Bayesian Inference for the 2-states Markovian Arrival process

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Motivation: teletraffic data and queueing systems.

The Markovian Arrival Process (MAP) and the Effective Markovian Arrival Process (E-MAP).

Identifiability of the MAP.

Bayesian Inference for the MAP.

Conclusions & Extensions.
MOTIVATION
Motivation: teletraffic data

Unusual features: High variability, Heavy-tails, Self-similarity, Dependence and correlation.
Motivation: Queueing systems

- Interest: congestion problems, waiting times, system size...
- Basic assumptions (Poisson arrivals, exponential service times) differs from reality: need for appropriate arrivals and service models.
- The Markovian Arrival process captures the dependence between arrivals $\rightarrow MAP/G/1$.
- The $BMAP/G/1$ queueing system (Lucantoni, 1993): Matrix-Analytic approach + transform inversion routines $\rightarrow$ Stationary and Transient distributions for the queue and waiting times.
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THE MARKOVIAN ARRIVAL PROCESS
Introduction

- Versatile Markovian point process (Neuts, 1979).

- Convenient representation: Batch Markovian Arrival process or BMAP (Lucanoni et al. 1990).

  1. Stationary BMAPs are dense in the family of stationary point processes.
  2. Keeps the tractability of the Poisson process.
  3. Allows the inclusion of dependent interarrival times.
  4. Non-exponential interarrival times.
  5. Correlated batch sizes.

- Special cases:
  1. Phase-type renewal processes (Erlang and Hyperexponential),
  2. Markov-modulated Markov process: MMPP.
  3. When all arrivals are of size 1, Markovian Arrival Process: MAP.
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- Special cases:
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  3. When all arrivals are of size 1, Markovian Arrival Process: MAP.
Definition

- Continuous Markov chain \( J(t) \), state space \( S = \{1, \ldots, m\} \) and generator matrix \( D \).
- Initial state \( i_0 \in S \) given by an initial probability \( \alpha \).
- At the end of a sojourn time in state \( i \), exponentially distributed with parameter \( \lambda_i > 0 \), two possible transitions:
  1. With probability \( p_{ij1} \) the MAP enters state \( j \in S \) and a single arrival occurs.
  2. With probability \( p_{ij0} \) the MAP enters state \( j \) without arrivals, \( j \neq i \)
- The MAP process is characterized by the set \( \{\alpha, \lambda, P_0, P_1\} \), where \( \lambda = (\lambda_1, \ldots, \lambda_m) \), where

\[
P_0 = \begin{pmatrix}
0 & p_{120} & \cdots & p_{1m0} \\
p_{210} & 0 & \cdots & p_{2m0} \\
\vdots & \vdots & \ddots & \vdots \\
p_{m10} & p_{m20} & \cdots & 0
\end{pmatrix}, \quad P_1 = \begin{pmatrix}
p_{111} & \cdots & p_{1m1} \\
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- Continuous Markov chain $J(t)$, state space $S = \{1, \ldots, m\}$ and generator matrix $D$.
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Graphical Illustration: $MAP_2$
Simulation of a $\text{MAP}_2$

Simulation of 6 arrivals of a $\text{MAP}_2$ characterized by

$$\lambda = (0.5, 4)$$

$$P_0 = \begin{pmatrix} 0 & 0.3 \\ 0.3 & 0 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 0.4 & 0.3 \\ 0.2 & 0.5 \end{pmatrix}$$
Alternative characterization

- Rate matrices

\[
D_0 = \begin{pmatrix}
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- \(D_0\) governs the transitions with no arrivals. \(D_1\) those with a single arrival.

- Then, \(D = D_0 + D_1\) is the generator of \(J(t)\).

- The MAP process is also characterized by the set \(\{\alpha, \lambda, D_0, D_1\}\).

