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PHASE RESETTING, CLUSTERS, AND WAVES IN A LATTICE OF COUPLED OSCILLATORY UNITS

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A one-dimensional chain consisting of many locally coupled oscillatory units is considered when both diffusive (linear) and nonlinear interactions exist. The dynamics of units may also be altered by a weak white Gaussian noise. Two different collective evolutionary paths have been identified in the chain: (i) the onset of temporal phase clusters and their subsequent breakdown or reconstruction through phase resetting, and (ii) the formation of a phase wave during a process of selforganization. Such a wave may originate from an initially disordered pattern after appropriate reorganization of phases and amplitudes occurring on a time lapse that can be very long.

1. Introduction

Synchronization of oscillations and, consequently, uniform rhythmic activity in coupled systems is a phenomenon widespread in nature and various branches of engineering [Haken, 1983a, 1983b, 1996; Murray, 1993]. The simplest coupling leading to synchronous oscillations is "diffusion" or a resistive type of coupling. Here we study a model-problem that combines the processes of synchronization and its destruction in an ensemble of diffusively coupled, identical oscillatory units.

As the unit or element of the lattice we choose a simple albeit nonlinear oscillator whose dynamics in polar coordinates is described by the system

$$\dot{r} = rF(r)$$

 $\dot{\varphi} = \omega,$ (1)

where r, ω and φ denote amplitude, frequency and phase of the oscillations, and F is a suitable nonlinearity regulating the excitability of the unit. Thus the system (1) represents an *isochronous* oscillator. For illustration we take $F = -2ar^4 + ar^2 - 1$. When a < 8, in the phase plane of the system (1) there exists only a stable equilibrium point (focus) which corresponds to the rest state of the unit. For a > 8 we have two attractors. They are a stable focus at the origin and a stable limit cycle. The latter corresponds to the oscillatory activity of the unit. Hence, by changing a we can switch from a monostable to a bistable regime.

To describe the cooperative behavior of an ensemble of such units we consider a 1D lattice consisting of n units (1) diffusively coupled with nearest-neighbor interaction. We assume periodic boundary conditions

$$\dot{z}_{j} = z_{j}[F(z_{j}) + i\omega + \delta(\Phi(\chi_{j}) + \Phi(\chi_{j-1}))] + d(z_{j-1} - 2z_{j} + z_{j+1}) + \epsilon\xi_{j}(t)$$
(2)
$$j = 1, 2, \dots, n, \quad z_{j+n} = z_{j},$$

where $z_j = r_j e^{i\varphi_j}$, d and δ account for the diffusive (linear) and nonlinear coupling between units, respectively. We have also introduced a functional, $\Phi(\chi)$, characterizing nonlinear phase coupling as we shall see below. In particular $\Phi(\chi)$ would be

responsible for desynchronization of oscillations in the vicinity of a given unit. Thus we choose the variable χ depending on phase differences between neighboring units,

$$\chi_j = \varphi_{j+1} - \varphi_j. \tag{3}$$

Hence $\Phi(\chi_i)$ and $\Phi(\chi_{i-1})$ define levels of synchronization of the unit j relative to its right and left neighbors, respectively. Low values of χ correspond to a high level of synchronization ($\chi = 0$ corresponds to strict in-phase oscillations) and high values of χ correspond to a low synchronization level. Such cooperative effect is best described by a sigmoidal function. Since phase differences can be either positive or negative, this function must be symmetric around $\chi = 0$. An example of such function is shown in Fig. 1. For values of χ around the origin it is almost equal to zero while with increasing $|\chi|$ the function Φ approaches a saturation value. As a concrete application, let us take for illustration and for all our numerical calculations the Boltzmann function

$$\Phi(\chi) = \frac{1}{1 + e^{-\alpha \tilde{\chi}}} - \frac{1}{1 + e^{\alpha(|\chi| - \tilde{\chi})}}$$
(4)

where α and $\tilde{\chi}$ are positive constants. This function has the shape shown in Fig. 1. and it is reminiscent of the Michaelis-Menten/Hinshelwood-Langmuir/Holling law found in biochemistry/chemical kinetics/or biology. Further we shall consider the system (2) working in an noisy environment that we model by $\xi_j(t)$. We choose it as a white Gaussian noise whose strength is controlled by the parameter ϵ , [Eq. (2)],

$$\langle \xi_j(t) \rangle = 0, \quad \langle \xi_j(t) \xi_k(t') \rangle = \delta_{jk} \delta(t - t') \,. \tag{5}$$

