International Journal of Bifurcation and Chaos, Vol. 11, No. 1 (2001) 109–122

PHASE RESETTING, CLUSTERS, AND WAVES IN A LATTICE OF COUPLED OSCILLATORY UNITS

V. A. MAKAROV*, V. I. NEKORKIN* and M. G. VELARDE
Instituto Pluridisciplinar, Universidad Complutense,
Paseo Juan XXIII, nº 1, Madrid 28 040, Spain
**Radiophysical Department, Nizhny Novgorod State University,*
Gagarin Av., 23, Nizhny Novgorod 603600, Russia

Received February 17, 2000; Revised March 9, 2000

A one-dimensional chain consisting of many locally coupled oscillatory units is considered when both diffusive (linear) and nonlinear interactions exist. The dynamics of units may also be altered by a weak white Gaussian noise. Two different collective evolutionary paths have been identified in the chain: (i) the onset of temporal phase clusters and their subsequent breakdown or reconstruction through phase resetting, and (ii) the formation of a phase wave during a process of selforganization. Such a wave may originate from an initially disordered pattern after appropriate reorganization of phases and amplitudes occurring on a time lapse that can be very long.

1. Introduction

Synchronization of oscillations and, consequently, uniform rhythmic activity in coupled systems is a phenomenon widespread in nature and various branches of engineering [Haken, 1983a, 1983b, 1996; Murray, 1993]. The simplest coupling leading to synchronous oscillations is "diffusion" or a resistive type of coupling. Here we study a model-problem that combines the processes of *synchronization and* its destruction in an ensemble of diffusively coupled, identical oscillatory units.

As the unit or element of the lattice we choose a simple albeit nonlinear oscillator whose dynamics in polar coordinates is described by the system

$$\begin{aligned} \dot{r} &= rF(r) \\ \dot{\varphi} &= \omega, \end{aligned} \tag{1}$$

where r, ω and φ denote amplitude, frequency and phase of the oscillations, and F is a suitable nonlinearity regulating the excitability of the unit. Thus the system (1) represents an *isochronous* oscillator. For illustration we take $F = -2ar^4 + ar^2 - 1$. When $a < 8$, in the phase plane of the system (1) there exists only a stable equilibrium point (focus) which corresponds to the rest state of the unit. For $a > 8$ we have two attractors. They are a stable focus at the origin and a stable limit cycle. The latter corresponds to the oscillatory activity of the unit. Hence, by changing a we can switch from a monostable to a bistable regime.

To describe the cooperative behavior of an ensemble of such units we consider a 1D lattice consisting of n units (1) diffusively coupled with nearest-neighbor interaction. We assume periodic boundary conditions

$$\begin{aligned} \dot{z}_j = z_j[F(z_j) + i\omega + \delta(\Phi(\chi_j) + \Phi(\chi_{j-1}))] + \\ + d(z_{j-1} - 2z_j + z_{j+1}) + \epsilon\xi_j(t) \\ j = 1, 2, \dots, n, \quad z_{j+n} = z_j, \end{aligned} \tag{2}$$

where $z_j = r_j e^{i\varphi_j}$, d and δ account for the diffusive (linear) and nonlinear coupling between units, respectively. We have also introduced a functional, $\Phi(\chi)$, characterizing nonlinear phase coupling as we shall see below. In particular $\Phi(\chi)$ would be responsible for desynchronization of oscillations in the vicinity of a given unit. Thus we choose the variable χ depending on phase differences between neighboring units,

$$\chi_j = \varphi_{j+1} - \varphi_j. \qquad (3)$$

Hence $\Phi(\chi_j)$ and $\Phi(\chi_{j-1})$ define levels of synchronization of the unit j relative to its right and left neighbors, respectively. Low values of χ correspond to a high level of synchronization ($\chi = 0$ corresponds to strict in-phase oscillations) and high values of χ correspond to a low synchronization level. Such cooperative effect is best described by a sigmoidal function. Since phase differences can be either positive or negative, this function must be symmetric around $\chi = 0$. An example of such function is shown in Fig. 1. For values of χ around the origin it is almost equal to zero while with increasing $|\chi|$ the function Φ approaches a saturation value. As a concrete application, let us take for illustration and for all our numerical calculations the Boltzmann function

$$\Phi(\chi) = \frac{1}{1+e^{-\alpha\tilde{\chi}}} - \frac{1}{1+e^{\alpha(|\chi|-\tilde{\chi})}} \qquad (4)$$

where α and $\tilde{\chi}$ are positive constants. This function has the shape shown in Fig. 1. and it is reminiscent of the Michaelis–Menten/Hinshelwood–Langmuir/Holling law found in biochemistry/chemical kinetics/or biology. Further we shall consider the system (2) working in an noisy environment that we model by $\xi_j(t)$. We choose it as a white Gaussian noise whose strength is controlled by the parameter ϵ, [Eq. (2)],

$$\langle \xi_j(t) \rangle = 0, \quad \langle \xi_j(t)\xi_k(t') \rangle = \delta_{jk}\delta(t-t'). \qquad (5)$$

