Bayesian Inference for the 2-states Markovian Arrival process

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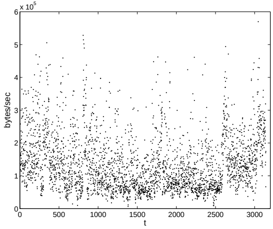
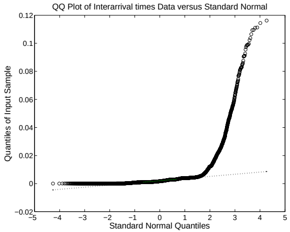
Contents

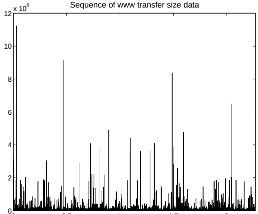
MOTIVATION

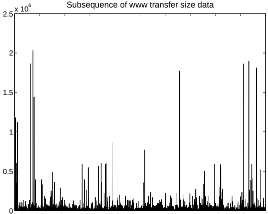
Motivation: teletraffic data

<u>Unusual features</u>: High variability, Heavy-tails, Self-similarity, Dependence and correlation.









Motivation: Queueing systems

- Interest: congestion problems, waiting times, system size...
- Basic assumptions (Poisson arrivals, exponential service times) differs from reality: need for appropriate arrivals and service models.
- The Markovian Arrival process captures the dependence between arrivals $\rightarrow MAP/G/1$.
- The $BMAP/G/1$ queueing system (Lucantoni, 1993): *Matrix-Analytic* approach + transform inversion routines $\rightarrow$ Stationary and Transient distributions for the queue and waiting times.

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THE MARKOVIAN ARRIVAL PROCESS

Introduction

- *Versatile Markovian point process* (Neuts, 1979).
- Convenient representation: *Batch Markovian Arrival process* or *BMAP* (Lucanoni et al. 1990).
  1. Stationary *BMAP*s are **dense** in the family of stationary point processes.
  2. Keeps the **tractability** of the Poisson process.
  3. Allows the inclusion of **dependent** interarrival times.
  4. **Non-exponential** interarrival times.
  5. **Correlated** batch sizes.
- Special cases:
  1. Phase-type renewal processes (Erlang and Hyperexponential),
  2. Markov-modulated Markov process: *MMPP*.
  3. When all arrivals are of size 1, Markovian Arrival Process: *MAP*.

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Definition

- Continuous Markov chain $J(t)$, state space $\mathcal{S} = \{1, \dots, m\}$ and generator matrix D.
- Initial state $i_0 \in \mathcal{S}$ given by an initial probability α.
- At the end of a sojourn time in state i, exponentially distributed with parameter $\lambda_i > 0$, two possible transitions:
  1. With probability p_{ij1} the *MAP* enters state $j \in \mathcal{S}$ and a **single arrival** occurs.
  2. With probability p_{ij0} the *MAP* enters state j **without arrivals**, $j \neq i$
- The *MAP* process is characterized by the set $\{\alpha, \boldsymbol{\lambda}, P_0, P_1\}$, where $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m)$, where

$$P_0 = \begin{pmatrix} 0 & p_{120} & \cdots & p_{1m0} \\ p_{210} & 0 & \cdots & p_{2m0} \\ \cdots & \cdots & \cdots & \cdots \\ p_{m10} & p_{m20} & \cdots & 0 \end{pmatrix}, \qquad P_1 = \begin{pmatrix} p_{111} & \cdots & p_{1m1} \\ p_{211} & \cdots & p_{2m1} \\ \cdots & \cdots & \cdots \\ p_{m11} & \cdots & p_{mm1} \end{pmatrix}$$

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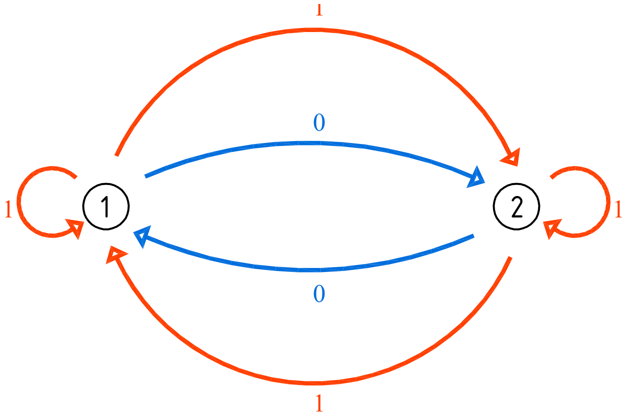
Graphical Illustration: MAP_2