Thus model (2) is a spatially distributed oscillatory system with local couplings perturbed by noise. In the absence of noise and depending on the properties of the unit we have two possibilities. One is the case of a network of limit cycle oscillators when its dynamics can be reduced to that of a system of phase oscillators [Winfree, 1967; Ermentrout & Rinzel, 1981; Kuramoto, 1984]. Lattice systems of phase oscillators have been studied in a number of papers [Ermentrout, 1985; Daido, 1988; Strogatz & Mirollo, 1988; Sakaguchi *et al.*, 1988; Niebur *et al.*, 1991]. The other case is that of a chain consisting of multistable units when the amplitude dynamics and vice versa [Defontaines *et al.*, 1990; Nekorkin &



Fig. 1. Qualitative shape of the nonlinear phase coupling functinal Φ .

Makarov, 1995; Nekorkin et al., 1996, 1997, 1998, 1999; Sepulchre & MacKay, 1997] or each unit is a chaotic oscillator [Heagy et al., 1994; Dolnik & Epstein 1996; Huerta et al., 1998]. Multistable systems with rather strong noise intensity can exhibit stochastic resonance [Wiesenfeld & Jaramillo, 1998]. This phenomenon may be observed in an array of coupled multistable units [Lindner et al., 1995; Gang et al., 1996] where the signal-to-noise ratio can be significantly enhanced. Here we shall not investigate such cases and we shall consider only low noise intensity (unable to overcome the potential barrier).

The paper is organized as follows. In Sec. 2 we first study the noise-free, deterministic case, and successively consider in detail the various possibilities by varying d and δ . Then based on the results obtained in Sec. 3 we study the influence of the noise on the known possible regimes. Qualitative results have been cross-checked by direct numerical integration of (2). In Sec. 4 we summarize the results obtained.

2. The Deterministic Case

In this section we investigate the dynamics of the system (2) in the absence of noise. By putting $\epsilon = 0$ we get from (2)

$$\dot{r}_{j} = r_{j} [F(r_{j}, a) + \delta(\Phi(\chi_{j}) + \Phi(\chi_{j-1}))] + d(r_{j-1} \cos(\varphi_{j} - \varphi_{j-1})) - 2r_{j} + r_{j+1} \cos(\varphi_{j+1} - \varphi_{j})), \qquad (6)$$
$$\dot{\varphi}_{j} = \omega + \frac{d}{r_{j}} (r_{j+1} \sin(\varphi_{j+1} - \varphi_{j})) - r_{j-1} \sin(\varphi_{j} - \varphi_{j-1})).$$

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2.1. Dynamics of the system (6) when there is no diffusion, d = 0

Let us first study the case d = 0. Accordingly, we have an array of phase coupled units (via $\Phi(\chi)$). As it consists of isochronous oscillators with a common frequency the phase differences between units do not change in time and only depend on initial conditions. Hence the phase-difference variables, χ_j , are constants and we can substitute the influence of the nonlinear phase coupling functional, Φ , in the system (6) by some constants $\Delta_j = \delta[\Phi(\chi_j^0) + \Phi(\chi_{j-1}^0)]$. Then the system (6) reduces to

$$\dot{r}_j = r_j (F(r_j, a) + \Delta_j)$$

$$\dot{\varphi}_j = \omega .$$
(7)

Thus we have n independent units phase-shifted by constant values Δ_j . Figures 2(a) and 2(b) show domains of parameter values (a, Δ) , and corresponding phase portraits of the system (7). Figure 2(c) shows a bifurcation diagram.

For a fixed parameter value, a < 8, depending on Δ , system (7) can exhibit three different dynamical regimes. For $\Delta < 1-a/8$ there is only one stable focus. For $1-a/8 < \Delta < 1$ there is a stable focus as well as a stable limit cycle separated by an unstable limit cycle. For $\Delta > 1$ there is an unstable focus and a stable limit cycle (Fig. 2).

Since $\Delta_j \in [0, 2\delta]$, for a < 8 and $\delta > 1/2$, depending on initial phase distribution the values of Δ_j along the chain can be found in all parameter domains shown in Fig. 2(a). Hence for random initial conditions along the rather long chain we get units which are in different regimes (motionless, bistable or oscillatory). Further we shall demand that $\delta > 1/2$. For a > 8 we have only bistable and oscillatory regimes.

