Problema 1

$$\begin{cases} t^2 u' = u^2 + tu \\ u(1) = u_0 \end{cases} \qquad u_0 \in \mathbb{R}$$

(a) Factor integrante $\mu(t,u) = \dfrac{u^\alpha}{t}$

(*) $\underbrace{u^\alpha t\, u'}_{P = \frac{\partial V}{\partial u}} \underbrace{- (u^2 + tu)\dfrac{u^\alpha}{t}}_{Q = \frac{\partial V}{\partial t}} = 0$ CONDICIÓN NECES. & sufic. $\boxed{\dfrac{\partial P}{\partial t} = \dfrac{\partial Q}{\partial u}}$

$$\frac{\partial P}{\partial t} = u^\alpha = \frac{\partial Q}{\partial u} = -\left[\frac{(2+\alpha)}{t} u^{\alpha+1} + (\alpha+1) u^\alpha\right] = 0$$

$$u^\alpha \left[(\alpha+2) + (\alpha+2)\frac{u}{t}\right] = (\alpha+2) u^\alpha \left(1 + \frac{u}{t}\right) = 0$$

Para $\boxed{\alpha = -2}$ la EDO (*) es exacta. Factor $\boxed{\mu = \dfrac{1}{tu^2}}$

(b) Solución

$$\frac{t}{u^2} u' - \left(\frac{1}{t} + \frac{1}{u}\right) = 0$$

$$\frac{\partial V}{\partial u} = \frac{t}{u^2} \longrightarrow V = -\frac{t}{u} + \varphi(t)$$

$$\frac{\partial V}{\partial t} = -\frac{1}{t} - \frac{1}{u} = -\frac{1}{u} + \varphi' \Rightarrow \varphi' = -\frac{1}{t} \Rightarrow \varphi = -\ln t$$

$$V = -\frac{t}{u} - \ln t = V_0$$

C.I. $\quad -\dfrac{1}{u_0} = V_0 \Rightarrow \dfrac{t}{u} + \ln t = \dfrac{1}{u_0} \quad \left\{\!\left\{ u_0 \neq 0 \right.\right.$

$$\frac{t}{u} = \frac{1}{u_0} - \ln t$$

$$\boxed{u = \frac{t u_0}{1 - u_0 \ln t}}$$

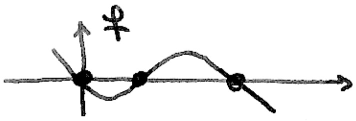
Si $u_0 = 0 \Rightarrow$
$u(t) = 0$ es solución
esta particular
está CONTENIDA EN
la GENERAL.

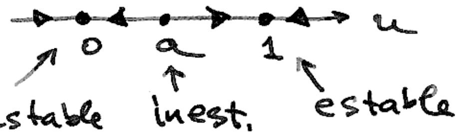
Problema 2

$$\begin{cases} u' = u(a-u)(u-1) \\ u(0) = u_0 \in \mathbb{R} \end{cases}$$

Puntos de equilibrio $\bar{u}_1 = 0$, $\bar{u}_2 = 1$, $\bar{u}_3 = a$

① $0 < a < 1$





si $u_0 \in (-\infty, a)$

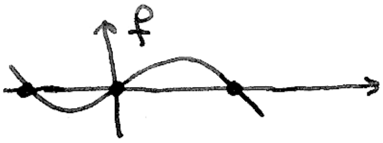
$\lim_{t\to\infty} u(t;u_0) = 0$

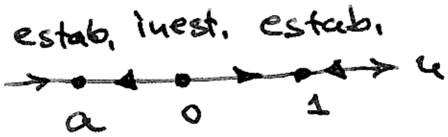
si $u_0 \in (a, +\infty)$

$\lim_{t\to\infty} u(t;u_0) = 1$

si $u_0 = a$ $u(t) = a$ (es única)

② $a < 0$



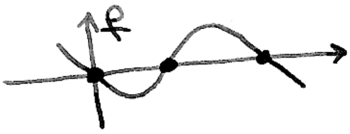


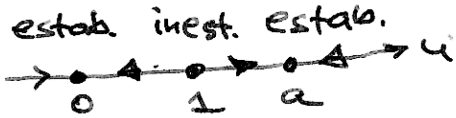
$u_0 \in (-\infty, 0)$ $\lim_{t\to\infty} u(t;u_0) = a$

si $u_0 = 0 \to u(t) = 0$

$u_0 \in (0, +\infty)$ $\lim_{t\to\infty} u(t;u_0) = 1$

③ $a > 1$





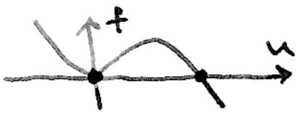
si $u_0 \in (-\infty, 1)$ $\lim_{t\to\infty} u(t;u_0) = 0$