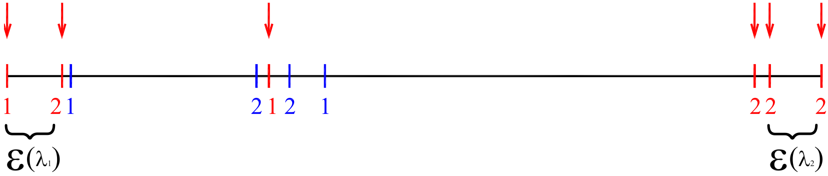


Simulation of a MAP_2

Simulation of 6 arrivals of a MAP_2 characterized by

$$\boldsymbol{\lambda} = (0.5, 4)$$

$$P_0 = \begin{pmatrix} 0 & 0.3 \\ 0.3 & 0 \end{pmatrix}, \qquad P_1 = \begin{pmatrix} 0.4 & 0.3 \\ 0.2 & 0.5 \end{pmatrix}$$



Alternative characterization

- Rate matrices

$$D_0 = \begin{pmatrix} -\lambda_1 & \lambda_1 p_{120} & \dots & \lambda_1 p_{1m0} \\ \lambda_2 p_{210} & -\lambda_2 & \dots & \lambda_2 p_{2m0} \\ \dots & \dots & \dots & \dots \\ \lambda_m p_{m10} & \lambda_m p_{m21} & \dots & -\lambda_m \end{pmatrix}, D_1 = \begin{pmatrix} \lambda_1 p_{111} & \dots & \lambda_1 p_{1m1} \\ \lambda_2 p_{211} & \dots & \lambda_2 p_{2m1} \\ \dots & \dots & \dots \\ \lambda_m p_{m11} & \dots & \lambda_m p_{mm1} \end{pmatrix}$$

- D_0 governs the transitions with no arrivals. D_1 those with a single arrival.
- Then, $D = D_0 + D_1$ is the generator of $J(t)$.
- The *MAP* process is also characterized by the set $\{\alpha, \lambda, D_0, D_1\}$.
- X_k = state of the *MAP* at the time of the kth arrival, Y_k = time between the $(k-1)$th and kth arrival. Then, $\{X_{k-1}, Y_k\}_{k=1}^{\infty}$ is a **Markov Renewal process**.

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Quantities of interest

- $\boldsymbol{\pi}$, stationary probability vector of the Markov process with generator D.
- Fundamental rate: $\lambda^\star = \boldsymbol{\pi} D_1 \mathbf{e}$.
- $1/\lambda^\star$ is the mean interarrival time in the stationary *MAP*.
- T = time between successive arrivals in the stationary version. Then,

$$F_T(t) = P(T \leq t) = (\boldsymbol{\pi} D_1 \mathbf{e})^{-1} \boldsymbol{\pi} D_1 (I - e^{D_0 t})(-D_0)^{-1} L, \quad t \geq 0,$$

where

$$L = \begin{pmatrix} \lambda_1 \left(1 - \sum_{j \neq 1} p_{1j0}\right) \\ \lambda_2 \left(1 - \sum_{j \neq 2} p_{2j0}\right) \\ \vdots \\ \lambda_m \left(1 - \sum_{j \neq m} p_{mj0}\right) \end{pmatrix}.$$

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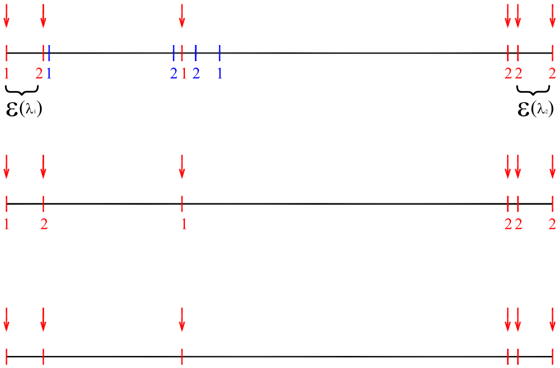
THE *EFFECTIVE* MARKOVIAN ARRIVAL PROCESS

Introduction to the $E-MAP$

MAP ⇒ *E-MAP* ⇒ **only times between arrivals** are assumed to be **observed**.