2.2. Dynamics of the system (6) when $\delta = 0$

Now let us put $\delta = 0$ in the system (6) while $d \neq 0$. Besides, by changing variables $\phi_j = \omega t + \varphi_j$ we can exclude from (6) a rotation with constant angular frequency ω . Then the system is gradient [Defontaines *et al.*, 1990]. The stable regimes correspond to local minima of the potential function in the form of either (i) in-phase oscillations [Nekorkin & Makarov, 1995] or (ii) phase waves [Nekorkin *et al.*, 1996]:

(i) In-phase oscillations. In this case a trajectory of the system (6) tends to a state with $\phi_{j+1} = \phi_j$



Fig. 2. (a) Parameter space of the system (7) exhibiting three possible dynamical regimes; (b) Phase planes illustrating these regimes; (c) Bifurcation diagram for fixed parameter value $a^* < 8$ (r is amplitude of oscillations). The solid lines correspond to stable solutions (fixed point, limit cycle) and the dotted line corresponds to an unstable limit cycle.

and, consequently, $\varphi_{j+1} = \varphi_j$. For the parameter region a > 8 and d small enough, there is *spatial amplitude disorder* [Nekorkin *et al.*, 1997]. In this case there are 2^n different possible stable amplitude distributions determined by suitable initial conditions. Each amplitude pattern can be coded by a sequence of "0" and "1". Symbols "0" and "1" correspond to small and high amplitudes of oscillations, respectively. For $n \to \infty$ such sequences can be periodic, with arbitrary period, or chaotic. For a < 8 the motionless state, $r_j = 0$, is the only global attractor.

(ii) *Phase waves.* They are characterized by a constant phase shift between neighboring oscillators,

and hence

$$\varphi_{j+1} - \varphi_j = \frac{2\pi k}{n} \tag{8}$$

with $k = \pm 1, \pm 2, \ldots, \pm n/2$. The value |n/k| can be treated as a wavelength and, accordingly, $K = 2\pi k/n$ is a wave vector on the 1D axis. Two types of phase waves can propagate in the system (6). The first one is a uniform wave. For this wave the oscillation amplitudes along the chain are one and the same constant. The second type is a nonuniform or patterned wave. For such waves the oscillation amplitude is a two-periodic function of the spatial coordinate j [Nekorkin *et al.*, 1996].

Thus depending on initial conditions and parameter values in the system (6) either a state of spatial disorder or phase waves appear. In the first case the spatial dynamics of the amplitude can be very complex and transitions from some ordered state to a disordered pattern can be observed. To characterize complexity in the spatial behavior of the system, we introduce a quantity which reflects the "level of disorder". Further this quantity can help us follow the time evolution of spatial patterns. There are several quantities characterizing chaotic time series, but usually the calculation of such quantities as entropy, Lyapunov exponents, Fourier power spectra and so on demands long timeconsuming evolutions. Here we introduce a simpler quantity, called the disorder parameter, that we think best suits to characterize rather short patterns. To do this let us consider some spatial pattern which can be described by variables u_i with $j = 1, 2, \ldots, n$, where n is not high. We assume that the disorder parameter has to be minimal (in fact zero) for patterns having purely periodic structure along the spatial coordinate and maximal for patterns without periodicity. For convenience, we continue considering periodic boundary conditions, $u_{i+n} = u_i$. Then our disorder parameter is:

$$\sigma(t) = \frac{2}{(n+1)U}$$
$$\min\left\{\sum_{j=1}^{n} |u_j - u_{j-k}|, \ k = 1, \ 2, \dots, \ n-1\right\}$$
(9)

where $U = u_{\text{max}} - u_{\text{min}}$ is a normalization constant and $u_{\text{max,min}}$ are the corresponding maximum and minimum possible values of $u_j(t)$. The quantity σ accounts for the shortest distance between the original pattern and patterns obtained by rotation of the original one for k nodes. Hence for periodic patterns $\sigma = 0$, because after rotating such pattern one period it will be identical to itself, and the distance will be equal to zero. With increasing pattern complexity σ grows and approaches its saturation value. It is equal to 1 and corresponds to a long, completely random pattern with u_j erratically jumping between u_{max} and u_{min} because for such pattern probability to find different (by U) values in the nodes of original and shifted patterns is maximal and equal to 1/2.

