\pm}\right] &= \pm 2 e_i^{\pm} \ , \\ \left[e_i^{+}, e_i^{-}\right] &= h_i \ . \end{aligned} \tag{7.30}$$

Thus to any simple root is associated an $\mathfrak{sl}(2,\mathbb{C})$ subalgebra. If we are considering the real form corresponding to the compact Lie algebra, to any root is associated an $\mathfrak{su}(2)$ subalgebra. This observation will be essential to obtain all unitary representations of the compact real form. Note however the different normalisation, which implies an overall 2 factor in the first equation. Note also that for $i \neq j$, the two $\mathfrak{sl}(2,\mathbb{C})$ (or $\mathfrak{su}(2)$) algebras do not necessarily commute.

It is also obvious that we have

$$\left[h_i, e_j^{\pm}\right] = A_{ij} e_j^{\pm} \ . \tag{7.31}$$

Finally, the property (7.26) translates into

$$\mathrm{ad}(e_i^{+})^{1-A_{ij}} \cdot e_j^{+} = 0, \quad \mathrm{ad}(e_i^{-}) \cdot e_j^{+} = 0 \ , \quad i \neq j \ ,$$

since $q = -A_{ij}$ and $p = 0$ in (7.24). This last identity will be essential in order to obtain all the generators of the Lie algebra $\mathfrak{g}$ from the generators associated to simple roots.

We finally summarise all the information concerning the Lie algebra $\mathfrak{g}$ in the Chevalley-Serre basis

$$\begin{aligned} &(1)\ \left[h_i, h_j\right] = 0 \ , \\ &(2)\ \left[e_i^{+}, e_i^{-}\right] = h_i \ , \\ &(3)\ \left[h_i, e_j^{+}\right] = A_{ij} e_j^{+} \ , \quad \left[h_i, e_j^{-}\right] = -A_{ij} e_j^{-} \ , \\ &(4)\ \left[e_i^{+}, e_j^{-}\right] = 0 \ , i \neq j \ , \\ &(5)\ \mathrm{ad}^{1-A_{ij}}(e_i^{+}) \cdot e_j^{+} = 0 \ , \ i \neq j \ , \\ &(6)\ \mathrm{ad}^{1-A_{ij}}(e_i^{-}) \cdot e_j^{-} = 0 \ , i \neq j \ . \end{aligned} \tag{7.32}$$

These relations are usually called the Chevalley-Serre relations.
Some remarks are in order before closing this subsection. As we have seen, the Cartan matrix is not necessarily a symmetric matrix.[4] In fact if $A_{ij} = A_{ji}$ then $A_{ij} = A_{ji} = -1$ and the roots $\beta_{(i)}$ and $\beta_{(j)}$ have the same length. Furthermore, we have seen that starting from a semisimple algebra we were able to associate a Cartan matrix. Conversely a Cartan matrix fully characterises the Lie algebra (see (7.32)). The Cartan matrix is associated with a choice of simple roots. In fact it is remarkable to observe that the three following problems are equivalent:

[4] It should be noted that some authors use the opposite definition for the Cartan matrix.

(1) The classification of simple Lie algebras;
(2) The classification of the possible sets of simple roots;
(3) The classification of Cartan matrices.

7.2.6 *Dynkin diagrams – Classification*

We have seen that a simple Lie algebra is completely specified by the Cartan matrix given in (7.27) and satisfying (7.28). In equivalent form, the properties of a simple complex Lie algebra can be characterised by means of a combinatorial structure called the Dynkin diagram. In this context, it is important to observe that if $A_{ij}A_{ji} \neq 1$, then the simple roots $\beta_{(i)}$ and $\beta_{(j)}$ do not have the same length. Anticipating the classification theorem, we note that there are at most two possible lengths for the roots of $\mathfrak{g}$: short and long roots. We can always normalise the roots such that the length of long roots is equal to one. This means that for any simple root

$$(\beta_{(i)}, \beta_{(i)}) \leq 1 \ .$$

This normalisation will be taken in this section for convenience, but will differ from that taken in forthcoming sections. The Dynkin diagram of $\mathfrak{g}$ is constructed as follows:

- If $\mathrm{rk}(\mathfrak{g}) = r$, then we consider a diagram (graph) with r circles (vertices).
- Any simple root corresponds to a circle.
- If there are two root lengths, short roots are denoted by a darkened circle;
- Any two circles associated with the roots $\beta_{(i)}$ and $\beta_{(j)}$ are connected by $A_{ij}A_{ji} = 0, 1, 2, 3$ lines (edges).

It is clear from this construction that the Dynkin diagram corresponds to a graphical method that codifies the simple roots and their relation with respect to the inner product defined on the root space. The classification theorem of simple complex Lie algebras is a consequence of several properties of the Dynkin diagrams. Assuming that $\mathrm{rk}(\mathfrak{g}) = r$, we have the following properties [Jacobson (1962)]

(1) The Dynkin diagram of a simple Lie algebra is connected.
(2) The Dynkin diagram of a semisimple non-simple Lie algebra is disconnected, each connected piece corresponding to a simple algebra.

(3) If we remove a circle from a Dynkin diagram of a rank r Lie algebra, we obtain the Dynkin diagram of a (semi)simple Lie algebra of rank $r-1$;
(4) If the lines between two connected roots are suppressed, we obtain the Dynkin diagram of a semisimple Lie algebra of the same rank;
(5) The number of pairs of circles connected by one line is at most $r-1$;
(6) A Dynkin diagram contains no closed loop;
(7) The number of lines connecting two circles is a most three;
(8) The only connected Dynkin diagram containing a triple line is

(9) A connected Dynkin diagram contains at most one double line;
(10) Replacing a linear chain of roots by a root generates a Dynkin diagram of a lower rank algebra.

To have a proof of these statements the reader is led to either [Jacobson (1962); Cahn (1984)] or [Gilmore (1974); Georgi (1999); Ramond (2010)] for a physicist's point of view. We just briefly comment on the proofs of some of these properties. To proceed with all the proof we replace all the roots $\beta_{(i)}$ by unit vectors u_i. Of course, u_i is a positive multiple of $\beta_{(i)}$, but now we have the conditions

$$\begin{array}{ll} (i) & (u_i,u_i)=1\ , i=1,\cdots,r\ , \\ (ii) & 4(u_i,u_j)^2=0,1,2,3,\ 1\le i\neq j\le r\ , \\ (iii) & (u_i,u_j)<0\ , 1\le i<j\le r\ . \end{array} \tag{7.33}$$

We now prove some of the properties above.

We observe that, albeit its similarity, Properties 3 and 4 are distinct as they give rise to algebras of different rank. However, they become equivalent if the suppression of an edge also eliminates the vertex it is attached to.

– (5): Consider a Dynkin diagram with only roots connected by one line. Then $(\beta_{(i)},\beta_{(i)})=1$ for $i=1,\cdots,r$ and $(\beta_{(i)},\beta_{(j)})=0,-1/2$ for $i<j$, $(\beta_{(i)},\beta_{(j)})=-1/2$ if $\beta_{(i)}$ is connected to $\beta_{(j)}$ and $(\beta_{(i)},\beta_{(j)})=0$ if not. In we denote N the number of connected pairs we have

$$2\sum_{i<j}(\beta_{(i)},\beta_{(j)})=-N\ .$$

Considering $\beta=\sum_{i=1}^r\beta_{(i)}$ since $(\beta,\beta)=\sum_{i=1}^r(\beta_{(i)},\beta_{(i)})+2\sum_{i<j}(\beta_{(i)},\beta_{(j)})>0$ we have

$$-2\sum_{i<j}(\beta_{(i)},\beta_{(j)})=N<\sum_{i=1}^r(\beta_{(i)},\beta_{(i)})=r\ .$$

– (6): Consider a Dynkin diagram with a closed loop. If we remove all the roots which are not in the loop, by (3) we obtain a Dynkin diagram of say r' roots connected by r' lines. Which contradicts (5).

– (7): Assume that we have a root v connected to the roots $v_1, \cdots, v_n$. Since by (6) there is no loop we have $(v_i, v_j) = 0$ for $i \neq j$. Consider now

$$E = \mathrm{Span}(v, v_1, \cdots, v_n) \ ,$$

and complete the orthonormal set of vectors $(v_1, \cdots, v_n)$ to an orthonormal basis $(v_0, v_1, \cdots, v_n)$ of E. We have

$$v = \sum_{i=0}^{n} (v, v_i) v_i \ ,$$

with $(v, v_0) \neq 0$ because $v \in E$ but $v \notin \mathrm{Span}(v_1, \cdots, v_n)$. Because of (i) in (7.33)

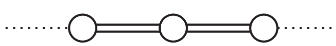
$$1 = (v, v) = \sum_{i=0}^{n} (v, v_i)^2 \quad \Rightarrow \quad 4 \sum_{i=1}^{n} (v, v_i)^2 < 4 \ .$$

Since the number of lines connecting v and v_i is precisely given by $\sum_{i=1}^{n} (u, u_i)^2$, this ends the proof.

– Property (8) is an obvious consequence of (7).

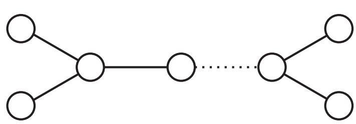
To prove (9) consider the diagram

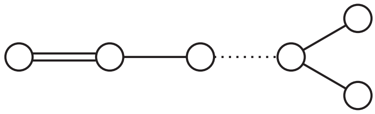
which can be shrinked to



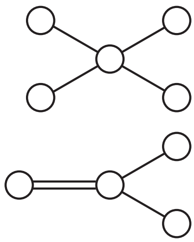
by removing the intermediate circles and lines. The resulting diagram however contradicts (7).

– The property (10) drastically reduces the possible diagrams. For instance the two following diagrams are excluded





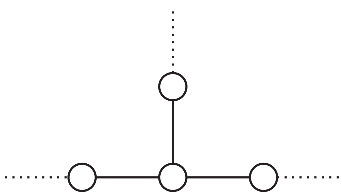
since they can be shrinked respectively to the diagrams



contradicting property (7). Thus the only possible diagrams with one double lines are linear and of the type

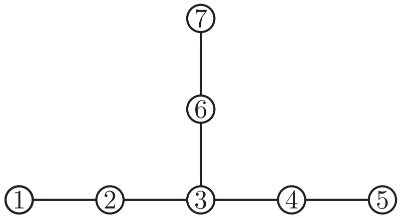
and the only possible diagrams with only simple lines are either linear

or have three branches



where the dots indicate a linear chain.