Definition & Properties

- *Effective* transitions in a *MAP* $\sim$ transitions in the corresponding *E-MAP*.
- Inference for the *MAP* | the *E-MAP* is *partially* observed.
- At the end of a sojourn time in i, (which is distributed as a **sum of exponentials**) there are m possible transitions: with probability $p^\star_{ij}$, for $j = 1, \ldots, m$, an arrival occurs and the process is instantaneously restarted in state j.
- The *E-MAP* is characterized by $\{\boldsymbol{\alpha}, \boldsymbol{\lambda}, P^\star\}$.

- The following properties are satisfied (Ramirez et al. 2008):

  P1. (Transition probability matrix).

$$P^\star = (I - P_0)^{-1} P_1.$$

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Definition & Properties

P2. (Holding times).

Let H_k represent the **holding time** in state k in a E-MAP. Then,

$$F_{H_k}(t) = P(H_k \leq t) = \xi_k (I - e^{D_0 t})(-D_0)^{-1} L,$$

where ξ_k is a vector of zeros with a single 1 in the kth position.

Definition & Properties

P3. (Holding times).

Let H_{ij} be defined as the **holding time** in state i *given that* j is the next visited state, in a *E-MAP*. Then,

$$F_{H_{ij}}(t) = P(H_{ij} \leq t) = \xi_i (I - e^{D_0 t})(-D_0)^{-1} D_1 \xi_j' \left(\xi_i P^\star \xi_j'\right)^{-1}.$$

Definition & Properties

Remark.

The densities of H_k and H_{ij} can be numerically approximated by

$$
\begin{aligned}
f_{H_i}^{(h)}(t) &\approx \frac{F_{H_i}(t+h) - F_{H_i}(t-h)}{2h}, \\
f_{H_{ij}}^{(\tilde{h})}(t) &\approx \frac{F_{H_{ij}}(t+\tilde{h}) - F_{H_{ij}}(t-\tilde{h})}{2\tilde{h}},
\end{aligned}
$$

for some $h, \tilde{h} \approx 0$ so that $f_{H_i}^{(h)}(t) = f_{H_i}^{(h')}(t)$ and $f_{H_{ij}}^{(\tilde{h})}(t) = f_{H_{ij}}^{(h'')}(t)$, for all $h' \leq h$, $h'' \leq \tilde{h}$.

Definition & Properties

P4. (Stationary distribution).

Let ϕ be the stationary distribution associated with the matrix $P^{\star}$. Then ϕ is related to $\boldsymbol{\pi}$ by

$$\phi = (\boldsymbol{\pi} D_1 \mathbf{e})^{-1} \boldsymbol{\pi} D_1.$$

Thus,

$$F_T(t) = P(T \leq t) = \phi(I - e^{D_0 t})(-D_0)^{-1} L, \quad t \geq 0,$$

ON IDENTIFIABILITY OF THE *MAP*

Introduction

- Inference & *identifiability* problems.

$$\text{Generator } MAP\ \{\boldsymbol{\alpha}, \boldsymbol{\lambda}, P_0, P_1\}$$
$$\downarrow$$
$$t_1, \ldots, t_n$$
$$\downarrow$$
$$\text{Estimated } MAP\ \{\widetilde{\boldsymbol{\alpha}}, \widetilde{\boldsymbol{\lambda}}, \widetilde{P}_0, \widetilde{P}_1\}$$

Q1. Is the MAP_2 identifiable?

A1. Only if there does not exist another *equivalent* MAP_2.

Q2. When are two MAP_2s *equivalent*?

A2. When the corresponding *effective* processes or *E-MAP*s are *equivalent*.

Q3. When are two *E-MAP*s *equivalent*?

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Q2. When are two MAP_2s *equivalent*?

A2. When the corresponding *effective* processes or *E-MAP*s are *equivalent*.

Q3. When are two *E-MAP*s *equivalent*?

