Problema 1:

$$u'' + u = at \qquad a \geq 0$$

1. Solución general:

Ec. caract.: $\lambda^2 + 1 = 0 \Rightarrow \lambda_{1,2} = \pm i$

Solución general. Probl. Homogeneo: $u_h = C_1 \cos t + C_2 \operatorname{sen} t$

Soluc. particular problema completo $u_p = at$.

En efecto: $u_p' = a$, $u_p'' = 0 \Rightarrow at = at$ OK.

Entonces

$$\boxed{u = u_h + u_p = C_1 \cos t + C_2 \operatorname{sen} t + at}$$

2. Dato inicial $u(0) = 1$, $u'(0) = 2$

$u(0) = C_1 = 1$, $u' = -\operatorname{sen} t + C_2 \cos t + a$

$u'(0) = C_2 + a = 2 \Rightarrow C_2 = 2 - a$

$$\boxed{u(t) = \cos t + (2-a) \operatorname{sen} t + at}$$

3. $a = 0$, condiciones de contorno: $u(0) = u(L) = 0$

$u(0) = C_1 = 0$, $u(L) = C_2 \operatorname{sen} L = 0$

Soluciones no triviales $\Rightarrow C_2 \neq 0 \Rightarrow L = \pi n$

$n = 1, 2, \ldots$

$$\boxed{u(t) = C_2 \operatorname{sen} t, \quad L = \pi n, \; n = 1, 2, 3, \ldots}$$

Problema 2:

$$u' = Au$$

$$A = \begin{pmatrix} 5 & 2 \\ 1 & 4 \end{pmatrix}$$

a) autovalores $\begin{vmatrix} 5-\lambda & 2 \\ 1 & 4-\lambda \end{vmatrix} = (5-\lambda)(4-\lambda) - 2 =$

$$= 20 - 9\lambda + \lambda^2 - 2 = \lambda^2 - 9\lambda + 18 = 0$$

$$\lambda_{1,2} = \frac{9 \pm \sqrt{81-72}}{2} = \{6, 3\}$$

b) Jordan: $J = \begin{pmatrix} 6 & 0 \\ 0 & 3 \end{pmatrix} \to e^{Jt} = \begin{pmatrix} e^{6t} & 0 \\ 0 & e^{3t} \end{pmatrix}$

c) Matriz de Paso:

$$u = Px \quad \to \quad Px' = APx \quad \to x' = P^{-1}APx = Jx$$

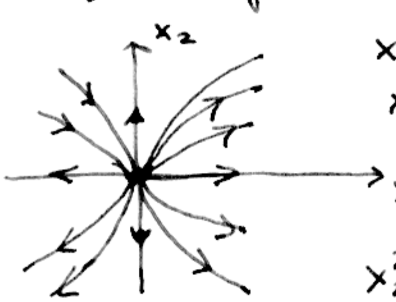
$$A\,[p_1\ p_2] = [p_1\ p_2]\begin{pmatrix} 6 & 0 \\ 0 & 3 \end{pmatrix} = [6p_1, 3p_2]$$

$(A-6I)p_1 = 0$ $\qquad \begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix} p_1 = 0 \Rightarrow p_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$(A-3I)p_2 = 0$ $\qquad \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} p_2 = 0 \Rightarrow p_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$P = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$$

d) Diagrama de fases en coordenadas x:



$x_1 = x_{10} e^{6t}$

$x_2 = x_{20} e^{3t}$

$$\frac{x_2^2}{x_{20}^2} = \frac{x_1}{x_{10}}$$

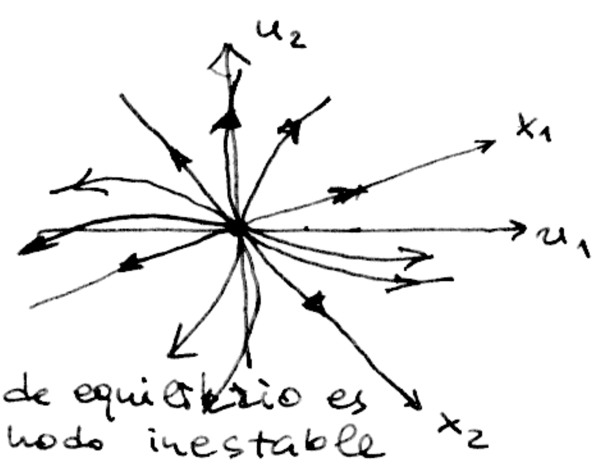
$u_1 = 2x_1 + x_2$

$u_2 = x_1 - x_2$

$x_2 = 0$ (eje x_1): $u_2 = \frac{1}{2}u_1$

$x_1 = 0$ (eje x_2): $u_2 = -u_1$

e) Diagrama de fases en u



El punto de equilibrio es un nodo inestable

Problema 3

$$(*) \quad a_2 u'' + a_1 u' + a_0 u = 0$$

(a) Sea u_1 solución del $(*)$, entonces:

$$a_2 u_1'' + a_1 u_1' + a_0 u_1 = 0$$

Buscamos $\quad u_2 = b(t)\, u_1(t)$

$$u_2' = b' u_1 + b u_1' \quad ; \quad u_2'' = b'' u_1 + 2 b' u_1' + b u_1''$$

Sustituimos:

$$a_2 u_1 b'' + 2 b' a_2 u_1' + \underline{a_2 b u_1''} + a_1 b' u_1 + \underline{a_1 b u_1'} + \underline{a_0 b u_1} = 0$$

$$a_2 u_1 b'' + (a_1 u_1 + 2 a_2 u_1') b' = 0$$

Como $a_2 \neq 0$ y $u_1 \neq 0$ en (t_0, t_1) este problema tiene solución.

$$b' = v \quad \rightarrow \quad v' = -\frac{a_1 u_1 + 2 a_2 u_1'}{a_2 u_1} v$$

Podemos tomar $\quad v = e^{-\int \left(\frac{a_1}{a_2} + 2 \frac{u_1'}{u_1}\right) dt} = e^{-\int \frac{a_1}{a_2} dt} e^{-2 \ln u_1} = \dfrac{e^{-\int \frac{a_1}{a_2}}}{u_1^2}$

$$b = \int v(t)\, dt + b_0$$

. Podemos tomar $b_0 = 0$ ($b_0 u_1$ este termino no hace falta tener)

Finalmente
$$u_2 = u_1 \int v(t)\, dt$$

veamos que $\{u_1, u_2\}$ es un sistema fundamental (determinante de Wronsky $\neq 0$)

$$|W| = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} = \begin{vmatrix} u_1 & b u_1 \\ u_1' & b' u_1 + b u_1' \end{vmatrix} = b' u_1^2 \neq 0$$

$u_1 \neq 0$ por definición

Tenemos que comprobar $b' \neq 0$. En efecto

$$b' = v = e^{-\int \left[\frac{a_1}{a_2} + 2 \frac{u_1'}{u_1}\right] dt} > 0$$

(b) Aplicar: $tu'' - (2t+1)u' + (t+1)u = 0$

$u_1 = e^t$, $a_1 = -(2t+1)$, $a_2 = t$

$$v = e^{-\int\left[-\frac{2t+1}{t} + 2\frac{e^t}{e^t}\right]dt} = e^{-(-2t - \ln|t| + 2t)} = t$$

$$b = \int v\,dt = \frac{t^2}{2}$$

Por lo tanto podemos tomar $\boxed{u_2 = t^2 e^t}$

Comprobemos: $u_2' = e^t(t^2+2t)$, $u_2'' = e^t(t^2+2t+2t+2)$

$$te^t(t^2+4t+2) - (2t+1)e^t t(t+2) + (t+1)t^2e^t = 0$$

$$t^2+4t+2-2t^2-\cancel{t}-4t-2+t^2+t = 0 \quad \text{ok}$$

$$|W| = \begin{vmatrix} e^t & t^2e^t \\ e^t & e^t(t^2+2t) \end{vmatrix} = 2te^t \neq 0 \quad \text{para } t>0$$

Finalmente la solución generar

$$\boxed{u = c_1u_1 + c_2u_2 = e^t(c_1 + c_2t^2)}$$

Problema 4

$$u' = a(t)u^p \,, \quad u(0) = u_0 > 0 \,, \quad p > 1$$

(a) Existencia y unicidad.

$$f(t,u) = a(t)\,u^p \qquad \frac{\partial f}{\partial u} = p\,a(t)\,u^{p-1}$$

Como $a \in C[0,\infty)$ f es Localmente Lipschitz

Por el teorema de Picard-Lindelof $\exists\, \delta > 0$ t.q.