The last step in the reduction process is to exclude some diagrams that cannot be excluded by the properties above. In fact one more important property of simple Lie algebra is that their Cartan matrix is non-singular or the determinant of the Cartan matrix is not zero. Imposing this latter condition excludes more diagrams. For instance the Cartan matrix of the diagram (where the numbers in the circles indicate the corresponding roots)



is given by

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix} ,$$

and the determinant clearly vanishes. Thus this diagram is excluded.[5]

Manipulating these properties leads to all possible simple complex Lie algebras [Jacobson (1962); Gilmore (1974); Georgi (1999); Ramond (2010)]. We obtain four series of Lie algebras $A_n, B_n, C_n, D_{n+1}, n \geq 1$ and five exceptional Lie algebras G_2, F_4, E_6, E_7, E_8 which are given in Table 7.2. The series are also called the classical Lie algebras.

The series correspond to some of the algebras introduced in Chapter 2, Sec. 2.6

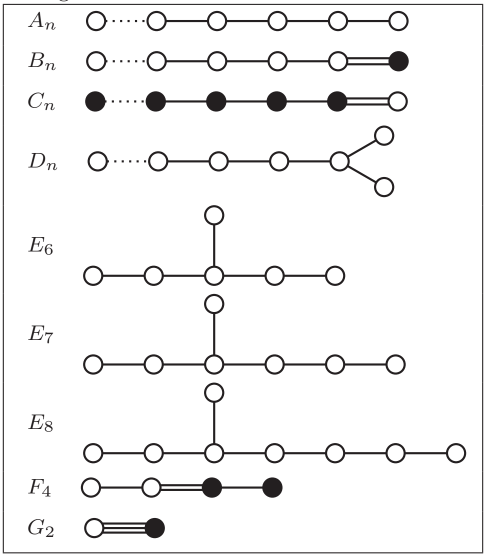
$$\begin{aligned} A_n &\cong \mathfrak{sl}(n+1, \mathbb{C}) \ , \\ B_n &\cong \mathfrak{so}(2n+1, \mathbb{C}) \ , \\ C_n &\cong \mathfrak{sp}(2n, \mathbb{C}) \ , \\ D_n &\cong \mathfrak{so}(2n, \mathbb{C}) \ , \end{aligned}$$

as we will see in the following Chapters. The Lie algebras E_6, E_7, E_8, F_4, G_2 are the so-called exceptional Lie algebras.

We finish this subsection with some important remarks

[5]This algebra is of rank 6 and isomorphic to $\hat{E}_6$, the affine extension of E_6, see Table 7.4.

Table 7.2 Dynkin diagrams of simple complex Lie algebras.



(1) There are accidental isomorphisms in small rank

$$\begin{aligned} &A_1 \cong B_1 \cong C_1 \ , \\ &B_2 \cong C_2 \ , \\ &D_2 \cong A_1 \times A_1 \ , \\ &A_3 \cong D_3 \ . \end{aligned}$$

(2) The algebra D_2 is not simple but semisimple.
(3) If for all $1 \leq i < j \leq r$ we have $A_{ij} = A_{ji}$, corresponding to the algebras A_n, D_n and E_6, E_7, E_8 all roots have the same length and the algebras are called simply-laced.
(4) There are only two possible lengths for the roots: short and long roots.
(5) All simple Lie algebras are finite-dimensional.
(6) Generalisation of simple complex Lie algebras was considered by Moody (1968); Macdonald (1986); Kac (1990) by means of a generalised Cartan matrix A. However, in this case the generalised Cartan matrix A has a vanishing determinant. Two types of algebras can be defined: (i) the affine Lie algebras, which can be easily handled and hyperbolic algebras, which require a complicated formalism for their description.

7.2.7 *Classification of simple real Lie algebras*

As can be expected, the classification of simple real Lie algebras is more involved than the classification of simple complex Lie algebras. Given an n-dimensional complex Lie algebra with basis $(T_1, \cdots, T_n)$, if we perform a change of basis $T'_a = C_a{}^b T_b$ with $C_a{}^b \in \mathbb{C}$, in the new basis the commutation relations read[6]

$$\left[T'_a, T'_b\right] = i f'_{ab}{}^c T'_c \ ,$$

with $f'_{ab}{}^c \in \mathbb{R}$, the Lie algebra generated by $(T'_1, \cdots, T'_n)$ is a real Lie algebra called a real form of the complex Lie algebra generated by $(T_1, \cdots, T_n)$ (see Sec. 2.9.1). Even if the classification of real Lie algebras is not obvious there is always, two extreme cases, the compact and the split real forms, can be easily defined. These two real forms play a relevant rôle within the real classification of simple algebras.

7.2.7.1 *The compact Lie algebras*

Considering the Cartan-Weyl basis, the compact Lie algebras is generated by

$$\begin{cases} H^i \ , & i = 1, \cdots, r \ , \\ X_\alpha = E_\alpha + E_{-\alpha} \ , & \alpha \in \Sigma_+ \ , \\ Y_\alpha = -i(E_\alpha - E_{-\alpha}) \ , & \alpha \in \Sigma_+ \ . \end{cases} \tag{7.34}$$

We observe that

$$(H^i)^\dagger = H^i, \quad X^\dagger_\alpha = X_\alpha \ , \quad Y^\dagger_\alpha = Y_\alpha \ .$$

In order to have closed commutation relations we extend (7.34) for $\alpha \in \Sigma_-$ with the obvious relations $X_{-\alpha} = X_\alpha$ and $Y_{-\alpha} = -Y_\alpha$. The commutation relations can be easily obtained form (7.23) and using (7.15) and the fact that 2α is not a root

$$\begin{aligned}
\left[H^i, X_\alpha\right] &= i\alpha^i Y_\alpha \ , \\
\left[H^i, Y_\alpha\right] &= -i\alpha^i X_\alpha \ , \\
\left[X_\alpha, X_\beta\right] &= i\mathcal{N}_{\alpha,\beta} Y_{\alpha+\beta} + i\mathcal{N}_{\alpha,-\beta} Y_{\alpha-\beta} \ , \\
\left[Y_\alpha, Y_\beta\right] &= -i\mathcal{N}_{\alpha,\beta} Y_{\alpha+\beta} + i\mathcal{N}_{\alpha,-\beta} Y_{\alpha-\beta} \ , \\
\left[X_\alpha, Y_\beta\right] &= \begin{cases} 2i\alpha_i H^i & \text{if } \alpha+\beta = 0 \ \text{ or } \alpha - \beta = 0 \\ -i\mathcal{N}_{\alpha,\beta} X_{\alpha+\beta} + i\mathcal{N}_{\alpha,-\beta} X_{\alpha-\beta} & \text{if } \alpha+\beta \neq 0 \ \text{ and } \alpha-\beta \neq 0 \ , \end{cases}
\end{aligned}$$

with the convention that $\mathcal{N}_{\alpha,\beta} = 0$, $\mathcal{N}_{\alpha,-\beta} = 0$ if $\alpha + \beta$, $\alpha - \beta \notin \Sigma$.

[6] In general the constant $f'_{ab}{}^c$ are not real.

The compact real forms corresponding to the classical Lie algebras are respectively

$$\begin{aligned} A_n &\to \mathfrak{su}(n+1) \ , \\ B_n &\to \mathfrak{so}(2n+1) \ , \\ C_n &\to \mathfrak{usp}(2n) \ , \\ D_n &\to \mathfrak{so}(2n) \ , \end{aligned}$$

with the geometrical interpretations given in Chapter 2, Sec. 2.6.

7.2.7.2 *The split Lie algebras*

The split Lie algebras are generated,[7] in the Cartan-Weyl basis by

$$\begin{cases} Z^j = iH^j \ , & j = 1, \cdots, r \ , \\ X_\alpha = i(E_\alpha + E_{-\alpha}) \ , & \alpha \in \Sigma_+ \ , \\ Y_\alpha = -i(E_\alpha - E_{-\alpha}) \ , & \alpha \in \Sigma_+ \ . \end{cases}$$

We observe that

$$(Z^i)^\dagger = -Z^i, \quad X_\alpha^\dagger = -X_\alpha \ , \quad Y_\alpha^\dagger = Y_\alpha \ .$$

As for the compact real forms, for later convenience we extend the definitions above to any $\alpha \in \Sigma$. The commutation relations can be easily obtained form (7.23), using (7.15) and the fact that 2α is not a root

$$\begin{aligned} \left[Z^i, X_\alpha\right] &= -i\alpha^i Y_\alpha \ , \\ \left[Z^i, Y_\alpha\right] &= -i\alpha^i X_\alpha \ , \\ \left[X_\alpha, X_\beta\right] &= -i\mathcal{N}_{\alpha,\beta} Y_{\alpha+\beta} - i\mathcal{N}_{\alpha,-\beta} Y_{\alpha-\beta} \ , \\ \left[Y_\alpha, Y_\beta\right] &= -i\mathcal{N}_{\alpha,\beta} Y_{\alpha+\beta} + i\mathcal{N}_{\alpha,-\beta} Y_{\alpha-\beta} \ , \\ \left[X_\alpha, Y_\beta\right] &= \begin{cases} 2i\alpha_i Z^i & \text{if } \alpha+\beta = 0 \\ -i\mathcal{N}_{\alpha,\beta} X_{\alpha+\beta} + i\mathcal{N}_{\alpha,-\beta} X_{\alpha-\beta} & \text{if } \alpha+\beta \neq 0 \ , \end{cases} \end{aligned}$$

with the convention that $\mathcal{N}_{\alpha,\beta} = 0, \mathcal{N}_{\alpha,-\beta} = 0$ if $\alpha+\beta, \alpha-\beta \notin \Sigma$.

The split real forms corresponding to the classical algebras are

$$\begin{aligned} A_n &\to \mathfrak{sl}(n+1, \mathbb{R}) \ , \\ B_n &\to \mathfrak{so}(n+1, n) \ , \\ C_n &\to \mathfrak{usp}(2n-2p, 2p) \ , p = \left[\frac{1}{2}n\right] \\ D_n &\to \mathfrak{so}(n,n) \ , \end{aligned}$$

[7] Sometimes the split form is also referred to as the normal real form.

with $[a]$ the integer part of a. The geometrical interpretation of these algebras is given in Chapter 2, Sec. 2.6.

In the two real forms constructed so far, a direct inspection to the commutation relations clearly shows that the structure constants are purely imaginary, as it is expected for an appropriate real form.