Formal definition

- T_n= *holding time in the* $(n-1)$*th transition in a E-MAP*
  = time between the $(n-1)$th and nth arrival in a *MAP*.
- **Definition 1.**
  Two *MAP*s $\{\alpha, \lambda, P_0, P_1\}$ and $\{\widetilde{\alpha}, \widetilde{\lambda}, \widetilde{P}_0, \widetilde{P}_1\}$ are equivalent if and only if the corresponding *E-MAP*s $\{\alpha, \lambda, P^\star\}$ and $\{\widetilde{\alpha}, \widetilde{\lambda}, \widetilde{P}^\star\}$ are equivalent.
- **Definition 2.**
  Two *E-MAP*s $\{\alpha, \lambda, P^\star\}$ and $\{\alpha, \widetilde{\lambda}, \widetilde{P}^\star\}$ are equivalent if and only if

$$T_n \stackrel{d}{=} \widetilde{T}_n, \quad \forall n \geq 1,$$

- **Definition 3.**
  A *MAP* $\{\alpha, \lambda, P_0, P_1\}$ with corresponding *E-MAP* $\{\alpha, \lambda, P^\star\}$ is identifiable if there does not exist a different *MAP* whose associated *E-MAP* $\{\widetilde{\alpha}, \widetilde{\lambda}, \widetilde{P}^\star\}$ is equivalent to $\{\alpha, \lambda, P^\star\}$.

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Remark

- Equivalence is expressed in a *weak* sense.
- Definition based on the **marginal** interarrival time distribution.
- However, for *strong* equivalence,

$$f(t_1, \ldots, t_n | \alpha, \lambda, P_0, P_1) = f(t_1, \ldots, t_n | \widetilde{\alpha}, \widetilde{\lambda}, \widetilde{P}_0, \widetilde{P}_1), \qquad \forall n.$$

- In a *MAP* the interarrival times are not *independent* (although they are *conditionally independent* given the sequence of visited states), and thus,

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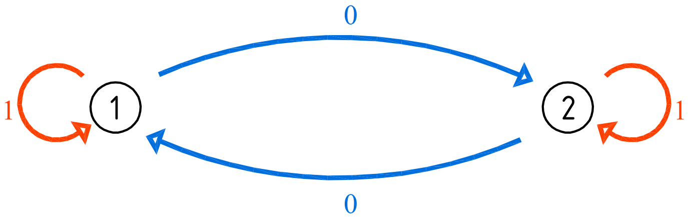
Remark: *MMPP*



Rydén (1996): the *MMPP* is identifiable (in strong sense) if and only if the exponential rates are ordered.

Two general results

- $\varphi_{T_{n+1}}(s) = \sum_{i=1}^{m} \alpha_i^{(n)} \varphi_{H_i}(s) = \boldsymbol{\alpha}^{(n)} \varphi_{\mathbf{H}}(s)$, where $\boldsymbol{\alpha}^{(n)} = \boldsymbol{\alpha}(P^\star)^n$.
- **Result 1**.

$$T_n \overset{d}{=} \widetilde{T}_n, \quad \forall n \geq 1$$

$$\iff$$

$$\boldsymbol{\alpha}(P^\star)^n \varphi_{\mathbf{H}}(s) = \widetilde{\boldsymbol{\alpha}}(\widetilde{P}^\star)^n \varphi_{\widetilde{\mathbf{H}}}(s), \quad \forall s, \quad \forall n \geq 0$$

- **Result 2**.
A necessary condition for two *MAP*s to be equivalent is

$$\phi \varphi_{\mathbf{H}}(s) = \widetilde{\phi} \varphi_{\widetilde{\mathbf{H}}}(s), \quad \forall s,$$

where ϕ is the stationary probability vector of $P^\star$, governing the state transitions in the *E-MAP*.

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General result for $m = 2$.

Let $\{\boldsymbol{\alpha}, \boldsymbol{\lambda}, P_0, P_1\}$ and $\{\widetilde{\boldsymbol{\alpha}}, \widetilde{\boldsymbol{\lambda}}, \widetilde{P}_0, \widetilde{P}_1\}$ define two MAP$_2$s, with corresponding E-MAP$_2$s $\{\boldsymbol{\alpha}, \boldsymbol{\lambda}, P^\star\}$ and $\{\widetilde{\boldsymbol{\alpha}}, \widetilde{\boldsymbol{\lambda}}, \widetilde{P}^\star\}$, where ϕ and $\widetilde{\phi}$ are the stationary probabilities associated to $P^\star$ and $\widetilde{P}^\star$. Assume,

(i) $P^\star \neq \boldsymbol{\Phi}$ or $\widetilde{P}^\star \neq \widetilde{\boldsymbol{\Phi}}$,