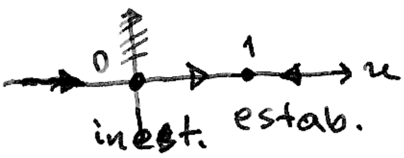
$u_0 = 1 \to u(t) = 1$

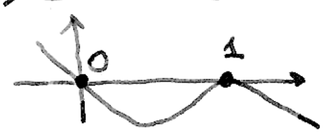
$u_0 \in (1, +\infty)$ $\lim_{t\to\infty} u(t;u_0) = a$

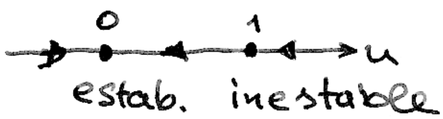
Los casos ①, ② y ③ representan sistemas equivalentes

④a $a = 0$ (estructuralmente inestable) ④b $a = 1$









$u_0 \in (-\infty, 0]$ $\lim_{t\to\infty} u(t;u_0) = 0$ | $u_0 \in (0, +\infty)$ $\lim_{t\to\infty} u(t;u_0) = 1$

$u_0 \in (-\infty, 1)$ $\lim_{t\to\infty} u(t;u_0) = 0$ | $u_0 \in [1, +\infty)$ $\lim_{t\to\infty} u(t;u_0) = 1$

(P3) $\begin{cases} tuu' + t^2 + u^2 = 0 \\ u(1) = 1 \end{cases}$

$$u' = -\frac{t^2+u^2}{tu} \qquad \text{Homogénea}$$

$$u = vt \quad \leftarrow \text{cambio}$$

$$u' = v't + v = -\frac{1+v^2}{v}$$

$$v't = -\frac{1+2v^2}{v} \qquad \text{Variab. separ.}$$

$$\frac{vv'}{1+2v^2} = -\frac{1}{t}$$

$$\int \frac{v\,dv}{1+2v^2} = -\ln|t| + C$$

$$\int \frac{v\,dv}{1+2v^2} = \frac{1}{2}\int \frac{dv^2}{1+2v^2} = \frac{1}{4}\int \frac{d(2v^2+1)}{1+2v^2} = \frac{1}{4}\ln(2v^2+1)$$

Por lo tanto

$$\frac{1}{4}\ln(2v^2+1) = \ln\frac{1}{|t|} + C$$

$$(2v^2+1)^{1/4} = \frac{B}{t}$$

$$2v^2 = \frac{B^4}{t^4} - 1 \rightarrow v = \pm\sqrt{\frac{B^4}{2t^4} - \frac{1}{2}}$$

$$u = \pm t\sqrt{\frac{B^4}{2t^4} - \frac{1}{2}}$$

COND. INIC.: $\quad 1 = \pm\sqrt{\frac{B^4-1}{2}} \rightarrow$ "+" y $B^4 = 3$

$$\boxed{u = t\sqrt{\frac{3-t^4}{2t^4}} = \frac{\sqrt{3-t^4}}{\sqrt{2}\,t}}$$

La solución existe para $t \in [1, 3^{1/4}]$

Problema 4

$$\begin{cases} u' = f(u) \\ u(t_0) = u_0 \end{cases}$$

tiene >1 solución entonces $f(u_0) = 0$.

f continua en entorno de u_0

Demostración

Supongamos que $f(u_0) \neq 0$, entonces, $f(u(t)) \neq 0$ en un entorno. Por lo tanto podemos escribir

$$\frac{u'}{f(u)} = 1$$

$$\frac{d}{dt}\int \frac{du}{f(u)} = 1 \Rightarrow \int_{u_0}^{u} \frac{ds}{f(s)} = \int_{t_0}^{t} ds = (t - t_0)$$

$$H(t,u) = \int_{u_0}^{u} \frac{ds}{f(s)} - (t - t_0) = 0$$ solución en forma implicita

a) $H \subset C^1(\Omega)$ donde Ω es un entorno de (t_0, u_0)

b) $H(t_0, u_0) = \int_{u_0}^{u_0} \frac{ds}{f(s)} - (t_0 - t_0) = 0$

c) $\left.\frac{\partial H}{\partial u}\right|_{(t_0,u_0)} = \frac{1}{f(u_0)} \neq 0$ ($f(u)$ es CONTINUA en Ω y $f(u_0) \neq 0$)

Por lo tanto H cumple todas condiciones del teorema de la función implicita.

$\Rightarrow$ existe una única solución $u(t)$ que pasa por (t_0, u_0). CONTRADICCIÓN CON la existencia de 2 soluciones. Por lo tanto necesariam. $f(u_0) = 0$ ∎