7.2.7.3 *General real Lie algebras*

The classifications of real non-compact Lie algebras can be deduced from the compact real forms, by classifying its involutive automorphisms. See for instance [Cornwell (1984b)]. Denote $\mathfrak{g}_c$ the real compact form of the complex Lie algebra $\mathfrak{g}$. We recall that Ψ is an automorphism of $\mathfrak{g}_c$ if for all $x, y \in \mathfrak{g}_c$ and all $\lambda, \mu \in \mathbb{R}$ we have

$$\begin{aligned} \Psi(\lambda x + \mu y) &= \lambda \Psi(x) + \mu \Psi(y) \ , \\ \Psi\Big(\big[x, y\big]\Big) &= \Big[\Psi(x), \Psi(y)\Big] \ , \end{aligned}$$

and is involutive if for all $x \in \mathfrak{g}_c$

$$\Psi \circ \Psi(x) = x \ .$$

Because of the equation above, the eigenvalues of Ψ are $1, -1$, and there exists a basis of $\mathfrak{g}_c$, $\left\{U_1^+, \cdots, U_{n_+}^+, U_1^-, \cdots, U_{n_-}^-\right\}$ with $n_+ + n_- = n$ such that

$$\begin{aligned} \Psi(U_a^+) &= U_a^+ \ , \quad a = 1, \cdots, n_+ \ , \\ \Psi(U_a^-) &= -U_a^- \ , \quad a = 1, \cdots, n_- \ . \end{aligned}$$

It can be shown that the Lie algebra $\mathfrak{g}_\Psi$ generated by

$$\begin{aligned} V_a^+ &= U_a^+ \ , \quad a = 1, \cdots, n_+ \ , \\ V_a^- &= iU_a^- \ , \quad a = 1, \cdots, n_- \ , \end{aligned}$$

is a real form of the complex Lie algebra. We denote

$$\begin{aligned} \mathfrak{k} &= \text{Span}(V_1^+, \cdots, V_{n_+}^+) \ , \\ \mathfrak{p} &= \text{Span}(V_1^-, \cdots, V_{n_-}^-). \end{aligned}$$

Obviously we have

$$\mathfrak{g}_\Psi = \mathfrak{k} \oplus \mathfrak{p} \ ,$$

and

$$\big[\mathfrak{k}, \mathfrak{k}\big] \subseteq \mathfrak{k} \ , \big[\mathfrak{k}, \mathfrak{p}\big] \subseteq \mathfrak{p} \ , \big[\mathfrak{p}, \mathfrak{p}\big] \subseteq \mathfrak{k} \ . \tag{7.35}$$

Then $\mathfrak{k}$ is a subalgebra of the $\mathfrak{g}_\Psi$ called the maximal compact subalgebra. It is straightforward to verify that the Killing form of $\mathfrak{g}_\Psi$ is diagonal and given by

$$\begin{aligned}\kappa(V_a^+, V_b^+) &= \delta_{ab} \ , \\ \kappa(V_a^-, V_b^-) &= -\delta_{ab} \ , \\ \kappa(V_a^+, V_b^-) &= 0 \ .\end{aligned}$$

The Lie algebra $\mathfrak{g}_\Psi$ has a signature (n_+, n_-) and it turns out that any real form is fully characterised by the signature. We define then the character of a Lie algebra by

$$\sigma = n_- - n_+ \ .$$

The real forms $\mathfrak{g}_\mathbb{R}$ associated to a simple complex Lie algebra $\mathfrak{g}_\mathbb{C}$ are given in Table 7.3. The Lie algebras given in this Table are the real forms obtained from each simple complex Lie algebra $\mathfrak{g}$. Since the character completely specifies the real form of a given complex Lie algebra, for the exceptional Lie algebras the second number in parenthesis in Table 7.3 denotes the character of the corresponding real form. This number reduces to minus the dimension of the Lie algebra for compact real forms and to its rank for split real forms. Note that the compact real form is the most compact real form (*i.e.*, all the generators correspond to compact directions) and the split real form is the less compact real form (the number of compact directions is equal to $(n-r)/2$ with n the dimension of the Lie algebra and r its rank). Note that in addition to this list there also exist non-compact Lie algebras associated to $\mathfrak{g} \times \mathfrak{g}$, *i.e.*, two copies of the same Lie algebra. These latter Lie algebras have also a vanishing character. See *e.g.* [Cornwell (1984b)].

Not all of the algebras are non-isomorphic, as follows easily from the already observed equivalence of some of the complex simple algebras through their Dynkin diagram. For the real forms, we have the following isomorphisms:

$$\begin{aligned}\mathfrak{su}(2,\mathbb{C}) \cong \mathfrak{so}(3,\mathbb{C}) \cong \mathfrak{sp}(2,\mathbb{C}) &\Rightarrow \begin{cases}\mathfrak{su}(2) \cong \mathfrak{so}(3) \cong \mathfrak{usp}(2) \ , \\ \mathfrak{su}(1,1) \cong \mathfrak{so}(1,2) \cong \mathfrak{sp}(2,\mathbb{R}) \ ,\end{cases} \\ \mathfrak{so}(5,\mathbb{C}) \cong \mathfrak{sp}(4,\mathbb{C}) &\Rightarrow \begin{cases}\mathfrak{so}(5) \cong \mathfrak{usp}(4) \ , \\ \mathfrak{so}(2,3) \cong \mathfrak{sp}(4,\mathbb{R}) \ , \\ \mathfrak{so}(1,4) \cong \mathfrak{usp}(2,2)\end{cases}\end{aligned}$$

Table 7.3 Simple real Lie algebras and their maximal compact Lie subalgebras.

Complex Lie algebras $\mathfrak{g}_{\mathbb{C}}$	Real forms $\mathfrak{g}_{\mathbb{R}}$	characters σ	maximal compact subalgebras $\mathfrak{k}$
A_n	$\mathfrak{su}(n+1)$	$-(n+1)^2+1$	$\mathfrak{su}(n+1)$
	$\mathfrak{su}(n+1-p,p)$ $p=1,\cdots,\left[\frac{1}{2}(n+1)\right]$	$-(n+1-2p)^2+1$	$\mathfrak{su}(n+1-p)\times\mathfrak{su}(p)\times\mathfrak{u}(1)$
	$\mathfrak{sl}(n+1,\mathbb{R})$	n	$\mathfrak{so}(n+1)$
	$\mathfrak{su}^*(n+1)$ n odd	$-n-2$	$\mathfrak{usp}(n+1)$
B_n	$\mathfrak{so}(2n+1)$	$-n(2n+1)$	$\mathfrak{so}(2n+1)$
	$\mathfrak{so}(2n+1-2p,2p)$ $p=1,\cdots,n$	$-2(n-2p)(n-2p+1)+n$	$\mathfrak{so}(2n+1-2p)\times\mathfrak{so}(2p)$
C_n	$\mathfrak{usp}(2n)$	$-n(2n+1)$	$\mathfrak{usp}(2n)$
	$\mathfrak{sp}(2n)$	n	$\mathfrak{u}(n)$
	$\mathfrak{usp}(2n-2p,2p)$ $p=1,\cdots,\left[\frac{1}{2}n\right]$	$-n-2(n-2p)^2$	$\mathfrak{usp}(2n-2p)\times\mathfrak{usp}(2p)$
D_n	$\mathfrak{so}(2n)$	$-n(2n-1)$	$\mathfrak{so}(2n)$
	$\mathfrak{so}(2n-p,p)$ $p=1,\cdots,\left[\frac{1}{2}n\right]$	$-2(n-p)^2+n$	$\mathfrak{so}(2n-p)\times\mathfrak{so}(p)$
	$\mathfrak{so}^*(2n)$	$-n$	$\mathfrak{u}(n)$
G_2	$G_{2(-14)}$	-14	$G_{2(-14)}$
	$G_{2(2)}$	2	$\mathfrak{su}(2)\times\mathfrak{su}(2)$
F_4	$F_{4(-52)}$	-52	$F_{4(-52)}$
	$F_{4(4)}$	4	$\mathfrak{usp}(6)\times\mathfrak{su}(2)$
	$F_{4(-20)}$	-20	$\mathfrak{so}(9)$
E_6	$E_{6(-78)}$	-78	$E_{6(-78)}$
	$E_{6(6)}$	6	$\mathfrak{usp}(8)$
	$E_{6(2)}$	2	$\mathfrak{su}(6)\times\mathfrak{su}(2)$
	$E_{6(-14)}$	-14	$\mathfrak{so}(10)\times\mathfrak{so}(2)$
	$E_{6(-26)}$	-26	$F_{4(-52)}$
E_7	$E_{7(-133)}$	-133	$E_{7(-133)}$
	$E_{7(7)}$	7	$\mathfrak{su}(8)$
	$E_{7(-5)}$	-5	$\mathfrak{so}(12)\times\mathfrak{so}(3)$
	$E_{7(-25)}$	-25	$E_{6(-78)}\times\mathfrak{so}(2)$
E_8	$E_{8(-248)}$	-248	$E_{8(-248)}$
	$E_{8(8)}$	8	$\mathfrak{so}(16)$
	$E_{8(-24)}$	-24	$E_{7(-133)}\times\mathfrak{su}(2)$

$$\mathfrak{su}(4,\mathbb{C})\cong\mathfrak{so}(6,\mathbb{C})\Rightarrow\begin{cases}\mathfrak{su}(4)\cong\mathfrak{so}(6)\ ,\\ \mathfrak{su}(2,2)\cong\mathfrak{so}(2,4)\ ,\\ \mathfrak{su}(1,3)\cong\mathfrak{so}^*(6)\ .\end{cases}$$

In addition to these, there is another important isomorphism, the origin of which is related to the so-called triality [Cornwell (1971)]

$$\mathfrak{so}(2,6)\cong\mathfrak{so}^*(8)\ .$$

7.3 Reconstruction of the algebra

In the previous sections we have given the classification of the simple complex and simple real Lie algebras. The former were fully defined by their Cartan matrix or Dynkin diagram and the latter by considering an appropriate involutive automorphism of the compact real form. In both cases the construction of the algebra is thoroughly associated to its presentation in the Chevalley-Serre basis (7.32), defined only for the generators associated with the simple roots. We now give the general rule to obtain all the operators of simple complex or real Lie algebras. Since both cases go along the same lines, we suppose now that $\mathfrak{g}$ is a simple complex Lie algebra of rank r. Denote $\beta_{(1)}, \cdots, \beta_{(r)}$ its simple roots and recall that to each simple root we have two types of operators: the semisimple operators $h_i, i = 1 \cdots, r$ which are diagonal and the nilpotent operators $e_i^{\pm}, i = 1, \cdots, r$, that vanishes at some power (in the adjoint representation). We identify further the operators e_i^+ to creation operators and the operators e_i^- to annihilation operators. The key observation in the reconstruction of the whole algebra is the relation (7.30), which means that to obtain all the generators associated to $\mathfrak{g}$ we only have to deal with the representation theory of $\mathfrak{sl}(2, \mathbb{C})$. For completeness, recall that if[8]

$$h_i \Big| -n \Big\rangle = -n \Big| -n \Big\rangle \text{ and } e_i^- \Big| -n \Big\rangle = 0 \ , \tag{7.36}$$

the full representation is $(n+1)$-dimensional and is obtained by acting n-times on $\Big| -n \Big\rangle$ with e_i^+

$$\underbrace{\Big| -n \Big\rangle \xrightarrow{e_i^+} \Big| -n+2 \Big\rangle \xrightarrow{e_i^+} \cdots \xrightarrow{e_i^+} \Big| n-2 \Big\rangle \xrightarrow{e_i^+} \Big| n \Big\rangle}_{e_i^{+\,n+1}|-n\rangle=0}$$