(ii) $\beta_1 \neq 0$, and $\widetilde{\beta}_1 \neq 0$, where

$$\begin{aligned}\beta_1 &= \lambda_1(p_{120} - 1) + \lambda_2(1 - p_{210}),\\ \widetilde{\beta}_1 &= \widetilde{\lambda}_1(1 - \widetilde{p}_{120}) + \widetilde{\lambda}_2(\widetilde{p}_{210} - 1).\end{aligned}$$

Then, the MAP$_2$s $\{\boldsymbol{\alpha}, \boldsymbol{\lambda}, P_0, P_1\}, \{\widetilde{\boldsymbol{\alpha}}, \widetilde{\boldsymbol{\lambda}}, \widetilde{P}_0, \widetilde{P}_1\}$ are (*weakly*) equivalent if and only if the following two conditions are fulfilled,

C1. $\phi\varphi_{\mathbf{H}}(s) = \widetilde{\phi}\varphi_{\widetilde{\mathbf{H}}}(s)$,

C2. $(\boldsymbol{\alpha}, \widetilde{\boldsymbol{\alpha}}) = (\phi, \widetilde{\phi})$. $\square$

Remarks

1. C1. is equivalent to $T \stackrel{d}{=} \widetilde{T}$.
2. C2. implies that $T_1 \stackrel{d}{=} T_2 \stackrel{d}{=} \ldots \stackrel{d}{=} T_n \stackrel{d}{=} \ldots \stackrel{d}{=} T$, and similarly with $\widetilde{T}_j$, $\forall j \geq 1$.
3. (Weak) equivalence between two MAP_2s can be established only if both MAP_2s are in the stationary version.
4. It can be shown that

$$\phi\varphi_{\mathbf{H}}(s) = \frac{a_1 s + d_0}{s^2 + d_1 s + d_0},$$

where

$$\begin{aligned} a_1 &= \phi\lambda_1(p_{120} - 1) + \lambda_2(\phi + p_{210} - 1 - \phi p_{210}), \\ d_1 &= -(\lambda_1 + \lambda_2), \\ d_0 &= \lambda_1\lambda_2(1 - p_{120}p_{210}), \end{aligned}$$

and thus, the result provides a simple way to test the weak equivalence of two MAP_2.

Example

Consider the MAP_2 defined by

$$\boldsymbol{\lambda} = (0.5, 20), \qquad P_0 = \begin{pmatrix} 0 & 0.3 \\ 0.3 & 0 \end{pmatrix}, \qquad P_1 = \begin{pmatrix} 0.6148 & 0.0852 \\ 0.0886 & 0.6114 \end{pmatrix}$$

and initial probability $\alpha = \phi = 0.504$.

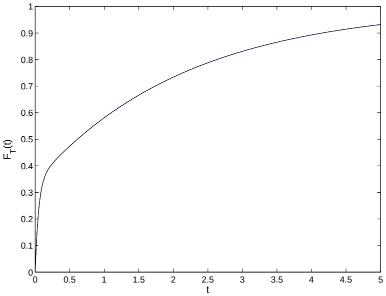
Consider another MAP_2 with parameters

$$\boldsymbol{\lambda} = (0.8, 19.7), \qquad P_0 = \begin{pmatrix} 0 & 0.7683 \\ 0.55 & 0 \end{pmatrix}, \qquad P_1 = \begin{pmatrix} 0.0513 & 0.1804 \\ 0.0873 & 0.3627 \end{pmatrix}$$

and initial probability $\alpha = \phi = 0.201$.

Example

- It can be seen that $\phi\varphi_H(s) = \widetilde{\phi}\varphi_{\widetilde{H}}(s)$, for all s.
- We are thus in the assumptions of the Theorem. This assures that the processes are weakly equivalent.
- Figure: CDF of T, time until next arrival in the stationary version of both MAP_2s.



BAYESIAN INFERENCE FOR THE MAP_2

Introduction

- Performance analysis for models incorporating *MAP*s: well-developed area.
- Less progress on statistical estimation for such models.
- *MMPP*:
  - Frequentist approaches: Heffes (1980), Rydén (1996), Salvador et al. (2003).
  - Bayesian approach: Fearnhead and Sherlock (2006). Methodology based on the construction of the unobserved components.
- *BMAP*: Klemm et al. (2003), EM to estimate the *BMAP*.
- Aim: Bayesian inference for the MAP_2 using theoretical results obtained for the *E-MAP*.