$\exists!\, u(t;u_0)$ $t \in [0,\delta]$. Luego esta solución la podemos prolongar hasta que 1) explote o 2) ∞.

$$\frac{u'}{u^p} = a(t) \Rightarrow \int_{u_0}^{u} \frac{ds}{s^p} = \int_0^t a(s)\,ds = I(t)$$

$$\left.\frac{s^{1-p}}{1-p}\right|_{u_0}^{u} = -\frac{1}{p-1}\left(\frac{1}{u^{p-1}} - \frac{1}{u_0^{p-1}}\right) = I(t)$$

$$\frac{1}{u^{p-1}} = \frac{1}{u_0^{p-1}} - (p-1)I(t)$$

$$\boxed{u = \frac{1}{\left(\frac{1}{u_0^{p-1}} - (p-1)I(t)\right)^{\frac{1}{p-1}}}}$$

(b) 1) la solución está definida en $[0,+\infty)$

$$(*)\quad \frac{1}{u_0^{p-1}} - (p-1)\,I(t) > 0 \quad \text{para } t \in [0,+\infty)$$

Como $a \geq 0$ $I(t)$ es una función creciente entonces la $(*)$ se debe cumplir para $t \to \infty$.

Sea $I_\infty = \lim_{t\to\infty} I(t) < \infty$.

$$\boxed{\frac{1}{u_0^{p-1}} > (p-1)\,I_\infty}$$

② En el caso contrario, es decir si

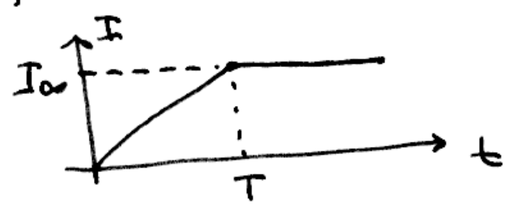
$$\frac{1}{u_0^{p-1}} < (p-1) I_\infty$$

existe $T > 0$ t. q. $\frac{1}{u_0^{p-1}} = (p-1) I(T)$

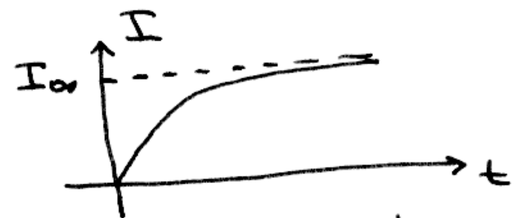
y por lo tanto $\lim_{t \to T} u(t) = +\infty$

la solución explota en tiempo finito

<u>Nota:</u> si $\frac{1}{u_0^{p-1}} = (p-1) I_\infty$ todo depende de la forma de $I(t)$



explota
(si $I(T) = I_\infty$, $T < \infty$)

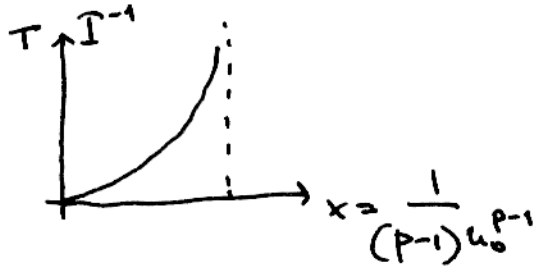


No explota

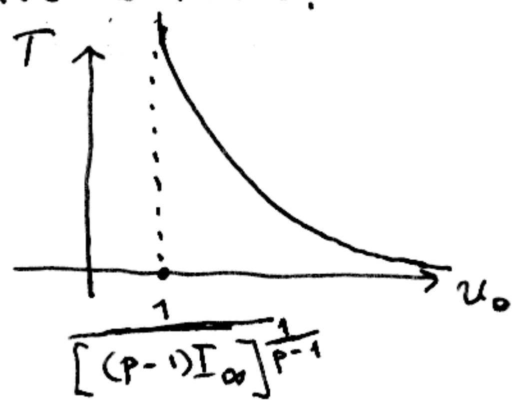
ⓒ Tiempo de explosión vs u_0

$$\boxed{T = I^{-1}\left(\frac{1}{(p-1)\, u_0^{p-1}}\right)}$$

$I(t)$ es creciente y continua. Existe I^{-1} que también es creciente y continua