Thus the relation

$$\big[h_k, e_i^+\big] = A_{ki} e_i^+ \ , \quad \big[e_k^-, e_i^+\big] = 0 \ ,$$

of (7.32) is in a direct correspondence with (7.36). Moreover, from (7.31), if we assume that $\beta_{(i)} + \beta_{(j)}$ is a root we have, using the Jacobi identity

$$\big[h_k, \big[e_i^+, e_j^+\big]\big] = (A_{ki} + A_{kj}) \big[e_i^+, e_j^+\big] \ .$$

With these notions recalled, we are now in position to describe an algorithmic procedure to obtain all the generators of $\mathfrak{g}$ from the Cartan matrix (7.27) and the Chevalley-Serre relations (7.32):

(1) To each simple root $\beta_{(i)}$ associate an r-dimensional vector $|A_i\rangle = |A_{1i}, \cdots, A_{ri}\rangle$ where the k^{th} entry represents the eigenvalue of h_k on the vector e_i^+.

[8] Note the overall additional factor two in the commutation relations (7.30).

(2) For each $|A_i\rangle$ identify the negative entries. If, say $A_{ki} = -q$, then act q times on the vector with e_k^+

$$\begin{array}{ccccccc}
|A_i\rangle & \xrightarrow{e_k^+} & |A_i\rangle + |A_k\rangle & \xrightarrow{e_k^+} \cdots & |A_i\rangle + (q-1)\,|A_k\rangle & \xrightarrow{e_k^+} & |A_i\rangle + q\,|A_k\rangle \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
E_{\beta_{(i)}} & & E_{\beta_{(i)}+\beta_{(k)}} & \cdots & E_{\beta_{(i)}+(q-1)\beta_{(k)}} & & E_{\beta_{(i)}+q\beta_{(k)}}
\end{array}$$

and identify the corresponding operators.

(3) In the series of vectors $|A_i\rangle + \ell\,|A_k\rangle\,, 1 \leq \ell \leq q$ constructed in the step 2, identify the entries, except the k^{th}-entry which is negative. If the k'^{th} entry is negative ($= -q'$) then act q'-times with $e_{k'}^+$.

(4) Reiterate the process until there are no more vectors with negative entries.

(5) The normalisation of the nilpotent operators is now fixed by (7.18).

Considering a nilpotent generator e_α associated with a non-necessarily simple root α, *i.e.*, obtained from multiple commutators of the nilpotent operators associated with simple roots as described in the procedure above (the operator e_α can of course be associated with a simple root $\beta_{(j)}$ with $e_\alpha = e^\pm_{\beta_{(j)}}$) we have

$$\begin{aligned}
\text{ad}(e_i^+)^{q+1} \cdot e_\alpha &= 0\ , \\
\text{ad}(e_i^-)^{p+1} \cdot e_\alpha &= 0\ .
\end{aligned}$$

This means that with respect to the $\mathfrak{sl}(2,\mathbb{C})$ subalgebra generated by $(h_i, e_i^\pm)$, e_α span a spin $1/2(p+q)$-representation of the $\mathfrak{sl}(2,\mathbb{C})$ (or of the $\mathfrak{su}(2)-$ for the real compact form) algebra associated with the i^{th}-simple root. In particular we have the identification $e_\alpha = |s,m\rangle = |1/2(p+q), 1/2(p-q)\rangle$ with $s = 1/2(p+q)$ and $m = 1/2(p-q)$. Thus the representation theory of $\mathfrak{su}(2)$ (see (5.9)) gives

$$\text{ad}(e_i^+)\,|s,m\rangle = \sqrt{(s-m)(s+m+1)}\,|s,m+1\rangle\,. \tag{7.37}$$

This latter relation is utterly compatible with the relation (7.18) when the operators $E_{\beta_{(i)}}$ are renormalised as in (7.29). We shall come to this point latter on. With this spin-interpretation note that (7.17) reduces to[9]

$$2\frac{(\alpha, \beta_{(i)})}{(\beta_{(i)}, \beta_{(i)})} = p - q = 2m\ , \tag{7.38}$$

[9] The factor two comes from the normalisation in (7.30).

and the scalar product of the two roots $\beta_{(i)}$ and α is directly related to the eigenvalue of the semisimple element h_i.

We now illustrate the procedure for the small rank Lie algebras. For the rank one Lie algebras nothing has to be done. For the rank two Lie algebras only G_2 has to be considered since $\mathfrak{su}(3)$ was already studied in Chapter 6 and $\mathfrak{so}(5)$ will be analysed in details in a latter chapter. Recall the Dynkin diagram, the Cartan matrix

$$A = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

and the simple roots

$$\beta_{(1)} = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ -\frac{3}{2} \end{pmatrix} \ , \beta_{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ . \tag{7.39}$$

of G_2. We obviously have

$$(\beta_{(1)}, \beta_{(1)}) = 3 \ , \quad (\beta_{(2)}, \beta_{(2)}) = 1 \ , \quad (\beta_{(1)}, \beta_{(2)}) = -\frac{3}{2}$$

The vectors of the point 1 above are given by

$$\begin{aligned} |A_1\rangle &= |A_{11}, A_{21}\rangle = |2, -3\rangle \ , \\ |A_2\rangle &= |A_{12}, A_{22}\rangle = |-1, 2\rangle \end{aligned}$$

and the procedure leads to the following set of positive roots

$$\begin{array}{ccccccc} & & |-1,2\rangle & & & & \\ & & \downarrow e_1^+ & & & & \\ |2,-3\rangle & \xrightarrow{e_2^+} & |1,-1\rangle & \xrightarrow{e_2^+} & |0,1\rangle & \xrightarrow{e_2^+} & |-1,3\rangle \\ & & & & & & \downarrow e_1^+ \\ & & & & & & |1,0\rangle \end{array}$$

In the Cartan-Weyl basis the commutation relations are given by (pay attention to the relationship between e_i^+ and $E_{\beta_{(i)}}$ together with (7.38))

$$\begin{aligned} \left[E_{\beta_{(1)}}, E_{\beta_{(2)}}\right] &= \sqrt{\frac{3}{2}} E_{\beta_{(1)}+\beta_{(2)}} \ , \\ \left[E_{\beta_{(1)}+\beta_{(2)}}, E_{\beta_{(2)}}\right] &= \sqrt{2} E_{\beta_{(1)}+2\beta_{(2)}} \ , \\ \left[E_{\beta_{(1)}+2\beta_{(2)}}, E_{\beta_{(2)}}\right] &= \sqrt{\frac{3}{2}} E_{\beta_{(1)}+3\beta_{(2)}} \ , \\ \left[E_{\beta_{(1)}+3\beta_{(2)}}, E_{\beta_{(1)}}\right] &= \sqrt{\frac{3}{2}} E_{2\beta_{(1)}+3\beta_{(2)}} \ , \end{aligned} \tag{7.40}$$

for the nilpotent operators associated to the positive roots. In the same way we obtain the commutation relations for the nilpotent operators associated to the negative roots. If for $\alpha, \beta \in \Sigma_+$ we have

$$\left[E_\alpha, E_\beta\right] = \mathcal{N}_{\alpha,\beta} E_{\alpha+\beta} \ ,$$

then by Hermitean conjugation we obtain

$$\left[E_{-\alpha}, E_{-\beta}\right] = -\mathcal{N}_{\alpha,\beta} E_{-\alpha-\beta} \ .$$

The commutation relations for operator involving positive and negative roots are obtained by means of the Jacoby identity. For instance

$$\begin{aligned}\left[E_{\beta_{(1)}+\beta_{(2)}}, E_{-\beta_{(1)}}\right] &= \sqrt{\frac{2}{3}}\left[\left[E_{\beta_{(1)}}, E_{\beta_{(2)}}\right], E_{-\beta_{(1)}}\right] \\ &= \sqrt{\frac{2}{3}}\left[\left[E_{\beta_{(1)}}, E_{-\beta_{(1)}}\right], E_{\beta_{(2)}}\right] + \sqrt{\frac{2}{3}}\left[\left[E_{-\beta_{(1)}}, E_{\beta_{(2)}}\right], E_{\beta_{(1)}}\right] \\ &= (\beta_{(1)}, \beta_{(2)})\sqrt{\frac{2}{3}}\ E_{\beta_{(2)}} = -\sqrt{\frac{3}{2}} E_{\beta_{(2)}} \ .\end{aligned}$$

Considering negative roots together with the Cartan subalgebra, we observe that G_2 is a 14-dimensional Lie algebra. The root diagram is given in Fig. 7.1.

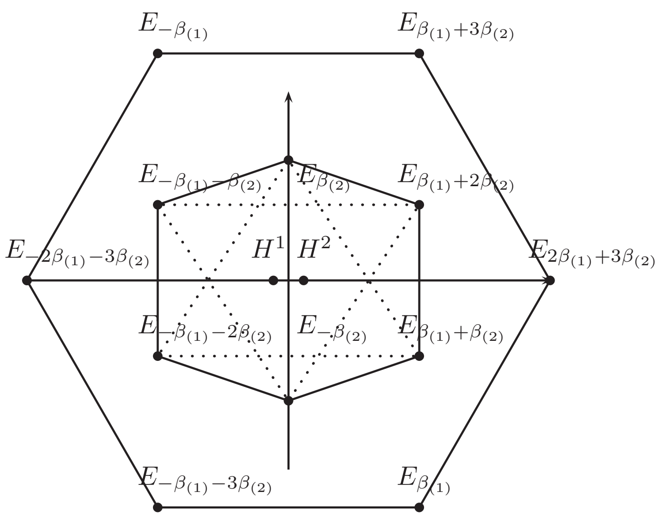


Fig. 7.1 Roots of G_2.