For a particular illustration of the concept just introduced we have integrated the equations for a ring chain (6) with 40 units and parameter values taken from the region of spatial amplitude disorder. We have chosen a periodic pattern as initial amplitude distribution along the ring [Fig. 3(a)]. The disorder parameter, σ , for such distribution vanishes though in practice due to the discreteness of a pattern it can slightly deviate from zero. The initial phase distribution have been chosen random [phase differences along the ring are shown in Fig. 3(c). Hence, its disorder parameter must be high. As expected, we finally obtain some rather complex spatial amplitude pattern which does not change in time [Fig. 3(b)]. Figure 3(d) shows the final distribution of phase differences. All of them vanish. Hence, finally, we get oscillations with homogeneous phase distribution and complex amplitude distribution. Figure 3(e) shows the evolution of the disorder parameter, σ , calculated for the amplitude distribution (solid line) and the distribution of phase differences (dashed line). The level of amplitude disorder starts from zero for the initial pattern and grows fast up to some fixed value. Hence the amplitude distributions soon gets rather complex and after about $t \approx 50$ we have a stationary amplitude pattern with a higher level of complexity than the initial one. At the same time the disorder level of phase differences starts from a high value, which corresponds to a random initial distribution and slowly goes down to zero. Hence the phase distribution is not stationary and keeps changing slowly while the amplitude distribution is practically unaltered. In the experiment, we observed formation and evolution of temporal phase clusters as earlier described [Nekorkin & Makarov, 1995]. Phase differences between clusters decrease in time and, finally, beyond $t \approx 2 \cdot 10^5$ we have the zero level of disorder in phase-difference distribution [Fig. 3(d)] corresponding to the in-phase oscillations of all units.







Fig. 3. Formation of spatial amplitude disorder. (a) Initial amplitude distribution; (b) Final steady amplitude distribution; (c) Initial distribution of phase differences; (d) Final steady distribution of phase differences; (e) Time evolution of disorder parameter, σ , for evolution of amplitude pattern (solid line) and for phase-difference pattern (dotted line) (a = 11, d = 0.1, $\delta = \epsilon = 0$).

Thus for a > 8, the diffusive coupling allows well-defined phase relations between oscillators in the lattice: (i) for rather weak coupling in the system there exists amplitude disorder with inphase oscillations; and (ii) choosing appropriately the initial conditions we can excite either uniform or nonuniform phase waves [Nekorkin *et al.*, 1996]. For a < 8 all oscillators decay to the state of rest.

2.3. Dynamics of the system (6) in the general case

Since for in-phase oscillations $\chi_j = 0$ and $\Phi(0) = \Phi'(0) \equiv 0$, the introduction of a new term $\delta[\Phi(\chi_j) + \Phi(\chi_{j-1})]$ into the system (6) does not change the existence conditions and local stability of such solutions. Thus, this term does not affect the disorder in spatial amplitude. We just get a different transient process to the behavior described in Sec. 2.2.

Let us now consider phase waves following the methodology described by earlier authors [Li & Erneux, 1994; Nekorkin *et al.*, 1996]. Seeking a stationary solution of the system (6) satisfying (8) we obtain the amplitude of the uniform waves

$$\rho^{\pm} = \sqrt{\frac{a \pm [a^2 - 8a\{1 - 2[\delta\Phi(K)] - d(1 - \cos K)]\}]^{1/2}}{4a}}$$
(10)

Two waves with different amplitudes are possible for a single wave vector K. From (10) we get the conditions for the existence of a wave with amplitude ρ^+ ,

$$\delta > \frac{8 - a + 16d(1 - \cos K)}{16\Phi(K)}$$
(11)

and for wave with amplitude ρ^-

$$\frac{8-a+16d(1-\cos K)}{16\Phi(K)} < \delta < \frac{1+2d(1-\cos K)}{2\Phi(K)}$$
(12)

Let us now investigate the (linear) stability of the wave solutions (8), (10) in the system (6). Introducing perturbations to a wave solution with amplitude ρ^+ or ρ^- and wave vector K, $\zeta_j = r_j - \rho^{\pm}$, $\eta = \varphi_j - (Kj + \varphi^0)$, and using their Fourier representation

$$\zeta_j = \sum_{m=1}^n \tilde{\zeta}_m \exp(iMj),$$

$$\eta_j = \sum_{m=1}^n \tilde{\eta}_m \exp(iMj),$$
(13)

with $M = 2\pi m/n$, we get n second-order linear systems for the Fourier modes

$$\tilde{\zeta}_{m} = C_{1}(\rho^{\pm}, K, M)\tilde{\zeta}_{m} + iC_{2}(\rho^{\pm}, K, M)\tilde{\eta}_{m}
\dot{\tilde{\eta}}_{m} = iC_{3}(\rho^{\pm}, K, M)\tilde{\zeta}_{m} + C_{4}(\rho^{\pm}, K, M)\tilde{\eta}_{m},$$
(14)

with

$$C_1 = \rho^{\pm} F'(\rho^{\pm}) + F(\rho^{\pm}) + 2[\delta \Phi(K) - d(1 - \cos K \cos M)]$$
$$C_2 = 2\rho^{\pm} (\delta \Phi'(K) - d \sin K) \sin M$$
$$C_3 = \frac{2d}{\rho^{\pm}} \sin K \sin M$$
$$C_4 = -2d \cos K(1 - \cos M).$$