Por lo tanto:

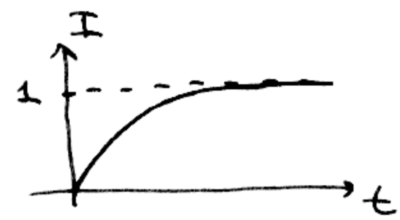


Si $u_0 \to \infty$ entonces $T \to 0$

si $u_0 \to \frac{1}{[(p-1) I_\infty]^{\frac{1}{p-1}}}$ entonces $T \to \infty$
(caso crecimiento estricto)

(d) Sea $a = e^{-t}$

$$I = \int_0^t e^{-s}\,ds = e^{-s}\Big|_t^0 = 1 - e^{-t}$$



$$I_\infty = 1$$

$$u = \frac{1}{\left[\frac{1}{u_0^{p-1}} - (p-1)(1-e^{-t})\right]^{\frac{1}{p-1}}}$$

El tiempo de explosión:

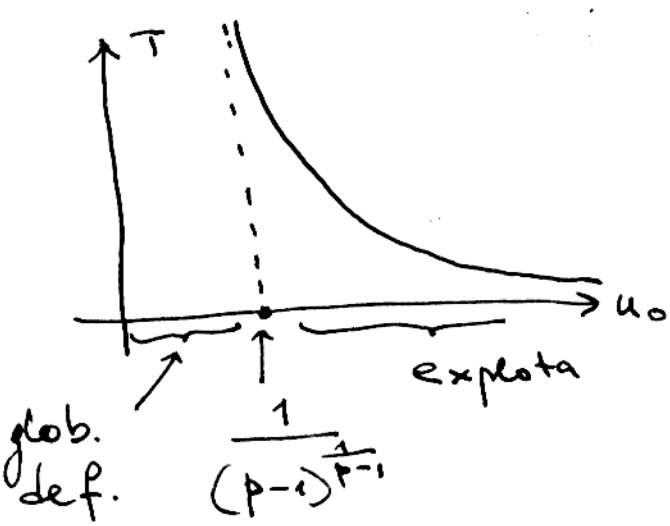
$$\frac{1}{u_0^{p-1}} = (p-1)(1-e^{-T})$$

$$1 - \frac{1}{(p-1)\,u_0^{p-1}} = e^{-T} \Rightarrow T = -\ln\left[\frac{(p-1)u_0^{p-1} - 1}{(p-1)\,u_0^{p-1}}\right]$$

$$\boxed{T = \ln\left[\frac{(p-1)\,u_0^{p-1}}{(p-1)u_0^{p-1} - 1}\right]}$$

si $\quad \frac{1}{u_0^{p-1}} > (p-1) \quad$ ó $\quad \frac{1}{(p-1)^{\frac{1}{p-1}}} \geq u_0 \quad$ entonces la solución está glob. def. en $[0,+\infty)$

si $\quad u_0 > \frac{1}{(p-1)^{\frac{1}{p-1}}} \quad$ entonces explota



para $\quad u_0 = \frac{1}{(p-1)^{\frac{1}{p-1}}} \quad$ $u(t)$ está glob. def. en $[0,+\infty)$

Problema 4

$$-u'' = f(u) = \frac{d\varphi}{du} \qquad \varphi = u^2 e^{-u^2}$$

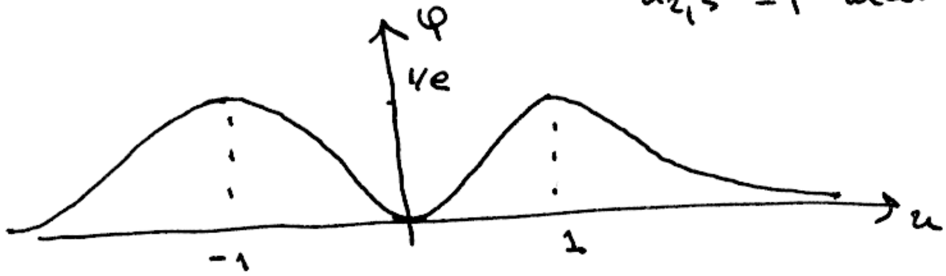
(a) $\varphi(-u) = \varphi(u)$ simetría , $\varphi(u) \geq 0$