An important remark with respect to the normalisations for the commutation relations of G_2 and in fact of any Lie algebra $\mathfrak{g}$ is in order. The various coefficients in the commutation relations are a direct consequence of relations (7.22). Recall that these relations where obtained considering the compact real form $\mathfrak{g}_c$ of the complex Lie algebra $\mathfrak{g}$ and are perfectly compatible with the unitarity of the adjoint representation and lead to an orthonormal basis. However, if we are considering different real forms or even the complex Lie algebra itself, since the adjoint representation is not unitary there is no need to normalise $\mathcal{N}_{\alpha,\beta}$ as in (7.22). Nevertheless for convenience this normalisation might be used.

For the rank three Lie algebras three cases must be studied. For $\mathfrak{su}(4)$ the Dynkin diagram and the Cartan matrix are given by

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

and the simple roots read

$$\beta_{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad , \quad \beta_{(2)} = \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ 0 \end{pmatrix} \quad , \quad \beta_{(3)} = \begin{pmatrix} 0 \\ -\frac{\sqrt{3}}{3} \\ \frac{\sqrt{6}}{3} \end{pmatrix} \quad ,$$

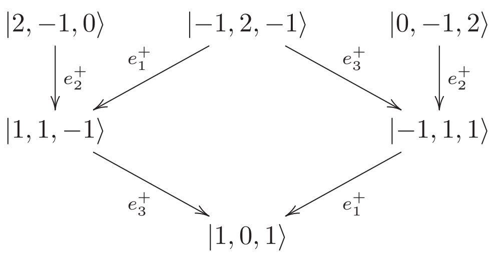
where

$$(\beta_{(i)}, \beta_{(i)}) = 1 \ , \quad i = 1, 2, 3$$
$$(\beta_{(i)}, \beta_{(i+1)}) = -\frac{1}{2} \ , \quad i = 1, 2 \ .$$

The vectors of point 1 are given by

$$|A_1\rangle = |A_{11}, A_{21}, A_{31}\rangle = |2, -1, 0\rangle \ ,$$
$$|A_2\rangle = |A_{12}, A_{22}, A_{32}\rangle = |-1, 2, -1\rangle \ ,$$
$$|A_3\rangle = |A_{13}, A_{23}, A_{33}\rangle = |0, -1, 2\rangle \ ,$$

and the reconstruction of the algebra gives

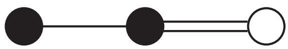


we thus have six creation operators

$$\left[E_{\beta_{(1)}}, E_{\beta_{(2)}}\right] = \frac{1}{\sqrt{2}} E_{\beta_{(1)}+\beta_{(2)}}$$
$$\left[E_{\beta_{(2)}}, E_{\beta_{(3)}}\right] = \frac{1}{\sqrt{2}} E_{\beta_{(2)}+\beta_{(3)}}$$
$$\left[E_{\beta_{(2)}+\beta_{(3)}}, E_{\beta_{(1)}}\right] = \frac{1}{\sqrt{2}} E_{\beta_{(1)}+\beta_{(2)}+\beta_{(3)}} \ .$$

Hence $\mathfrak{su}(4)$ is a 15-dimensional algebra.

For $\mathfrak{usp}(6)$ the Dynkin diagram and the Cartan matrix are given by

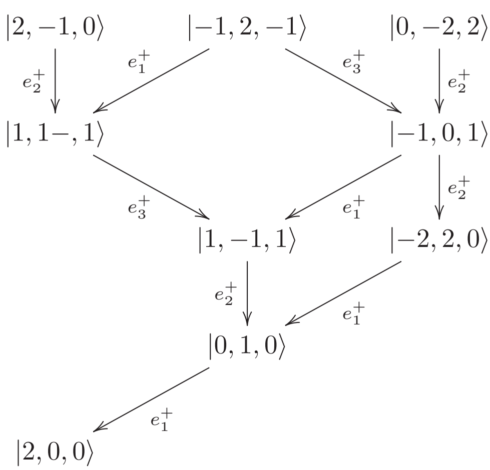


$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}$$

and the vectors of point 1 by

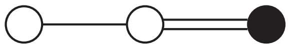
$$|A_1\rangle = |A_{11}, A_{21}, A_{31}\rangle = |2, -1, 0\rangle \ ,$$
$$|A_2\rangle = |A_{12}, A_{22}, A_{32}\rangle = |-1, 2, -1\rangle \ ,$$
$$|A_3\rangle = |A_{13}, A_{23}, A_{33}\rangle = |0, -2, 2\rangle \ .$$

The reconstruction of the algebra gives



and $\mathfrak{usp}(6)$ is 21-dimensional.

For $\mathfrak{so}(7)$ the Dynkin diagram and the Cartan matrix are given by

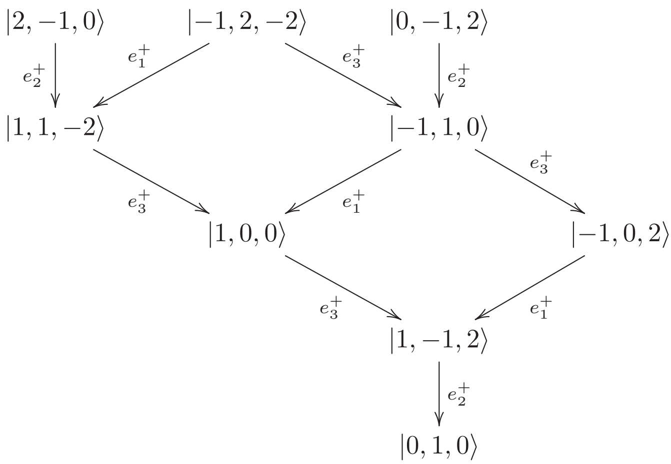


$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}$$

and the vectors of point 1 by

$$\begin{aligned} |A_1\rangle &= |A_{11}, A_{21}, A_{31}\rangle = |2, -1, 0\rangle \ , \\ |A_2\rangle &= |A_{12}, A_{22}, A_{32}\rangle = |-1, 2, -2\rangle \ , \\ |A_3\rangle &= |A_{13}, A_{23}, A_{33}\rangle = |0, -1, 2\rangle \ . \end{aligned}$$

The reconstruction of the algebra gives



and we find that $\mathfrak{so}(7)$ is of dimension 21.

7.4 Subalgebras of simple Lie algebras

Having obtained all simple complex and simple real Lie algebras an important related question is to obtain all possible Lie subalgebras of a given Lie algebra $\mathfrak{g}$. Given $\mathfrak{g}$ a rank r Lie algebra, fewer rank subalgebras of $\mathfrak{g}$ can be obtained straightforwardly, due to Property 3 of Sec. 7.2.6. Indeed, if we remove a circle (vertex) from the Dynkin diagram of $\mathfrak{g}$, we get the Dynkin diagram of a Lie algebra of lower rank $\mathfrak{g}'$ but also $\mathfrak{g}' \subset \mathfrak{g}$. If the corresponding Dynkin diagram is constituted of one connected (two disconnected) part(s), the corresponding Lie algebra is simple (semisimple).

For instance it is easy to see that $D_5 \subset E_6 \subset E_7 \subset E_8$ or $B_3 \subset F_4$ or $A_n \times A_m \subset A_{n+m+1}$, *etc.*

There is an nice diagrammatic way to obtain all the subalgebras of the same rank of a given simple Lie algebra. Denote Ψ the highest root of a given simple Lie algebra of rank r, that is, satisfying $\Psi - \alpha > 0$ for any $\alpha \in \Sigma$ different from Ψ and introduce

$$\beta_{(0)} = -\Psi \ ,$$

since Ψ is the highest root $\beta_{(0)}$ is the lowest root. Therefore, since for any simple roots $\beta_{(i)}, i = 1, \cdots, r$ we know that $\beta_{(0)} - \beta_{(i)}$ is not a root, the relation (7.17) shows that

$$2\frac{(\beta_{(i)}, \beta_{(0)})}{(\beta_{(i)}, \beta_{(i)})} \ , \quad 2\frac{(\beta_{(i)}, \beta_{(i)})}{(\beta_{(0)}, \beta_{(0)})} \ \text{ are negative integers .}$$

If we define now the extended Cartan matrix

$$A_{ij} = 2\frac{(\beta_{(i)}, \beta_{(j)})}{(\beta_{(i)}, \beta_{(i)})} \ , \quad 0 \leq i \leq j \leq r \ ,$$

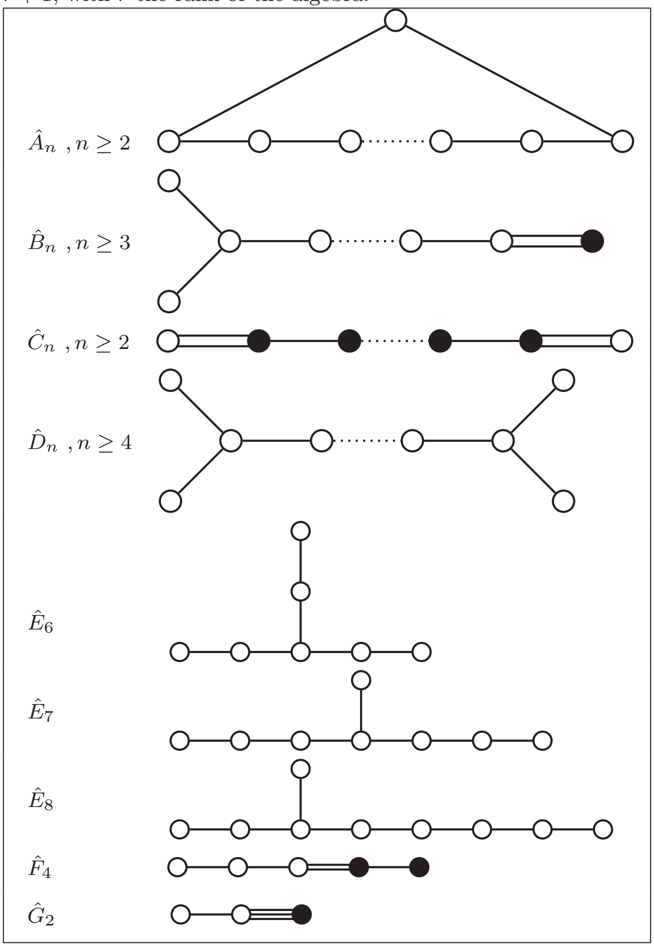
we obtain an $(r+1) \times (r+1)$ matrix which satisfies all the hypothesis of (7.28) but the last one (vi). To this extended Cartan matrix we associate a Dynkin diagram. The Dynkin diagram of the extended system of roots $(\beta_{(0)}, \cdots, \beta_{(r)})$ is called the extended Dynkin diagram of $\mathfrak{g}$. In this general construction, some care must be taken for the small rank Lie algebras. The case of $\mathfrak{g} = A_1 \cong B_1 \cong C_1$ must be treated differently since there is only one positive root α and $\alpha_{(0)} = -\alpha$. Note also that we need enough roots to put the additional lines (see in Table 7.4 the extended Dynkin diagrams). Correspondingly a special attention is needed for the small rank Lie algebra, taking under consideration the accidental isomorphisms. In particular

(1) Since $B_2 \cong C_2$, the extended Dynkin diagram of B_2 is given by the extended Dynkin diagram of C_2;
(2) Since $D_2 \cong A_1 \times A_1$, D_2 is not simple but semisimple, no new extended Dynkin diagram is defined (see the case of A_1);
(3) Since $D_3 \cong A_3$ the extended Dynkin diagram of D_3 is given by the extended Dynkin diagram of A_3.