- \(X_k = \) state of the MAP at the time of the \(k\)th arrival, \(Y_k = \) time between the \((k-1)\)th and \(k\)th arrival. Then, \(\{X_{k-1}, Y_k\}_{k=1}^{\infty}\) is a Markov Renewal process.
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Quantities of interest

- $\pi$, stationary probability vector of the Markov process with generator $D$.
- Fundamental rate: $\lambda^* = \pi D_1 e$.
- $1/\lambda^*$ is the mean interarrival time in the stationary MAP.
- $T =$ time between successive arrivals in the stationary version. Then,
  
  $$F_T(t) = P(T \leq t) = (\pi D_1 e)^{-1} \pi D_1 (I - e^{D_0 t})(-D_0)^{-1} L, \quad t \geq 0,$$

  where

  $$L = \begin{pmatrix}
  \lambda_1 \left(1 - \sum_{j \neq 1} p_{1j0}\right) \\
  \lambda_2 \left(1 - \sum_{j \neq 2} p_{2j0}\right) \\
  \vdots \\
  \lambda_m \left(1 - \sum_{j \neq m} p_{mj0}\right)
  \end{pmatrix}.$$
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THE EFFECTIVE MARKOVIAN ARRIVAL PROCESS
Introduction to the $E – MAP$

$MAP \Rightarrow E-MAP \Rightarrow$ only times between arrivals are assumed to be observed.
Definition & Properties

- *Effective* transitions in a MAP $\sim$ transitions in the corresponding E-MAP.
- Inference for the MAP | the E-MAP is partially observed.
- At the end of a sojourn time in $i$, (which is distributed as a sum of exponentials) there are $m$ possible transitions: with probability $p_{ij}^*$, for $j = 1, \ldots, m$, an arrival occurs and the process is instantaneously restarted in state $j$.
- The E-MAP is characterized by $\{\alpha, \lambda, P^*\}$.

The following properties are satisfied (Ramirez et al. 2008):

P1. (Transition probability matrix).

\[ P^* = (I - P_0)^{-1}P_1. \]
Definition & Properties

- *Effective* transitions in a $MAP \sim$ transitions in the corresponding $E-MAP$.
- Inference for the $MAP$ | the $E-MAP$ is *partially* observed.
- At the end of a sojourn time in $i$, (which is distributed as a sum of exponentials) there are $m$ possible transitions: with probability $p_{ij}^*$, for $j = 1, \ldots, m$, an arrival occurs and the process is instantaneously restarted in state $j$.
- The $E-MAP$ is characterized by $\{\alpha, \lambda, P^*\}$.

- The following properties are satisfied (Ramirez et al. 2008):
  - **P1.** (Transition probability matrix).
    \[ P^* = (I - P_0)^{-1}P_1. \]
Definition & Properties

**P2.** (Holding times).

Let $H_k$ represent the **holding time** in state $k$ in a E-MAP. Then,

$$F_{H_k}(t) = P(H_k \leq t) = \xi_k (I - e^{D_0 t})(-D_0)^{-1}L,$$

where $\xi_k$ is a vector of zeros with a single 1 in the $k$th position.
Definition & Properties

**P3.** (Holding times).

Let $H_{ij}$ be defined as the **holding time** in state $i$ *given that* $j$ is the next visited state, in an $E$-MAP. Then,

$$
F_{H_{ij}}(t) = P(H_{ij} \leq t) = \xi_i(1 - e^{D_0 t})(-D_0)^{-1} D_1 \xi_j' \left( \xi_i P^* \xi_j' \right)^{-1}.
$$
Definition & Properties

Remark.

The densities of $H_k$ and $H_{ij}$ can be numerically approximated by

$$f_{H_i}^{(h)}(t) \approx \frac{F_{H_i}(t + h) - F_{H_i}(t - h)}{2h},$$

$$f_{H_{ij}}^{(\tilde{h})}(t) \approx \frac{F_{H_{ij}}(t + \tilde{h}) - F_{H_{ij}}(t - \tilde{h})}{2\tilde{h}},$$

for some $h, \tilde{h} \approx 0$ so that $f_{H_i}^{(h)}(t) = f_{H_i}^{(h')}(t)$ and $f_{H_{ij}}^{(\tilde{h})}(t) = f_{H_{ij}}^{(h'')}(t)$, for all $h' \leq h$, $h'' \leq \tilde{h}$. 
Definition & Properties

P4. (Stationary distribution).