The eigenvalues corresponding to (14) are

$$\lambda_{1,2}^{m} = \frac{C_1 + C_4 \pm \sqrt{(C_1 + C_4)^2 - 4(C_1C_4 + C_2C_3)}}{2}$$
(15)

One of the eigenvalues vanishes due to the invariance of the solution relative to a constant phase shift or translation symmetry. Hence for the stability of phase waves we must require that real parts of the remaining eigenvalues be negative. Figure 4 shows the domain of existence and (linear) stability of waves with high amplitude, $\rho = \rho^+$ in a ring of 45 units. The waves with |k| > 11 are unstable and the waves with $|k| \leq 11$ are stable in the full domains of their existence. The waves with $\rho = \rho^-$ are unstable. Thus for large enough δ one of the waves with amplitude $\rho = \rho^+$ and $k = \pm 1, \pm 2, \ldots, \pm 11$ can propagate around the ring. To get a prescribed wave we must choose appropriate initial conditions.

3. The Role of Noise

We restrict consideration to the case of weak noise. Here we follow the structure of the previous section and, sequentially, consider the various cases investigated above taking into account the influence of the noise.

3.1. Nonlinear phase coupling with noise

We start by neglecting the diffusive coupling between units, d = 0. Thus we have an array of units coupled together by the nonlinear phase coupling



Fig. 4. Stability diagram for phase waves with amplitude ρ^+ . The area above dark dots correspond to existence of waves with specified wave number $(n = 45, d = 0.02, a = 5, \alpha = 2, \bar{\chi} = 1)$.

functional, Φ , and seek the effect of noise. In polar coordinates, Eqs. (2) become

J-1

$$\dot{r}_j = r_j [(F(r_j) + \delta(\Phi(\chi_j) + \Phi(\chi_{j-1}))] + \epsilon(\eta_j^1(t) \cos \varphi_j + \eta_j^2(t) \sin \varphi_j)$$
(16)

J+1

(a)

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$$\dot{\varphi}_j = \omega + \frac{\epsilon}{r_j} (\eta_j^2(t) \cos \varphi_j - \eta_j^1(t) \sin \varphi_j),$$

where η_j^1 and η_j^2 account for the corresponding real and imaginary parts of the noise $\xi_j(t)$.

The noise alters the instantaneous frequencies of oscillations and, consequently, affects their phases. Note that the action of the noise is stronger near the origin, when the amplitude of oscillations, r_j , is close to zero because in the system (16) for $\dot{\varphi}$ the noisy term is proportional to ϵ/r_j . Hence a weak noise can drastically alter the phase differences between nearest neighbors while the amplitude of one of them is small. Phase differences remain almost unaffected while oscillation amplitudes are high. Thus now the values $\Delta_j = \delta[\Phi(\chi_j) + \Phi(\chi_{j-1})]$ are not constants but bounded functions of time, $0 \leq \Delta_j(t) < 2\delta$.

First we fix a < 8 and δ large enough ($\delta > 1/2$). Besides, as earlier said, we assume that the intensity of the noise is weak enough so that the transition probability to go over the amplitude potential barrier in the bistable case is negligible. As all Δ_j depend on time, each oscillator in the system (16) can alter its excitation regimes, by moving along a vertical line (a = const, $0 \le \Delta \le 2\delta$) on the plane (a, Δ) (Fig. 2). The behavior of an oscillator strongly depends on its amplitude and the phase difference with its neighbors:



Fig. 5. Qualitative illustration of the evolution of a unit starting from different initial conditions. (a) High value and (b) Low value of phase differences (solid circle corresponds to initial position of the unit, shaded circle corresponds to its position after phase resetting).



Fig. 6. Effect of nonlinear phase coupling on spatial amplitude and phase distributions. Sequences of snapshots for: (a) Amplitude distribution and (b) Distribution of phase differences (the first distribution at t = 0 falls into the x-axis, $\chi_j(0) \approx 0$). (c) Time evolution of the quantity, σ , characterizing disorder the amplitude pattern (solid line) and for the phase-difference pattern (dotted line) (a = 7, d = 0.0, $\delta = 0.87$, $\epsilon = 0.005$).