In conclusion, this means that $\hat{A}_n$ is defined for $n \geq 2$, $\hat{B}_n$ for $n \geq 3$, $\hat{C}_n$ for $n \geq 2$ and $\hat{D}_n$ for $n \geq 4$. The corresponding extended Dynkin diagrams are given in Table 7.4. In fact these extended Dynkin diagrams are associated with some possible extensions of simple Lie algebras called affine Lie algebras. To the list given in Table 7.4, one has to add the affine extension of A_1 and the twisted affine Lie algebras. See *e.g.* [Lorente and Gruber (1972); Goddard and Olive (1986)].

These extended Dynkin diagrams are powerful tools to obtain subalgebras of simple Lie algebras. Indeed, if we remove an arbitrary circle or equivalently one line and the corresponding column of the Cartan matrix, we obtain the Dynkin diagram (or the Cartan matrix) of a Lie algebra $\mathfrak{g}'$ of the same rank such that $\mathfrak{g}' \subset \mathfrak{g}$.

Table 7.4 Extended Dynkin diagrams. The numbers of roots is $r+1$, with r the rank of the algebra.



Several examples constructed along these lines are now given. Note firstly that two types of algebras can be obtained. If we remove a circle at the end of the diagram we obtain a simple subalgebra $\mathfrak{g}' \subset \mathfrak{g}$ of the same rank although if we remove a roots such that the corresponding Dynkin diagram is constituted by two disconnected diagrams we obtain a semisimple (but not simple) subalgebra $\mathfrak{g}' \subset \mathfrak{g}$. As explicit examples we only consider the simple subalgebras that can be obtained from the extended Dynkin diagrams:

(1) For B_n we obtain $D_n \subset B_n$.
(2) For E_8 we obtain $D_8 \subset E_8$.
(3) For G_2 we obtain $A_2 \subset G_2$.
(4) For F_4 we obtain $B_4 \subset F_4$.

Obviously semisimple algebras could also be obtained, but we skip this step as it is straightforward.

7.5 System of roots and Cartan matrices

Looking at the roots of either G_2 (7.39) or $\mathfrak{su}(3)$ (6.7), we observe that their components are irrational numbers. It is however possible to express all the roots, in a possibly higher dimensional space, in terms of orthogonal vectors. This construction is very useful for instance to obtain an oscillator or a differential realisation of simple Lie algebras. Introduce a set of N orthogonal vectors $e_1, \cdots, e_N$

$$e_i \cdot e_j = \delta_{ij} \ ,$$

where now the scalar product is the Euclidean scalar product in $\mathbb{R}^N$. Interestingly, it is possible to express the simple roots of all simple Lie algebras in terms of the vectors e_i for a given N. Two observations are in order. Firstly we have $N \geq r$ where r is the rank of the Lie algebra. Secondly, the metric in the root space g^{ij} is *a priori* not equal to the diagonal metric tensor δ^{ij}. However, considering the compact real form, its is known that the Killing form is positive definite. So g^{ij} is also definite positive. Thus in this case one can find a Cartan subalgebra $\mathfrak{h} = \big\{h_1, \cdots, h_r\big\}$ such that

$$\mathrm{Tr}\Big(\mathrm{ad}(h_i)\mathrm{ad}(h_j)\Big) = \delta_{ij} \ .$$

Note that this choice was implicitly done for the reconstruction of G_2 and $\mathfrak{su}(4)$ in Sec. 7.3.

Before giving all simple roots of all semisimple Lie algebras in terms of an orthogonal basis, we would like to mention how all the roots can be obtained from simple roots, just using geometrical properties. This procedure is an alternative way for the reconstruction of the full algebra and is based on the following properties. Considering α and β two different roots satisfying (7.17) since $\alpha + k\beta \in \mathfrak{g}_{\alpha,\beta}$ for $-p \le k \le q$

$$\alpha' = \sigma_\beta(\alpha) = \alpha - (p-q)\beta = \alpha - 2\frac{(\beta,\alpha)}{(\beta,\beta)}\beta \ , \tag{7.41}$$

belongs to $\mathfrak{g}_{\alpha,\beta}$ and is a root. This transformation has a nice geometrical interpretation if we write

$$\alpha = \alpha_\| + \alpha_\perp = \frac{(\beta,\alpha)}{(\beta,\beta)}\beta + \left(\alpha - \frac{(\beta,\alpha)}{(\beta,\beta)}\beta\right) \ ,$$

where $\alpha_\|$ (resp. $\alpha_\perp$) is parallel (resp. perpendicular) to β. Thus, σ_β can be seen as a reflection in the hyperplane orthogonal to β. The set of all the possible reflections with respect to all the roots constitute the so-called Weyl group. Since all the roots can be obtained from simple roots, the Weyl group can be obtained considering the group generated by the reflections with respect to all the simple roots. Alternatively this gives rise to a procedure to obtain all the roots of a given simple Lie algebra. The set of roots is obtained inductively. At the first step we consider the simple roots. At the second step we construct all the roots obtained by all possible Weyl reflections from all the simple roots. Having obtained the roots at the step n, the roots at the step $n+1$ are obtained by considering all the possible Weyl reflections from the roots obtained in the step n. The process ended when no more roots can be generated. We shall illustrate this process on G_2.

We suppose now that the Killing form is Euclidean, and we restrict ourselves to the compact real forms. The scalar product is then denoted by a dot. For completeness we also give the Cartan matrices of all simple Lie algebras in addition to their Dynkin diagrams.

(1) For $\mathfrak{su}(n+1)$ take $N = n+1$. The simple roots are given by

$$\beta_{(i)} = e_i - e_{i+1} \ , \quad i = 1, \cdots, n \ .$$

The non-vanishing scalar products between two roots is

$$\beta_{(i)} \cdot \beta_{(i)} = 2 \ , \ i = 1, \cdots, n \ ,$$
$$\beta_{(i)} \cdot \beta_{(i+1)} = -1 \ , \ i = 1, \cdots, n-1 \ ,$$

and the Cartan matrix reduces to

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ \\ \vdots & \cdots & \ddots & \ddots & \ddots & \vdots \\ \\ 0 & \cdots & \cdots & -1 & 2 & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 2 \end{pmatrix} .$$

Finally, all the $(n+1)n$ roots of $\mathfrak{su}(n+1)$ are given by

$$\beta_{(ij)} = e_i - e_j \ , \quad 1 \le i, j \le n+1$$

and $\beta_{(ij)}$ is a positive root when $i > j$ and obviously $\beta_{(ij)} = -\beta_{(ji)}$.

(2) For $\mathfrak{so}(2n+1)$ take $N = n$. The simple roots are given by

$$\begin{aligned} \beta_{(i)} &= e_i - e_{i+1} \ , \quad i = 1, \cdots, n-1 \\ \beta_{(n)} &= e_n \ . \end{aligned}$$

The non-vanishing scalar product between two roots

$$\begin{aligned} \beta_{(i)} \cdot \beta_{(i)} &= \quad 2 \ , \ i = 1, \cdots, n-1 \ , \\ \beta_{(n)} \cdot \beta_{(n)} &= \quad 1 \ , \\ \beta_{(i)} \cdot \beta_{(i+1)} &= -1 \ , i = 1, \cdots, n-1 \ , \end{aligned}$$

and $\beta_{(n)}$ is the shorter root. The Cartan matrix takes the form

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ \\ \vdots & \cdots & \ddots & \ddots & \ddots & \vdots \\ \\ 0 & \cdots & \cdots & -1 & 2 & -1 \\ 0 & \cdots & \cdots & 0 & -2 & 2 \end{pmatrix} .$$

Finally, the n^2 positive roots of $\mathfrak{so}(2n+1)$ are

$$\begin{aligned} \beta_{(ij)} &= e_i - e_j \ , \quad 1 \le i < j \le n \ , \\ \beta'_{(ij)} &= e_i + e_j , \ ,1 \le i < j \le n \ . \\ \beta''_{(i)} &= e_i \ , \quad i = 1, \cdots, n \ . \end{aligned}$$

(3) For $\mathfrak{usp}(2n)$ take $N = n$. The simple roots are given by

$$\begin{aligned} \beta_{(i)} &= e_i - e_{i+1}\ ,\ i = 1, \cdots, n-1\ , \\ \beta_{(n)} &= 2e_n\ . \end{aligned}$$

The non-vanishing scalar products between two roots is

$$\begin{aligned} \beta_{(i)} \cdot \beta_{(i)} &= \ \ 2\ ,\ i = 1, \cdots, n-1\ , \\ \beta_{(n)} \cdot \beta_{(n)} &= \ \ 4\ , \\ \beta_{(i)} \cdot \beta_{(i+1)} &= -1\ ,\ \ i = 1, \cdots, n-2\ , \\ \beta_{(n-1)} \cdot \beta_{(n)} &= -2\ , \end{aligned}$$

$\beta_{(n)}$ is the longer root. The Cartan matrix takes the form

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ & & & & & \\ \vdots & \cdots & \ddots & \ddots & \ddots & \vdots \\ & & & & & \\ 0 & \cdots & \cdots & -1 & 2 & -2 \\ 0 & \cdots & \cdots & 0 & -1 & 2 \end{pmatrix} .$$