Let $\phi$ be the stationary distribution associated with the matrix $P^*$. Then $\phi$ is related to $\pi$ by

$$
\phi = (\pi D_1 e)^{-1} \pi D_1.
$$

Thus,

$$
F_T(t) = P(T \leq t) = \phi(1 - e^{D_0 t})(-D_0)^{-1} L, \quad t \geq 0,
$$
## ON IDENTIFIABILITY OF THE MAP

<table>
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Introduction

- Inference & *identifiability* problems.

Generator $MAP\ \{\alpha, \lambda, P_0, P_1\}$

\[
\downarrow \\
\quad t_1, \ldots, t_n \\
\downarrow \\
\text{Estimated } MAP\ \{\tilde{\alpha}, \tilde{\lambda}, \tilde{P}_0, \tilde{P}_1\}
\]

Q1. Is the $MAP_2$ identifiable?
A1. Only if there does not exist another equivalent $MAP_2$.
Q2. When are two $MAP_2$s equivalent?
A2. When the corresponding *effective* processes or $E$-$MAP$s are equivalent.
Q3. When are two $E$-$MAP$s equivalent?
Introduction

- Inference & identifiability problems.

Generator $MAP \{\alpha, \lambda, P_0, P_1\}$

\[ \downarrow \]

$t_1, \ldots, t_n$

\[ \downarrow \]

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Introduction

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 Generator $\text{MAP} \{\alpha, \lambda, P_0, P_1\}$

 $\downarrow$

$t_1, \ldots, t_n$

 $\downarrow$

 Estimated $\text{MAP} \{\tilde{\alpha}, \tilde{\lambda}, \tilde{P}_0, \tilde{P}_1\}$

Q1. Is the $\text{MAP}_2$ identifiable?

A1. Only if there does not exist another *equivalent* $\text{MAP}_2$.

Q2. When are two $\text{MAP}_2$s equivalent?

A2. When the corresponding *effective* processes or $E$-MAPs are equivalent.

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Introduction

- Inference & identifiability problems.

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\begin{align*}
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\downarrow \\
\{t_1, \ldots, t_n\} \\
\downarrow \\
\text{Estimated MAP} \; \{\tilde{\alpha}, \tilde{\lambda}, \tilde{P}_0, \tilde{P}_1\}
\end{align*}
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- Inference & *identifiability* problems.

Generator MAP \( \{\alpha, \lambda, P_0, P_1\} \)

\[ \downarrow \]

\( t_1, \ldots, t_n \)

\[ \downarrow \]

Estimated MAP \( \{\tilde{\alpha}, \tilde{\lambda}, \tilde{P}_0, \tilde{P}_1\} \)

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- Inference & identifiability problems.

Generator \( MAP \{ \alpha, \lambda, P_0, P_1 \} \)
\[ \downarrow \]
\[ t_1, \ldots, t_n \]
\[ \downarrow \]
Estimated \( MAP \{ \tilde{\alpha}, \tilde{\lambda}, \tilde{P}_0, \tilde{P}_1 \} \)

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Formal definition

- $T_n = \text{holding time in the } (n - 1)\text{th transition in a E-MAP} = \text{time between the } (n - 1)\text{th and } n\text{th arrival in a MAP.}$

- **Definition 1.**
  Two MAPs $\{\alpha, \lambda, P_0, P_1\}$ and $\{\tilde{\alpha}, \tilde{\lambda}, \tilde{P}_0, \tilde{P}_1\}$ are equivalent if and only if the corresponding E-MAPs $\{\alpha, \lambda, P^*\}$ and $\{\tilde{\alpha}, \tilde{\lambda}, \tilde{P}^*\}$ are equivalent.