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Data & Parameters of the model

- We assume that the available data are the times between two successive arrivals, $\mathbf{t} = (t_1, \ldots, t_n)$ in a **stationary** MAP_2.
- The underlying Markov process governing the different states of the process, and the transition changes will be assumed to be unobservable.
- Parameters:

$$\boldsymbol{\lambda} = (\lambda_1, \lambda_2) : \qquad \text{Exponential rates}$$

$$\begin{aligned} \mathbf{p}_1 &= (p_{120}, p_{111}, p_{121}) : && \text{Transition probabilities from state 1} \\ \mathbf{p}_2 &= (p_{210}, p_{211}, p_{221}) : && \text{Transition probabilities from state 2} \end{aligned}$$

Prior distributions

- Independent gamma priors for λ_1 and λ_2,

$$\lambda_1, \lambda_2 \sim \mathcal{G}(\alpha, \beta),$$

  where we introduce the minimum order restriction $\lambda_1 < \lambda_2$ to reduce problems due to lack of identifiability of the model.

- Dirichlet priors for the vector of probabilities,

$$\mathbf{p}_1, \mathbf{p}_2 \sim D(c\mathbf{e}),$$

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Likelihood

$$f(\mathbf{t}|\boldsymbol{\lambda},\mathbf{p}_1,\mathbf{p}_2)=\sum_{i_n=1}^{2}\ldots\sum_{i_1=1}^{2}\phi_{i_1}p^{\star}_{i_1 i_2}f_{H_{i_1 i_2}}(t_1)p^{\star}_{i_2 i_3}f_{H_{i_2 i_3}}(t_2)\ldots p^{\star}_{i_{n-1} i_n}f_{H_{i_{n-1} i_n}}(t_{n-1})f_{H_{i_n}}(t_n)$$

where,

$$\begin{aligned}
\phi_i &= \text{Stationary probability that the E-MAP is in state } i.\\
p^{\star}_{ij} &= \text{Probability of a transition from } i \text{ to } j \text{ in the E-MAP.}\\
f_{H_{ij}}(t) &= \text{Density of the holding time in a transition } i\rightarrow j\text{, in the E-MAP.}\\
f_{H_i}(t) &= \text{Density of the holding time in state } i \text{ in the E-MAP.}
\end{aligned}$$

Likelihood

It can be shown that

$$f(\mathbf{t}|\boldsymbol{\lambda}, \mathbf{p}_1, \mathbf{p}_2) = \phi \prod_{i=1}^{n-1} \mathcal{F}(t_i)\mathcal{B}(t_n),$$

where

$$\mathcal{F}(t) = \begin{pmatrix} p^\star_{11} f_{H_{11}}(t) & p^\star_{12} f_{H_{12}}(t) \\ p^\star_{21} f_{H_{21}}(t) & p^\star_{22} f_{H_{22}}(t) \end{pmatrix} \quad \text{and} \quad \mathcal{B}(t) = \begin{pmatrix} f_{H_1}(t) \\ f_{H_2}(t) \end{pmatrix}.$$

Numerical complexity due to

1. Approximation of $f_{H_k}(t)$ and $f_{H_{ij}}(t)$.
2. Product of n matrices.

The posterior distribution

- Combining the likelihood & priors gives a non-conjugate posterior distribution:

$$f(\boldsymbol{\lambda}, \mathbf{p}_1, \mathbf{p}_2|\mathbf{t}) \propto \pi(\lambda_1)\pi(\lambda_2)\pi(\mathbf{p}_1)\pi(\mathbf{p}_2)f(\mathbf{t}|\boldsymbol{\lambda}, \mathbf{p}_1, \mathbf{p}_2).$$

- Metropolis-Hastings algorithm.

- Increase the acceptance rate: 3 blocks.