Note that the Cartan matrix of $\mathfrak{usp}(2n)$ is the transposed of the Cartan matrix of $\mathfrak{so}(2n+1)$. The positive n^2 roots are

$$\begin{aligned} \beta_{(ij)} &= e_i - e_j\ ,\ \ 1 \le i < j \le n\ , \\ \beta'_{(ij)} &= e_i + e_j\ ,\ \ 1 \le i \le j \le n\ . \end{aligned}$$

(4) For $\mathfrak{so}(2n)$ take $N = n$. The simple roots are given by

$$\begin{aligned} \beta_{(i)} &= e_i - e_{i+1}\ ,\ \ i = 1, \cdots, n-1\ , \\ \beta_{(n)} &= e_{n-1} + e_n\ . \end{aligned}$$

The non-vanishing scalar products between two roots is

$$\begin{aligned} \beta_{(i)} \cdot \beta_{(i)} &= 2\ , i = 1, \cdots, n\ , \\ \beta_{(i)} \cdot \beta_{(i+1)} &= -1\ ,\ i = 1, \cdots n-2\ , \\ \beta_{(n-2)} \cdot \beta_{(n)} &= -1\ , \end{aligned}$$

and the Cartan matrix is

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ \\ \vdots & \cdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \\ 0 & \cdots & \cdots & -1 & 2 & -1 & 0 & 0 \\ 0 & \cdots & \cdots & 0 & -1 & 2 & -1 & -1 \\ 0 & \cdots & \cdots & 0 & 0 & -1 & 2 & 0 \\ 0 & \cdots & \cdots & 0 & 0 & -1 & 0 & 2 \end{pmatrix} .$$

The $n(n-1)$ positive roots are given by

$$\beta_{(ij)} = e_i - e_j \ , 1 \leq i < j \leq n \ ,$$
$$\beta'_{(ij)} = e_i + e_j \ , 1 \leq i < j \leq n \ .$$

The set of roots of the classical Lie groups are related to their fundamental representation as we will see in the next chapters.

(5) For $G_{2(-14)}$ take $N = 3$. The simple roots are given by

$$\beta_{(1)} = -2e_1 + e_2 + e_3 \ ,$$
$$\beta_{(2)} = e_1 - e_2 \ ,$$

and we have

$$\beta_{(1)} \cdot \beta_{(1)} = 6 \ ,$$
$$\beta_{(2)} \cdot \beta_{(2)} = 2 \ ,$$
$$\beta_{(1)} \cdot \beta_{(2)} = -3 \ ,$$

recall the Cartan matrix already given

$$A = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} .$$

The set of positive roots can be deduced from the proceeding subsection, or alternatively can be obtained from Weyl reflections. We now illustrate how to obtain all the roots of G_2 in this way. We start from the simple roots $\beta_{(1)}$ and $\beta_{(2)}$ and proceed to all the possible Weyl reflections. At the first level we obtain

$$\sigma_{\beta_{(1)}}(\beta_{(2)}) = \beta_{(2)} + \beta_{(1)} \ ,$$
$$\sigma_{\beta_{(2)}}(\beta_{(1)}) = \beta_{(1)} + 3\beta_{(2)} \ .$$

At the second level we obtain

$$\begin{aligned}
\sigma_{\beta_{(1)}+3\beta_{(2)}}(\beta_{(1)}) &= 2\beta_{(1)} + 3\beta_{(2)} \ , \\
\sigma_{\beta_{(1)}+\beta_{(2)}}(\beta_{(2)}) &= \beta_{(1)} + 2\beta_{(2)} \ , \\
\sigma_{\beta_{(1)}+\beta_{(2)}}(\beta_{(1)}) &= -2\beta_{(1)} - 3\beta_{(2)} \ , \\
\sigma_{\beta_{(1)}+3\beta_{(2)}}(\beta_{(2)}) &= -\beta_{(1)} - 2\beta_{(2)} \ .
\end{aligned}$$

No more positive roots are obtained. We observe that the generation of positive roots is in one-to-one correspondence with the reconstruction of the algebra (see Sec. 7.3). Finally the positive roots are given by

$$\begin{aligned}
\beta_{(1)} + \beta_{(2)} &= -e_1 + e_3 \ , \\
\beta_{(1)} + 2\beta_{(2)} &= -e_2 + e_3 \ , \\
\beta_{(1)} + 3\beta_{(2)} &= e_1 - 2e_2 + e_3 \ , \\
2\beta_{(1)} + 3\beta_{(2)} &= -e_1 - e_2 + 2e_3 \ .
\end{aligned}$$

(6) For $F_{4(-52)}$ we take $N = 4$. From $\mathfrak{so}(5) \subset F_{4(-52)}$, to construct the root system of $F_{4(-52)}$ we just add one root to the root system of $\mathfrak{so}(5)$. The simple roots are then given by

$$\begin{aligned}
\beta_{(1)} &= e_2 - e_3 \ , \\
\beta_{(2)} &= e_3 - e_4 \ , \\
\beta_{(3)} &= e_4 \ , \\
\beta_{(4)} &= \frac{1}{2}\Big(e_1 - e_2 - e_3 - e_4\Big) \ ,
\end{aligned}$$

which satisfy

$$\begin{aligned}
\beta_{(i)} \cdot \beta_{(i)} &= 2 \ , i = 1, 2 \ , \\
\beta_{(i)} \cdot \beta_{(i)} &= 1 \ , i = 3, 4 \ , \\
\beta_{(1)} \cdot \beta_{(2)} &= -1 \ , \\
\beta_{(2)} \cdot \beta_{(3)} &= -1 \ , \\
\beta_{(3)} \cdot \beta_{(4)} &= -\frac{1}{2} \ ,
\end{aligned}$$

leading to

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \ .$$

To construct the roots associated to E_6, E_7 and E_8 we just use the embeddings

$$\mathfrak{so}(10) \subset \; E_{6(-78)} \subset E_{7(-133)} \subset E_{8(-248)} \ ,$$

and complete the system of simple roots of $\mathfrak{so}(10)$ to a system of simple roots of $E_{6(-78)}, E_{7(-133)}$ and finally $E_{8(-248)}$.

(7) Denote $\beta_{(i)}, i = 1, \cdots, 5$ the simple roots of $\mathfrak{so}(10)$ and set for $E_{6(-78)}$

$$\begin{aligned} \beta_{(i)} &= e_{i+1} - e_i \ , i = 1, \cdots, 4 \ , \\ \beta_{(5)} &= e_1 + e_2 \ , \\ \beta_{(6)} &= \frac{1}{2}\left(e_8 + e_1 - \sum_{i=2}^{7} e_i\right) \ , \end{aligned}$$

which satisfy

$$\begin{aligned} \beta_{(i)} \cdot \beta_{(i)} &= 2 \ , \quad i = 1, \cdots, 6 \ , \\ \beta_{(i)} \cdot \beta_{(i+1)} &= -1 \ , \quad i = 1, \cdots, 3 \ , \\ \beta_{(2)} \cdot \beta_{(5)} &= -1 \ , \\ \beta_{(1)} \cdot \beta_{(6)} &= -1 \ . \end{aligned}$$

Now reordering roots, namely in the basis $(\beta_{(6)}, \beta_{(1)}, \cdots, \beta_{(5)})$ the Cartan matrix reduces to

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix} .$$

Note that if we have chosen

$$\beta_{(6)} = \frac{1}{2}\left(e_1 - \sum_{2}^{5} e_i\right) - \sqrt{2 - \frac{6}{4}} e_6 \ ,$$

we would have reproduced the Cartan matrix above but with irrational coefficients. Furthermore the embedding $E_6 \subset E_7$ would have been less immediate.

(8) The roots of $E_{7(-133)}$ are deduced from those of $E_{6(-78)}$ adding one more root $\beta_{(7)}$

$$\beta_{(7)} = e_6 - e_5 \ ,$$

with the new non-vanishing scalar product (with the reordered vectors of $E_{8(-78)}$)

$$(\beta_{(7)}, \beta_{(4)}) = -1 \ , (\beta_{(7)}, \beta_{(7)}) = 2 \ .$$

Now reordering roots, namely in the basis $(\beta_{(1)}, \cdots, \beta_{(5)}, \beta_{(7)}, \beta_{(6)})$, the Cartan matrix reduces to

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 2 \end{pmatrix} .$$

(9) The roots of $E_{8(-248)}$ are deduced from those of $E_{7(-133)}$ adding one more root $\beta_{(8)}$

$$\beta_{(8)} = e_7 - e_6 \ ,$$

where the non-vanishing scalar product (with the reordered vectors of $E_{7(-133)}$) are

$$\beta_{(8)} \cdot \beta_{(6)} = -1 \ , \quad \beta_{(8)} \cdot \beta_{(8)} = 2$$

Now reordering roots, namely in the basis $(\beta_{(1)}, \cdots, \beta_{(6)}, \beta_{(8)}, \beta_{(7)})$ the Cartan matrix reduces to

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 2 \end{pmatrix} .$$

To end this section we collect in Table 7.5 the simple and positive roots of simple Lie algebras.

7.6 The Weyl group

We now focus briefly on the group of transformations on a root system defined by equation (7.41). To this extent, we reformulate the notion of root

Table 7.5 Simple roots and positive roots of simple Lie algebras. Dimension represents the dimension of the underlying Euclidean space.