- **Definition 2.**
  Two E-MAPs $\{\alpha, \lambda, P^*\}$ and $\{\tilde{\alpha}, \tilde{\lambda}, \tilde{P}^*\}$ are equivalent if and only if
  \[ T_n \overset{d}{=} \tilde{T}_n, \quad \forall n \geq 1, \]

- **Definition 3.**
  A MAP $\{\alpha, \lambda, P_0, P_1\}$ with corresponding E-MAP $\{\alpha, \lambda, P^*\}$ is identifiable if there does not exist a different MAP whose associated E-MAP $\{\tilde{\alpha}, \tilde{\lambda}, \tilde{P}^*\}$ is equivalent to $\{\alpha, \lambda, P^*\}$. 
Formal definition

- \( T_n = \) holding time in the \((n - 1)th\) transition in a \(E-MAP\)
  \(=\) time between the \((n - 1)th\) and \(n\)th arrival in a \(MAP\).

- **Definition 1.**
  Two \(MAPs\) \(\{\alpha, \lambda, P_0, P_1\}\) and \(\{\tilde{\alpha}, \tilde{\lambda}, \tilde{P}_0, \tilde{P}_1\}\) are equivalent if and only if the corresponding \(E-MAPs\) \(\{\alpha, \lambda, P^*\}\) and \(\{\tilde{\alpha}, \tilde{\lambda}, \tilde{P}^*\}\) are equivalent.

- **Definition 2.**
  Two \(E-MAPs\) \(\{\alpha, \lambda, P^*\}\) and \(\{\alpha, \tilde{\lambda}, \tilde{P}^*\}\) are equivalent if and only if
  \[ T_n \overset{d}{=} \tilde{T}_n, \quad \forall n \geq 1, \]

- **Definition 3.**
  A \(MAP\) \(\{\alpha, \lambda, P_0, P_1\}\) with corresponding \(E-MAP\) \(\{\alpha, \lambda, P^*\}\)
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Remark

- Equivalence is expressed in a weak sense.
- Definition based on the marginal interarrival time distribution.
- However, for strong equivalence,

$$f(t_1, \ldots, t_n | \alpha, \lambda, P_0, P_1) = f(t_1, \ldots, t_n | \tilde{\alpha}, \tilde{\lambda}, \tilde{P}_0, \tilde{P}_1), \quad \forall n.$$ 

- In a MAP the interarrival times are not independent (although they are conditionally independent given the sequence of visited states), and thus,

Weak equivalence $\not\sim$ Strong equivalence.
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  Weak equivalence $\not\sim$ Strong equivalence.
Remark: \textit{MMPP}

Rydén (1996): the \textit{MMPP} is identifiable (in strong sense) if and only if the exponential rates are ordered.
Two general results

- \( \varphi_{T_{n+1}}(s) = \sum_{i=1}^{m} \alpha_i^{(n)} \varphi_{H_i}(s) = \alpha^{(n)} \varphi_{H}(s) \), where \( \alpha^{(n)} = \alpha(P^*)^n \).

- Result 1.

\[
T_n \overset{d}{=} \tilde{T}_n, \quad \forall n \geq 1
\]

\[
\iff
\alpha(P^*)^n \varphi_{H}(s) = \tilde{\alpha}(\tilde{P}^*)^n \varphi_{\tilde{H}}(s), \quad \forall s, \quad \forall n \geq 0
\]

- Result 2.

A necessary condition for two MAPs to be equivalent is

\[
\phi \varphi_{H}(s) = \tilde{\phi} \varphi_{\tilde{H}}(s), \quad \forall s,
\]

where \( \phi \) is the stationary probability vector of \( P^* \), governing the state transitions in the E-MAP.
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General result for $m = 2$.