(i) High amplitudes and large phase differences [Fig. 5(a)]. If the values of χ_j and χ_{j-1} are rather high then $\Phi(\chi_j) + \Phi(\chi_{j-1}) \approx 2$. It corresponds to the third parameter domain in Fig. 2(a) and a highly excitable regime of the unit appears. Thus this unit will keep its oscillations with high am-

plitude and high phase differences with its neighbors because the coupling affects amplitudes only. Consequently, the noise plays negligible role.

(ii) High amplitudes and low phase differences [Fig. 5(b)]. As in this case the value $\Phi(\chi_j) + \Phi(\chi_{j-1})$ is close to zero and it corresponds to a high

synchronization level. The parameter values of such a unit are in the first domain in Fig. 2(a). We have decaying oscillations, with the amplitude of the unit going to zero [Fig. 5(b)]. However, as it becomes nearly vanishing, the noise drastically alters the phase of this unit. The values $\Phi(\chi_j)$ and $\Phi(\chi_{j-1})$ grow to unity and, consequently, the system enters into the third domain in Fig. 2(a). Hence the excitation of oscillations in the unit is possible but with a different phase. This new value of the phase must be significantly different from the initial one [Fig. 5(b)]. Thus we return to the case (i) earlier considered.

The above described processes are valid for all units in the lattice. Hence after some transient time all oscillators in the chain reorganize their phases which become arbitrary but with high differences between neighbors and with high oscillation amplitudes.

Figure 6 shows the results of direct integration of the system (16). We have chosen random initial conditions with rather low amplitude and almost uniform phase distribution [see first snapshots in Figs. 6(a) and 6(b)]. At the beginning, all units tend to the rest state. In this time interval the phase differences are so small that the functions $\Phi(\chi_j)$ are effectively switched-off. The disorder level decreases down to zero [Fig. 6(c)]. When the amplitude of oscillations are close to zero the phase differences drastically change. This corresponds to rapid oscillations in disorder level for phase differences around t = 10 [Fig. 6(c)]. Further oscillations with high amplitude are excited in the units [Fig. 6(a)]. Finally, the amplitude distribution becomes almost homogeneous and the distribution of phases becomes arbitrary with large phase shifts between neighboring oscillators [see latest snapshots on Figs. 6(a) and 6(b)].

For the parameter region a > 8 nothing new is expected. However, the fact that a stable limit cycle exists for any Δ yields the possibility of oscillations with high amplitude and small phase differences between neighbors. Figure 7 shows the almost final, practically constant distribution of amplitudes and phase differences. Initial conditions and parameter values for this calculation were the same (except a) as for the previous calculations (Fig. 6). As before we see that all oscillators go to a high amplitude regime [Fig. 7(a)] but some of them, e.g. oscillators 21, 22, have rather low phase differences with their neighbors and oscillators 20, 23 have low phase difference with right and left neighbors, respectively [the values of χ_{20} , χ_{21} , χ_{22} are low. See Fig. 7(b)]. Hence we have a phase cluster composed by these oscillators.

In conclusion, the nonlinear phase coupling described by the functional Φ acts like a *positive* feedback loop and enhances oscillations in the units which are in the low excited regime. For the case a < 8 (or monostable case) phase clusters can be



Fig. 7. Stationary high amplitude regime with both high and low phase differences between neighboring oscillators. (a) Snapshot of final amplitude distribution and (b) Phase-difference distribution (a = 11, d = 0.0, $\delta = 0.87$, $\epsilon = 0.005$).

broken. This process takes place through phase resetting [Nekorkin *et al.*, 1998] while oscillation amplitudes are practically vanishing and the phase drastically changes. New oscillations appear with largely different phases.

3.2. General case

We omit consideration of the case $\delta = 0$, because for small noise intensity we do not expect significant effects. Since the system is gradient in the deterministic case the dynamics of the ring must be about the same with additional small perturbations in oscillation amplitudes and phases, except for those units whose amplitudes are small. For these units the phases are drastically perturbed even with weak noise level. Thus, in general, we do not get in-phase oscillations in the whole chain as we had in Sec. 2.2 (Fig. 3), but rather we have spatial amplitude disorder, and phase waves occur for a > 8.