The Metropolis-Hastings to estimate the MAP_2

1. Draw a starting point $\boldsymbol{\lambda}^{(0)}$, $\mathbf{p}_1^{(0)}$ and $\mathbf{p}_2^{(0)}$ from the prior distributions.
2. For $t = 2, \ldots$:
   (a) Sample a proposal $\boldsymbol{\lambda}^\star$ from a *Log-Normal* distribution,
   $$\log(\boldsymbol{\lambda}^\star) \sim N\left(\log(\boldsymbol{\lambda}^{(t-1)}), \sigma\right).$$
   Accept or reject.
   (b) Sample a proposal $\mathbf{p}_1^\star$ from a *Dirichlet* distribution
   $$\mathbf{p}_1^\star \sim \mathcal{D}(d_1\mathbf{e}).$$
   Accept or reject.
   (c) Sample a proposal $\mathbf{p}_2^\star$ from a *Dirichlet* distribution
   $$\mathbf{p}_2^\star \sim \mathcal{D}(d_2\mathbf{e}).$$
   Accept or reject.

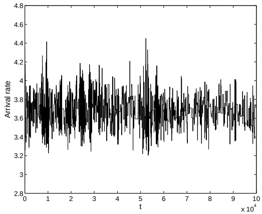
Performance: Simulated data 1

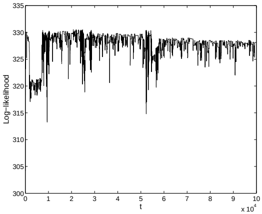
- 1000 simulated interarrival times from the stationary MAP_2

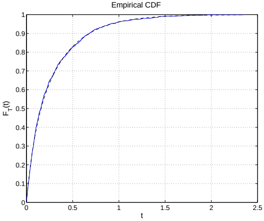
$$\boldsymbol{\lambda} = (3, 10), \qquad P_0 = \begin{pmatrix} 0 & 0.2 \\ 0.25 & 0 \end{pmatrix}, \qquad P_1 = \begin{pmatrix} 0.35 & 0.45 \\ 0.35 & 0.4 \end{pmatrix}$$

- $$\lambda^\star = 3.6509, \qquad \log(f(\mathbf{t}|\boldsymbol{\lambda}, \mathbf{p}_1, \mathbf{p}_2)) = 328.8059$$
- 100 000 iterations, 50 000 burn-in
- $d_1 = d_2 = 0.6$
- Initially, $\sigma = 1$; Within the *burn-in* period: $\sigma = 0.3$
- $$\boldsymbol{\lambda}^0 = (1, 5), \qquad P_0^0 = \begin{pmatrix} 0 & 0.0872 \\ 0.0270 & 0 \end{pmatrix}, \qquad P_1^0 = \begin{pmatrix} 0.1027 & 0.8101 \\ 0.6735 & 0.2995 \end{pmatrix}$$

Arrival rate, Log-Likelihood, $F_T(t)$







Results

- $$\lambda^\star = 3.6509$$
- $$E(\lambda^\star|\cdot) = 3.6712$$
- Acceptance rate for $\boldsymbol{\lambda}$: 14.63%
- Acceptance rate for $\mathbf{p}_1$, $\mathbf{p}_2$: 2.5%
- Computational time: $\approx$ 4h

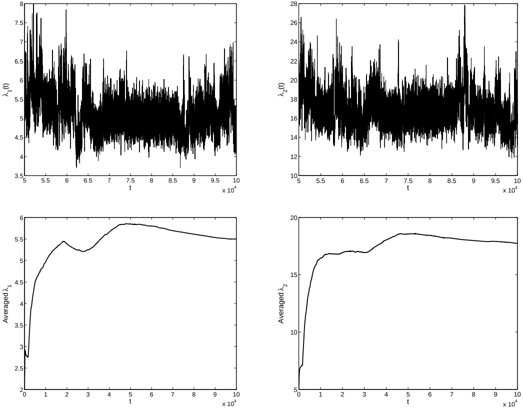
Performance: Simulated data 2

- 1000 simulated interarrival times from the stationary $MMPP_2$

$$\boldsymbol{\lambda} = (5, 20), \qquad P_0 = \begin{pmatrix} 0 & 0.7 \\ 0.4 & 0 \end{pmatrix}, \qquad P_1 = \begin{pmatrix} 0.3 & \mathbf{0} \\ \mathbf{0} & 0.6 \end{pmatrix}$$

- $$\lambda^\star = 4.6957, \qquad \log(f(\mathbf{t}|\boldsymbol{\lambda}, \mathbf{p}_1, \mathbf{p}_2)) = 618.5995$$
- 100 000 iterations, 50 000 burn-in
- $d_1 = d_2 = 0.6$
- Initially, $\sigma = 1$; Within the *burn-in* period: $\sigma = 0.3$
- $$\boldsymbol{\lambda}^0 = (1, 5), \qquad P_0^0 = \begin{pmatrix} 0 & 0.783 \\ 0.6739 & 0 \end{pmatrix}, \qquad P_1^0 = \begin{pmatrix} 0.217 & 0 \\ 0 & 0.3261 \end{pmatrix}$$