$\lim_{u\to\pm\infty} \varphi(u) = 0$, extremos: $\varphi' = e^{-u^2}(2u - 2u^3) =$

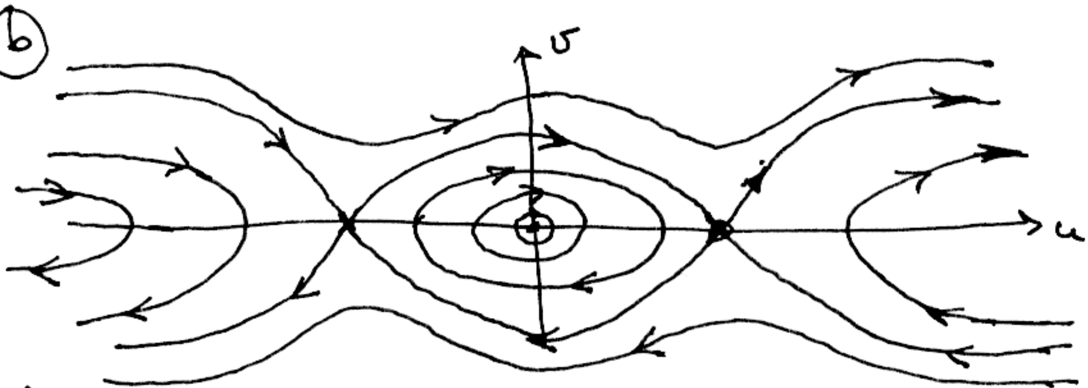
$= 2u\, e^{-u^2}(1-u^2) = 0 \Rightarrow u_1 = 0\,,\ u_{2,3} = \pm 1$

$\varphi(0) = 0 \qquad \varphi(\pm 1) = \frac{1}{e} \qquad u_1 = 0$ - mínimo

$u_{2,3} = 1$ máximos

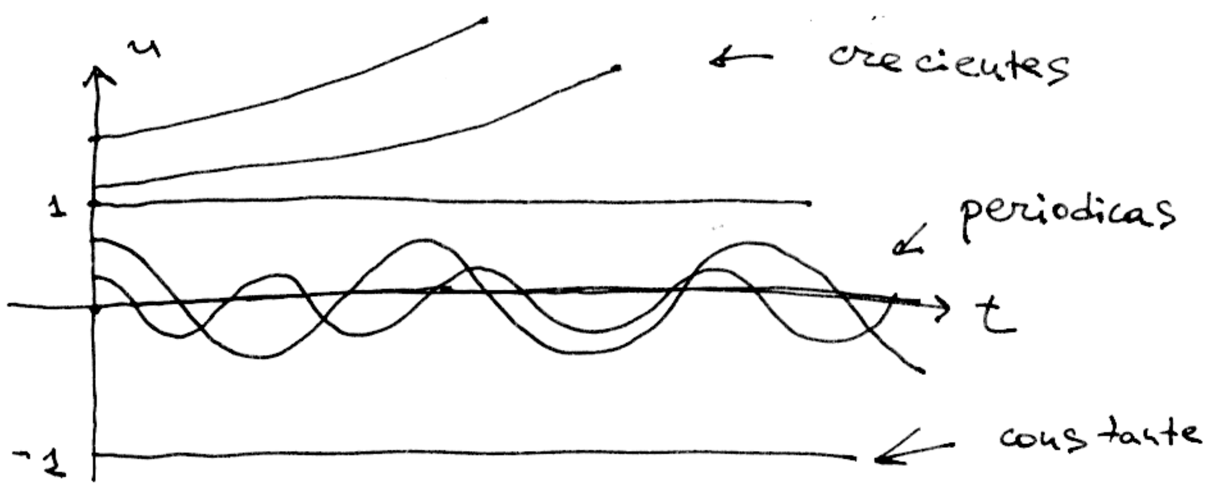


(b)



$$\begin{cases} u' = v \\ v' = -f(u) \end{cases} \qquad H = \frac{v^2}{2} + \varphi(u) = H_0$$

(c)



Si $0 \leq u_0 \leq 1$ las soluciones son periodicas y NO explotan