3.2.1. Phase waves

Let us first consider the phase waves described in Sec. 2.3. Without noise, in the phase space (r, ϕ) , each stable wave (Fig. 4) corresponds to a stable steady state with corresponding basin of attraction. If we start with initial conditions inside one of the basins, after a transient process we get phase waves with prescribed amplitude and wave vector. In general, to reduce time in obtaining a phase wave we should choose initial conditions as close as possible to its shape. At variance with this behavior, in the system with weak noise we can expect that starting with arbitrary initial conditions the trajectory in the phase space, after erratically "wandering" is finally attracted by a phase wave. Since we are restricting consideration to a low noise level we can neglect the transition over the potential barrier and safely say that this wave regime is stable. Figure 8 shows an example of such selforganizing process. We started from random initial conditions and followed the evolution of amplitudes and phases. After $t \approx 1800$, the phase relations between oscillators became practically the same as for the phase wave with k = -1 and we have a constant gradient of phase along the ring [Fig. 8(b)]. Although phases have become almost settled, amplitudes in a piece of the ring still drastically change [Fig. 8(a)] until $t \approx 2800$. The corresponding disorder levels as a function of time are shown in Fig. 8(c). At the beginning the disorder level in phase differ-

ences rapidly decreases. It corresponds to the for mation of a "wave" relation between phases. A ter $t \approx 2000$ this parameter becomes almost equa to zero. For the evolution of the amplitude we can distinguish three stages in the development of the spatial structure. At first, up to $t \approx 2000$ the disorder level oscillates around a rather high value $\sigma \approx 0.31$ [dashed line, stage 1 in Fig. 8(c)]. It con responds to amplitude waves erratically appearin and falling down due to collisions in a large portio of the ring [Fig. 8(a)]. Subsequently, the disorde level rapidly goes down and oscillates around a low value, $\sigma \approx 0.12$ [dashed line, stage 2 in Fig. 8(c)] For this time interval we have a narrow domain of inhomogeneous oscillations which disappears a $t \approx 2700$. Finally, a disorder-free state appears with a clear stable space-homogeneous wave [stage 3 in Fig. 8(c). Note that, for the same initial conditions and parameter values, without noise after a transient process a wave does not appear and the rest state takes over the whole ring. Hence, the noise helps overcoming the tendency to wave formation.

3.2.2. Phase clusters and phase resetting

Let us now consider another possible behavior of the system (2). To rule out a wave regime of the kind described above we make a suitable choice of parameter values. We take δ below the point with k = 11 in the curve shown in Fig. 4. Thus phase waves cannot exist in the ring.

To qualitatively describe the dynamics of the ring in this case we consider the "quasistationary" approximation. In the limit $d \rightarrow 0$ the oscillation amplitude in the *j*th oscillator does not depend on the amplitude of its neighbors. Besides, in this limit the process of transition between "quasistationary" states with different amplitudes (the motionless and oscillatory states) is much faster than the process of synchronization (changing χ) whose time scale is proportional to *d*. With these assumptions the "quasistationary" probability density of amplitude of oscillations of the *j*th oscillator has a narrow peak centered around zero or around the quasistationary amplitude value

$$r = \sqrt{\frac{a + (a^2 - 8a[1 - \delta(\Phi(\chi_j) + \Phi(\chi_{j-1}))])^{\frac{1}{2}}}{4a}}.$$
(17)

Figure 9 qualitatively shows the amplitude-phase dynamics in the case considered. The bistability of the units in the lattice makes possible the hysteresis

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Fig. Phase wave formation process from an initial disordered pattern. (a) Spatiotemporal evolution of oscillation amplitude (higher gray intensity corresponds to higher amplitude); (b) Spatiotemporal evolution of phases (gray intensity from black to white corresponds to the range from $-\pi$ to π); (c) Disorder level, σ , for amplitude and phase differences versus time (a = 0.02, $\delta = 11$, $\alpha = 2$, $\tilde{\chi} = 1$, $\epsilon = 0.005$).

shown in this figure. If, for example, at some instant of the several neighboring oscillators attain the upperbranch (high amplitude, r, and high phase difference, χ) then they are insensitive to the weak noise. In this stage the synchronization dominates. Hence the phase difference between neighboring oscillators, j and (j+1), decays rather slowly until the values $\chi = \chi_d$ and $r = \frac{1}{2}$ are reached. Then, suddenly, the oscillation amplitude vanishes, the phase becomes very sensitive to the noise and the system



Fig. 9. Qualitative evolution of the oscillation amplitude as a function of the phase difference χ .

jumps to a desynchronized state. After some time interval, that depends on the noise characteristics and can be rather long, the phase difference, χ , reaches the value χ_u and the amplitude of the oscillator jumps to the upper branch (Fig. 9). Then synchronization reappears. The values χ_d and χ_u for the *j*th oscillator depend on the left phase difference. Assuming that phase differences with left and right neighbors are the same we get from (17)

$$\chi_d = \Phi^{-1} \left(\frac{8-a}{16\delta} \right), \quad \chi_u = \Phi^{-1} \left(\frac{1}{2\delta} \right)$$
 (18)

where Φ^{-1} denotes the inverse functional of Φ .