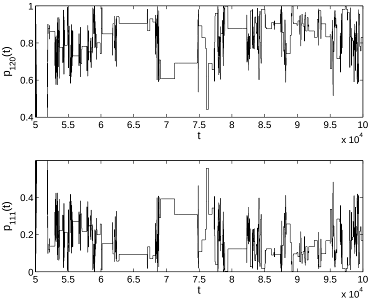
Exponential rates



$$E(\lambda_1|\cdot) = 5.14, \qquad E(\lambda_2|\cdot) = 17.21$$

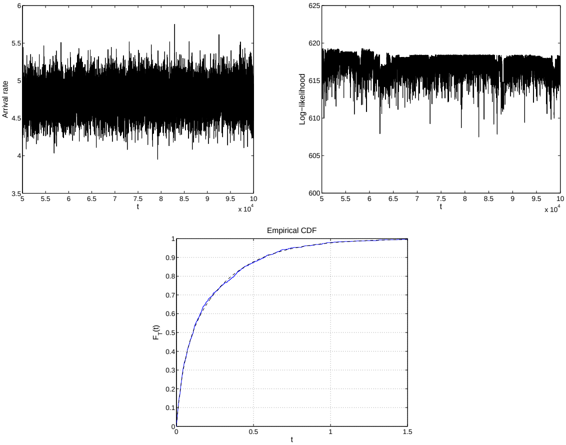
Transition probabilities

The $MMPP_2$ is identifiable, thus, small variability is expected for the values of $\mathbf{p}_1$ and $\mathbf{p}_2$.



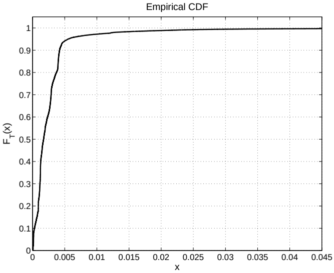
$$E(\mathbf{p}_1|\cdot) = (0.7866, 0.2134), \qquad E(\mathbf{p}_2|\cdot) = (0.3722, 0.6278).$$

Arrival rate, Log-Likelihood, $F_T(t)$

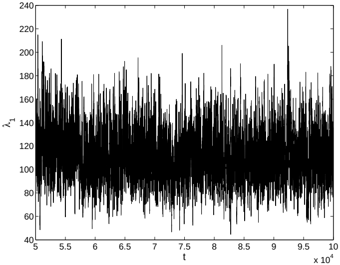


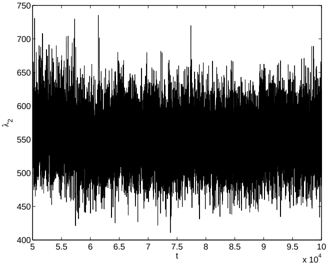
Real data set

50000 first interarrival times in seconds of a trace of 1 million ethernet packets. Source: `http://www.xtremes.de/xtremes/xtremes/download/download.htm`.

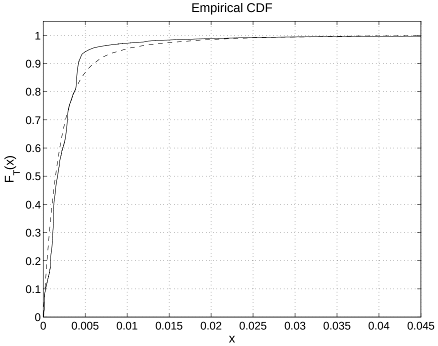


Exponential rates





CDF


Empirical CDF
$F_T(x)$
x
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0.005
0.01
0.015
0.02
0.025
0.03
0.035
0.04
0.045

CONCLUSIONS & EXTENSIONS

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- First step in the study of the identifiability of *MAP*s.
- Deep study of the *E-MAP*.
- Results that assures weak equivalence.

- Bayesian method to estimate the MAP_2.
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Extensions

- Study identifiability of *MAP*s in the strong sense.
- Get a better acceptance rate for $\mathbf{p}_1$ and $\mathbf{p}_2$: play with proposals.
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