Lie algebra	Dimension	Simple roots	Positive roots
A_n	$n+1$	$e_i - e_{i+1}$, $i = 1, \cdots, n$	$e_i - e_j$, $1 \le i < j \le n+1$
B_n	n	$e_i - e_{i+1}$, $i = 1, \cdots, n-1$ e_n	$e_i \pm e_j$, $1 \le i < j \le n$ e_i , $1 \le i \le n$
C_n	n	$e_i - e_{i+1}$, $i = 1, \cdots, n-1$ $2e_n$	$e_i \pm e_j$, $1 \le i < j \le n$ $2e_i$, $1 \le i \le n$
D_n	n	$e_i - e_{i+1}$, $i = 1, \cdots, n-1$ $e_{n-1} + e_n$	$e_i \pm e_j$, $1 \le i < j \le n$
E_6	8	$\frac{1}{2}(e_8 + e_1 - \sum_{i=2}^{7} e_i)$ $e_{i+1} - e_i$, $1 \le i \le 4$ $e_1 + e_2$	$\frac{1}{2}(e_8 - e_7 - e_6 + \sum_{i=1}^{5} \pm e_i)$ even number of signs $e_j \pm e_i$, $1 \le i < j \le 5$
E_7	8	$\frac{1}{2}(e_8 + e_1 - \sum_{i=2}^{7} e_i)$ $e_{i+1} \pm e_i$, $1 \le i \le 5$ $e_1 + e_2$	$\frac{1}{2}(e_8 - e_7 + \sum_{i=1}^{6} \pm e_i)$ odd number of signs $e_j \pm e_i$, $1 \le i < j \le 6$ $e_8 - e_7$
E_8	8	$\frac{1}{2}(e_8 + e_1 - \sum_{i=2}^{7} e_i)$ $e_{i+1} - e_i$, $1 \le i \le 6$ $e_1 + e_2$	$\frac{1}{2}(e_8 + \sum_{i=1}^{7} \pm e_i)$ even number of signs $e_j \pm e_i$, $1 \le i < j \le 8$
F_4	4	$e_2 - e_3$, $e_3 - e_4$ e_4 $\frac{1}{2}(e_1 - e_2 - e_3 - e_4)$	$e_i + \pm e_j$, $1 \le i < j \le 4$ e_i , $1 \le i \le 4$ $\frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)$
G_2	3	$e_1 - e_2$ $-2e_1 + e_2 + e_3$	$e_1 - e_2, e_3 - e_1, e_3 - e_2$ $e_2 + e_3 - 2e_1, e_1 + e_3 - 2e_2$ $-e_1 - e_2 + 2e_3$

system axiomatically and prove that the Weyl group actually corresponds to the point group of such systems.[10]

If E denotes an Euclidean vector space with inner product (α, β), the transformation

$$\sigma_\beta(\alpha) = \alpha - 2\frac{(\beta, \alpha)}{(\beta, \beta)}\beta \ , \tag{7.42}$$

determines geometrically a reflection[11] with reflecting hyperplane $P_\beta = \{\alpha \in E \mid (\alpha, \beta) = 0\}$.

[10]Recall that the symmetry group of a finite system (that is, which leaves invariant the system) is called the point group if all its elements leave at least one point of the system invariant.

[11]Also called an involution, since $\sigma_\beta^2 = \mathrm{Id}$.

Let R be a finite subset of E. It is called a root system if it satisfies the following axioms:

(1) R spans the vector space E and does not contain the zero vector
(2) If $\alpha \in R$, then the only multiples of α in R are $\pm\alpha$.
(3) For any α, the reflection σ_α leaves R invariant.
(4) If $\alpha, \beta \in R$, then $2\frac{(\beta,\alpha)}{(\beta,\beta)} \in \mathbb{Z}$.

We observe that the root system, as defined for semisimple Lie algebras, trivially satisfies the preceding requirements, as a consequence of the Cartan-Weyl decomposition. In fact, the root system can be deduced taking into account the linear forms associated to the generators of a Cartan subalgebra $\mathfrak{h}$ of a semisimple Lie algebra $\mathfrak{g}$, *i.e.*, considering the dual space $\mathfrak{h}^*$. This result in the usual presentation of the classification theorem as to be found in most mathematical texts [Humphreys (1980)]. Here, however, we have opted for a more direct approach.

The Weyl group $\mathcal{W}$ of a root system R is defined as the subgroup of $GL(E)$ (the group of invertible matrices acting on the vector space E) generated by the reflections σ_α. It is clear from this definition that $\mathcal{W}$ leaves the root system R invariant, thus it corresponds to the symmetries of R, and can be interpreted as the point group of the geometric polytope (see Secs. 3.3.2 and 3.5) spanned by the vectors in R. Since $\mathcal{W}$ permutes the roots of R, each reflection can be identified with a permutation of $|R|$ elements, showing that $\mathcal{W}$ is always a finite group.

Now consider an automorphism τ of the vector space E and suppose that it leaves invariant the root system R. If σ_α is an arbitrary element of $\mathcal{W}$, then

$$\tau\sigma_\alpha\tau^{-1}(\tau(\beta)) = \tau\sigma_\alpha(\beta) = \tau(\beta) - 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}\tau(\alpha) \in R. \tag{7.43}$$

It is clear from this equation that $\tau\sigma_\alpha\tau^{-1}$ leaves the root system invariant, sends $\tau(\alpha)$ to $-\tau(\alpha)$ and fixes pointwise the hyperplane $\tau(P_\alpha)$. It follows that the transformation $\tau\sigma_\alpha\tau^{-1}\sigma_{\tau(\alpha)}$ fixes $\tau(\alpha)$ and acts as the identity on the one-dimensional subspace $\mathbb{R}\tau(\alpha)$ of E, as well as on the quotient space $E/\mathbb{R}\tau(\alpha)$. In these conditions, it follows that $\tau\sigma_\alpha\tau^{-1}\sigma_{\tau(\alpha)} = \mathrm{Id}$, thus that $\tau\sigma_\alpha\tau^{-1} = \sigma_{\tau(\alpha)}$, as the latter is an involution. In particular, since

$$\sigma_{\tau(\alpha)}(\tau(\beta)) = \tau(\beta) - 2\frac{(\tau(\beta),\tau(\alpha))}{(\tau(\alpha),\tau(\alpha))}\tau(\alpha) \tag{7.44}$$

using (7.43) we conclude that

$$\frac{(\beta,\alpha)}{(\beta,\beta)} = \frac{(\tau(\beta),\tau(\alpha))}{(\tau(\beta),\tau(\beta))}. \tag{7.45}$$

This property shows that an automorphism of a root system R is the same as an automorphism of the Euclidean space E leaving R invariant. Therefore the Weyl group $\mathcal{W}$ is a normal subgroup of the automorphism group $\mathrm{Aut}(R)$.

The interest of the Weyl group for semisimple Lie algebras lies in the fact that it solves the ambiguity in the choice of simple roots. More specifically, if Σ is the root system associated to the semisimple Lie algebra $\mathfrak{g}$ and we denote by Δ a basis of simple roots, then the following properties are satisfied:[12]

(1) The Weyl group acts transitively on the bases of simple roots, *i.e.*, for two bases Δ, Δ' of Σ there exists some $\sigma \in \mathcal{W}$ such that $\sigma(\Delta) = \Delta'$.
(2) If $\sigma(\Delta) = \Delta$ for some $\sigma \in \mathcal{W}$, then $\sigma = 1$. Hence the Weyl group acts simply transitively on bases.
(3) If $\alpha \in \Sigma$, there exists some $\sigma \in \mathcal{W}$ such that $\sigma(\alpha) \in \Delta$.
(4) The Weyl group is generated by the reflections σ_α, $\alpha \in \Delta$.
(5) All roots of Σ of a given length are conjugate under $\mathcal{W}$.

We first observe that the Weyl group of a semisimple Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$ is isomorphic to the direct product of the Weyl groups associated to the simple algebras $\mathfrak{g}_i$ for $1 \leq i \leq r$. Now, as a consequence of the properties above, to find the Weyl group of a simple algebra it is sufficient to analyse the reflections corresponding to a basis Δ of simple roots. Computing the order of each product of different involutions enables to determine a presentation for the group as:

$$\mathcal{W} = \left\{ \sigma_{\alpha_i}, \alpha_i \in \Delta \mid \sigma_{\alpha_i}^2 = 1,\ \left(\sigma(\alpha_i)\sigma(\alpha_j)\right)^{k_{ij}} = 1 \right\}, \tag{7.46}$$

where k_{ij} denotes the order of the transformation $\sigma(\alpha_i)\sigma(\alpha_j)$. Equation (7.46) is nothing but a particular case of the study of finite groups generated by reflections, commented in a Chapter 3. The enumeration of finite reflection groups is well known, and can be found for example in [Coxeter (1935)]. In this context, however, we merely mention that if a group G is generated by two involutions, then it is isomorphic to a dihedral group. This implies in particular that the Weyl group of the simple Lie algebras $A_2, B_2 = C_2, D_2$ and G_2 is a dihedral group, D_3, D_4, D_2 and D_6, respectively, the Weyl group of $D_2 = C_2 \times C_2$ being the Klein Vierergruppe.

It can further be shown that the automorphism group of the root system R, whenever we fix a basis Δ, is given by the semidirect product of $\Gamma =$

[12] See *e.g.* [Humphreys (1980)] for the detailed proof, as well as for additional properties of the Weyl group.

$\{\tau \in \mathrm{Aut}(R) \mid \tau(\Delta) = \Delta\}$ and the Weyl group $\mathcal{W}$. The group Γ can be easily determined using the corresponding Dynkin diagram of $\mathfrak{g}$. The group Γ is sometimes also called the group of outer automorphisms. Note also that the outer automorphisms group of the extended Dynkin diagram is always larger than the outer automorphisms group of the corresponding Dynkin diagram. For the Weyl group itself, a presentation in terms of generators and relations can also be extracted from the Dynkin diagram [Coxeter (1935)]. The procedure is the following:

(1) Each vertex α of the Dynkin diagram corresponds to a reflection σ_α.
(2) If k is the number of edges joining the vertices α and β, then the transformation $\sigma_\alpha\sigma_\beta$ satisfies the relation $(\sigma_\alpha\sigma_\beta)^{k+2} = 1$.
(3) If the vertices α and β are not connected by an edge, then $\sigma_\alpha\sigma_\beta$ satisfies the relation $(\sigma_\alpha\sigma_\beta)^3 = 1$.

We remark that, specially in the case of the exceptional Lie algebras, their Weyl group shows some connection with the classification of simple Lie groups.

In the following Table we give the order and structure of the Weyl group for the classical complex simple Lie algebras. The precise structure for the Weyl group of exceptional algebras is quite involved, and we omit the details here. The interested reader can find a precise description in [Coxeter (1935)].

Table 7.6 Weyl groups of simple complex Lie algebras.

Lie algebra	Rank	$\mathcal{W}$	$\lvert\mathcal{W}\rvert$	Γ
A_ℓ	ℓ	$\Sigma_{\ell+1}$	$(\ell+1)!$	C_2, $\ell \geq 2$
B_ℓ	ℓ	$C_2^\ell \ltimes \Sigma_\ell$	$2^\ell \ell!$	1
C_ℓ	ℓ	$C_2^\ell \ltimes \Sigma_\ell$	$2^\ell \ell!$	1
D_4	4	$C_2^3 \ltimes \Sigma_4$	$2^3 4!$	C_{3v}
D_ℓ	$\ell > 4$	$C_2^{\ell-1} \ltimes \Sigma_\ell$	$2^{\ell-1}\ell!$	C_2
G_2	2	D_6	12	1
F_4	4		$2^7.3^2$	1
E_6	6		$2^7.3^4.5$	C_2
E_7	7		$2^{10}.3^4.5.7$	1
E_8	8		$2^{14}.3^5.5^2.7$	1