Let $\{\alpha, \lambda, P_0, P_1\}$ and $\{\tilde{\alpha}, \tilde{\lambda}, \tilde{P}_0, \tilde{P}_1\}$ define two MAP$_2$s, with corresponding E-MAP$_2$s $\{\alpha, \lambda, P^*\}$ and $\{\tilde{\alpha}, \tilde{\lambda}, \tilde{P}^*\}$, where $\phi$ and $\tilde{\phi}$ are the stationary probabilities associated to $P^*$ and $\tilde{P}^*$. Assume,

(i) $P^* \neq \Phi$ or $\tilde{P}^* \neq \tilde{\Phi}$,

(ii) $\beta_1 \neq 0$, and $\tilde{\beta}_1 \neq 0$, where

$$\beta_1 = \lambda_1(p_{120} - 1) + \lambda_2(1 - p_{210}),$$

$$\tilde{\beta}_1 = \tilde{\lambda}_1(1 - \tilde{p}_{120}) + \tilde{\lambda}_2(\tilde{p}_{210} - 1).$$

Then, the MAP$_2$s $\{\alpha, \lambda, P_0, P_1\}$, $\{\tilde{\alpha}, \tilde{\lambda}, \tilde{P}_0, \tilde{P}_1\}$ are (weakly) equivalent if and only if the following two conditions are fulfilled,

C1. $\phi \varphi_H(s) = \tilde{\phi} \tilde{\varphi}_H(s)$,

C2. $(\alpha, \tilde{\alpha}) = (\phi, \tilde{\phi})$. □
Remarks

1. C1. is equivalent to $T \overset{d}{=} \tilde{T}$.
2. C2. implies that $T_1 \overset{d}{=} T_2 \overset{d}{=} \ldots \overset{d}{=} T_n \overset{d}{=} \ldots \overset{d}{=} T$, and similarly with $\tilde{T}_j$, $\forall j \geq 1$.
3. (Weak) equivalence between two MAP$_2$s can be established only if both MAP$_2$s are in the stationary version.
4. It can be shown that

$$\phi \varphi_H(s) = \frac{a_1 s + d_0}{s^2 + d_1 s + d_0},$$

where

$$a_1 = \phi \lambda_1 (p_{120} - 1) + \lambda_2 (\phi + p_{210} - 1 - \phi p_{210}),$$
$$d_1 = - (\lambda_1 + \lambda_2),$$
$$d_0 = \lambda_1 \lambda_2 (1 - p_{120} p_{210}),$$

and thus, the result provides a simple way to test the weak equivalence of two MAP$_2$s.
Example

Consider the $MAP_2$ defined by

$$\lambda = (0.5, 20), \quad P_0 = \begin{pmatrix} 0 & 0.3 \\ 0.3 & 0 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 0.6148 & 0.0852 \\ 0.0886 & 0.6114 \end{pmatrix}$$

and initial probability $\alpha=\phi = 0.504$.

Consider another $MAP_2$ with parameters

$$\lambda = (0.8, 19.7), \quad P_0 = \begin{pmatrix} 0 & 0.7683 \\ 0.55 & 0 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 0.0513 & 0.1804 \\ 0.0873 & 0.3627 \end{pmatrix}$$

and initial probability $\alpha=\phi = 0.201$. 
Example

- It can be seen that $\phi \varphi_H(s) = \tilde{\phi} \varphi_{\tilde{H}}(s)$, for all $s$.
- We are thus in the assumptions of the Theorem. This assures that the processes are weakly equivalent.
- Figure: CDF of $T$, time until next arrival in the stationary version of both $MAP_2$s.
BAYESIAN INFERENCE FOR THE MAP$_2$
Introduction

- Performance analysis for models incorporating MAPs: well-developed area.
- Less progress on statistical estimation for such models.
- \textit{MMPP}:
- \textit{BMAP}: Klemm et al. (2003), EM to estimate the BMAP.
- Aim: Bayesian inference for the MAP\textsubscript{2} using theoretical results obtained for the E-MAP.
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- Aim: Bayesian inference for the MAP$_2$ using theoretical results obtained for the E-MAP.
Data & Parameters of the model

- We assume that the available data are the times between two successive arrivals, $t = (t_1, \ldots, t_n)$ in a stationary MAP$_2$.

- The underlying Markov process governing the different states of the process, and the transition changes will be assumed to be unobservable.