Let us take the value χ_d low enough while χ_u is high enough. Then in the ring we get phase clustering and phase resetting altering one another. Starting from an arbitrary initial distribution, due to the linear part of the coupling, neighboring units tend to synchronize their oscillations and form a phase cluster. Phase differences between units decay, however nonlinear phase coupling, via Φ , together with noise do not permit the formation of a perfect cluster for which phases are exactly the same. The deviation from a perfect phase cluster is defined by the value χ_d and can be rather small. When part of a cluster becomes too close to the perfect state (for some units inside, phase differences $\chi_i \leq \chi_d$), then the amplitude in a unit jumps to a low value (see



Fig. 10. Temporal clustering and resetting. (a) Spatiotemporal evolution of amplitudes (high gray intensity corresponds to high amplitude); (b) Spatiotemporal evolution of phases (gray intensity from white through black to white corresponds to the circle range from $-\pi$ to π); (c) Amplitude of oscillations at site 5 versus time (a = 7.97, d = 0.02, $\delta = 2$, $\alpha = 3$, $\bar{\chi} = 2.52$, $\epsilon = 0.001$).



Fig. 9) and the noise drastically changes its phase. Hence new oscillations with high amplitude and resetted phase appear again. The phase shift after this process is defined by the value χ_u . Since it was chosen high enough, the cluster previously formed is broken. An example of such spatiotemporal dynamics of the ring is shown in Fig. 10. At the synchronization stage (when oscillation amplitudes in some parts of the chain are high and the linear coupling plays a significant role) neighboring oscillators form phase clusters which further break in the desynchronization stage (when amplitude is low and noise resets phase). Then new clusters at different spatial locations appear [Figs. 10(a) and 10(b)]. Since there are no attractors which could attract the trajectory (for example, like phase waves described in the previous section), such motions will continue forever. Figure 10(c) shows the evolution of the amplitude of the fifth oscillator. Time intervals with high amplitude alternate with intervals with low amplitude when the amplitude reaches the value 0.5. The shape of "amplitude pulses" corresponds to the upper branch shown in Fig. 9. The duration of time intervals between pulses strongly depends on the characteristics of the noise in agreement with the qualitative analysis shown in Fig. 9.

Thus by varying the parameter values of the system we can get different values for χ_d and χ_u and hence we can control the synchronization stage. The value χ_u is responsible for the phase difference between neighboring oscillators after phase resetting or after the desynchronization stage. The value χ_d defines the average phase difference after the synchronization process.

4. Conclusion

We have studied the mechanisms of synchronization and desynchronization of oscillations in a lattice of oscillatory units perturbed by a weak additive noise, when the coupling combines two different interactions. One is "diffusive", which easily yields synchronization of oscillations in the array. The other is a nonlinear phase coupling acting on the oscillation amplitudes. We have given evidence that nonlinear phase coupling and noise acting together drastically alter the dynamics of the lattice even if the noise intensity is weak and the individual dynamics of units is rather simple. We have found that phase coupling and noise can bring the ring from collective erratic behavior to a periodic phase wave. The wave can spontaneously form during evolution from an initially random spatial pattern. The time scale of this process can be very long. We note that for the formation of such a wave, without coupling and noise, we must start from suitable initial conditions. A wave cannot be formed from initial conditions which are too different from its shape. We have also found that for certain conditions there are no stable attractors. Hence trajectories erratically wander in the phase space. Choosing appropriate values χ_d and χ_u we can obtain conditions such that the erratic wandering ends in the formation of temporal phase clusters. It is enough to demand that χ_d be as low as possible and χ_u rather high. Then during the synchronization stage, phases of neighboring units in some part of the chain become almost the same and hence they form a phase cluster. Subsequently, the cluster is destroyed, units reorganize and a new cluster appears at a different location in

the lattice. We have also introduced an algoritmically simple and attractive quantity characterizing spatial complexity of finite (intermediate and small length) patterns.

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