- Parameters:

  \[ \lambda = (\lambda_1, \lambda_2) : \quad \text{Exponential rates} \]

  \[ \mathbf{p}_1 = (p_{10}, p_{11}, p_{12}) : \quad \text{Transition probabilities from state 1} \]

  \[ \mathbf{p}_2 = (p_{20}, p_{21}, p_{22}) : \quad \text{Transition probabilities from state 2} \]
Prior distributions

- Independent gamma priors for $\lambda_1$ and $\lambda_2$,

$$\lambda_1, \lambda_2 \sim G(\alpha, \beta),$$

where we introduce the minimum order restriction $\lambda_1 < \lambda_2$ to reduce problems due to lack of identifiability of the model.

- Dirichlet priors for the vector of probabilities,

$$p_1, p_2 \sim D(e),$$

where $e$ is a unit vector of dimension $1 \times 3$. 
Prior distributions

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$$p_1, p_2 \sim \mathcal{D}(ce),$$

where $e$ is a unit vector of dimension $1 \times 3$. 
Likelihood

\[
f(t|\lambda, p_1, p_2) = \sum_{i_n=1}^2 \cdots \sum_{i_1=1}^2 \phi_{i_1} p_{i_1i_2}^* f_{H_{i_1i_2}}(t_1) p_{i_2i_3}^* f_{H_{i_2i_3}}(t_2) \cdots p_{i_{n-1}i_n}^* f_{H_{i_{n-1}i_n}}(t_{n-1}) f_{H_{i_n}}(t_n)
\]

where,

\[
\phi_i \quad \text{Stationary probability that the E-MAP is in state } i.
\]

\[
p_{ij}^* \quad \text{Probability of a transition from } i \text{ to } j \text{ in the E-MAP.}
\]

\[
f_{H_{ij}}(t) \quad \text{Density of the holding time in a transition } i \rightarrow j, \text{ in the E-MAP.}
\]

\[
f_{H_i}(t) \quad \text{Density of the holding time in state } i \text{ in the E-MAP.}
\]
Likelihood

It can be shown that

\[ f(t|\lambda, p_1, p_2) = \phi \prod_{i=1}^{n-1} \mathcal{F}(t_i)\mathcal{B}(t_n), \]

where

\[ \mathcal{F}(t) = \begin{pmatrix} p_{11}^* f_{H11}(t) & p_{12}^* f_{H12}(t) \\ p_{21}^* f_{H21}(t) & p_{22}^* f_{H22}(t) \end{pmatrix} \quad \text{and} \quad \mathcal{B}(t) = \begin{pmatrix} f_{H1}(t) \\ f_{H2}(t) \end{pmatrix}. \]

Numerical complexity due to

1. Approximation of \( f_{H_k}(t) \) and \( f_{H_{ij}}(t) \).
2. Product of \( n \) matrices.
The posterior distribution

- Combining the likelihood & priors gives a non-conjugate posterior distribution:

\[ f(\lambda, p_1, p_2 | t) \propto \pi(\lambda_1)\pi(\lambda_2)\pi(p_1)\pi(p_2)f(t | \lambda, p_1, p_2). \]

- Metropolis-Hastings algorithm.

- Increase the acceptance rate: 3 blocks.
The Metropolis-Hastings to estimate the MAP

1. Draw a starting point $\lambda^{(0)}$, $p_1^{(0)}$ and $p_2^{(0)}$ from the prior distributions.

2. For $t = 2,\ldots$:
   (a) Sample a proposal $\lambda^*$ from a Log-Normal distribution,
   $$\log(\lambda^*) \sim N\left(\log(\lambda^{(t-1)}), \sigma\right).$$
   Accept or reject.
   (b) Sample a proposal $p_1^*$ from a Dirichlet distribution
   $$p_1^* \sim \mathcal{D} (d_1e).$$
   Accept or reject.
   (c) Sample a proposal $p_2^*$ from a Dirichlet distribution
   $$p_2^* \sim \mathcal{D} (d_2e).$$
   Accept or reject.
Performance: Simulated data 1

- 1000 simulated interarrival times from the stationary $MAP_2$

\[ \lambda = (3, 10), \quad P_0 = \begin{pmatrix} 0 & 0.2 \\ 0.25 & 0 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 0.35 & 0.45 \\ 0.35 & 0.4 \end{pmatrix} \]

- $\lambda^* = 3.6509$, \quad $\log(f(t|\lambda, p_1, p_2)) = 328.8059$

- 100 000 iterations, 50 000 burn-in

- $d_1 = d_2 = 0.6$

- Initially, $\sigma = 1$; Within the burn-in period: $\sigma = 0.3$

\[ \lambda^0 = (1, 5), \quad P_0^0 = \begin{pmatrix} 0 & 0.0872 \\ 0.0270 & 0 \end{pmatrix}, \quad P_1^0 = \begin{pmatrix} 0.1027 & 0.8101 \\ 0.6735 & 0.2995 \end{pmatrix} \]
Arrival rate, Log-Likelihood, $F_T(t)$
Results

- \( \lambda^* = 3.6509 \)

- \( E(\lambda^*|\cdot) = 3.6712 \)

- Acceptance rate for \( \lambda \): 14.63%

- Acceptance rate for \( p_1, p_2 \): 2.5%

- Computational time: \( \approx 4h \)
Performance: Simulated data 2

- 1000 simulated interarrival times from the stationary MMPP

\[
\lambda = (5, 20), \quad P_0 = \begin{pmatrix} 0 & 0.7 \\ 0.4 & 0 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 0.3 & 0 \\ 0 & 0.6 \end{pmatrix}
\]

- \(\lambda^* = 4.6957, \quad \log(f(t|\lambda, p_1, p_2)) = 618.5995\)

- 100 000 iterations, 50 000 burn-in

- \(d_1 = d_2 = 0.6\)

- Initially, \(\sigma = 1\); Within the burn-in period: \(\sigma = 0.3\)

\[
\lambda^0 = (1, 5), \quad P_0^0 = \begin{pmatrix} 0 & 0.783 \\ 0.6739 & 0 \end{pmatrix}, \quad P_1^0 = \begin{pmatrix} 0.217 & 0 \\ 0 & 0.3261 \end{pmatrix}
\]
Exponential rates

\[ E(\lambda_1 | \cdot) = 5.14, \quad E(\lambda_2 | \cdot) = 17.21 \]
Transition probabilities

The $MMPP_2$ is identifiable, thus, small variability is expected for the values of $p_1$ and $p_2$.

$$E(p_1|\cdot) = (0.7866, 0.2134), \quad E(p_2|\cdot) = (0.3722, 0.6278).$$
Arrival rate, Log-Likelihood, $F_T(t)$
Real data set

50000 first interarrival times in seconds of a trace of 1 million ethernet packets. Source: http://www.xtremes.de/xtremes/xtremes/download/download.htm.
Exponential rates
CDF

Empirical CDF

\[ F_T(x) \]

\[ x \]

\[ 0 \quad 0.005 \quad 0.01 \quad 0.015 \quad 0.02 \quad 0.025 \quad 0.03 \quad 0.035 \quad 0.04 \quad 0.045 \]

\[ 0 \quad 0.1 \quad 0.2 \quad 0.3 \quad 0.4 \quad 0.5 \quad 0.6 \quad 0.7 \quad 0.8 \quad 0.9 \quad 1 \]
CONCLUSIONS & EXTENSIONS
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- First step in the study of the identifiability of MAPs.
- Deep study of the E-MAP.
- Results that assures weak equivalence.

- Bayesian method to estimate the MAP₂.
- Easy to implement, based on our theoretical results.
- Good estimation results, suitable for real teletraffic data.
Conclusions

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- Easy to implement, based on our theoretical results.
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## Extensions

- Study identifiability of \( MAPs \) in the strong sense.
- Get a better acceptance rate for \( p_1 \) and \( p_2 \): play with proposals.
- Compute the theoretical ACF of the \( E-MAP \) to test if the dependence is captured.
- Bayesian inference for the \( MAP_2/G/1 \) queueing system. (In process).
- Extension to the \( BMAP \).
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Bibliography


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