Thus model (2) is a spatially distributed oscillatory system with local couplings perturbed by noise. In the absence of noise and depending on the properties of the unit we have two possibilities. One is the case of a network of limit cycle oscillators when its dynamics can be reduced to that of a system of phase oscillators [Winfree, 1967; Ermentrout & Rinzel, 1981; Kuramoto, 1984]. Lattice systems of phase oscillators have been studied in a number of papers [Ermentrout, 1985; Daido, 1988; Strogatz & Mirollo, 1988; Sakaguchi *et al.*, 1988; Niebur *et al.*, 1991]. The other case is that of a chain consisting of multistable units when the amplitude dynamics drastically affects and regulates the phase dynamics and vice versa [Defontaines *et al.*, 1990; Nekorkin & Makarov, 1995; Nekorkin *et al.*, 1996, 1997, 1998, 1999; Sepulchre & MacKay, 1997] or each unit is a chaotic oscillator [Heagy *et al.*, 1994; Dolnik & Epstein 1996; Huerta *et al.*, 1998]. Multistable systems with rather strong noise intensity can exhibit stochastic resonance [Wiesenfeld & Jaramillo, 1998]. This phenomenon may be observed in an array of coupled multistable units [Lindner *et al.*, 1995; Gang *et al.*, 1996] where the signal-to-noise ratio can be significantly enhanced. Here we shall not investigate such cases and we shall consider only low noise intensity (unable to overcome the potential barrier).

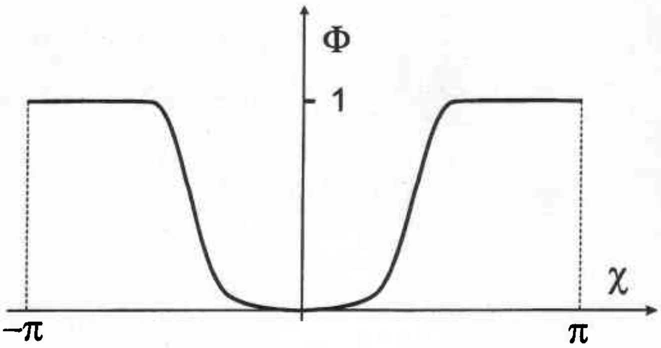


Fig. 1. Qualitative shape of the nonlinear phase coupling functinal Φ.

The paper is organized as follows. In Sec. 2 we first study the noise-free, deterministic case, and successively consider in detail the various possibilities by varying d and δ. Then based on the results obtained in Sec. 3 we study the influence of the noise on the known possible regimes. Qualitative results have been cross-checked by direct numerical integration of (2). In Sec. 4 we summarize the results obtained.

2. The Deterministic Case

In this section we investigate the dynamics of the system (2) in the absence of noise. By putting $\epsilon = 0$ we get from (2)

$$\begin{aligned}\dot{r}_j &= r_j[F(r_j,a) + \delta(\Phi(\chi_j) + \Phi(\chi_{j-1}))] \\ &\quad + d(r_{j-1}\cos(\varphi_j - \varphi_{j-1}) \\ &\quad - 2r_j + r_{j+1}\cos(\varphi_{j+1} - \varphi_j)), \qquad (6) \\ \dot{\varphi}_j &= \omega + \frac{d}{r_j}(r_{j+1}\sin(\varphi_{j+1} - \varphi_j) \\ &\quad - r_{j-1}\sin(\varphi_j - \varphi_{j-1})).\end{aligned}$$

2.1. *Dynamics of the system (6) when there is no diffusion, $d = 0$*

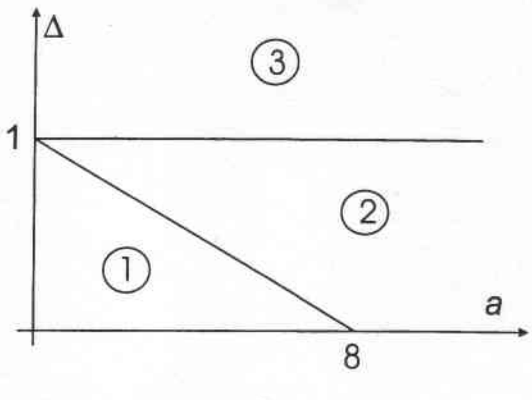
Let us first study the case $d = 0$. Accordingly, we have an array of phase coupled units (via $\Phi(\chi)$). As it consists of isochronous oscillators with a common frequency the phase differences between units do not change in time and only depend on initial conditions. Hence the phase-difference variables, χ_j, are constants and we can substitute the influence of the nonlinear phase coupling functional, Φ, in the system (6) by some constants $\Delta_j = \delta[\Phi(\chi_j^0)+\Phi(\chi_{j-1}^0)]$. Then the system (6) reduces to

$$\begin{aligned} \dot{r}_j &= r_j(F(r_j, a) + \Delta_j) \\ \dot{\varphi}_j &= \omega\,. \end{aligned} \tag{7}$$

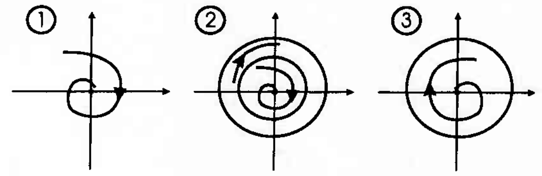
Thus we have n independent units phase-shifted by constant values Δ_j. Figures 2(a) and 2(b) show domains of parameter values (a, Δ), and corresponding phase portraits of the system (7). Figure 2(c) shows a bifurcation diagram.

For a fixed parameter value, $a < 8$, depending on Δ, system (7) can exhibit three different dynamical regimes. For $\Delta < 1-a/8$ there is only one stable focus. For $1-a/8 < \Delta < 1$ there is a stable focus as well as a stable limit cycle separated by an unstable limit cycle. For $\Delta > 1$ there is an unstable focus and a stable limit cycle (Fig. 2).

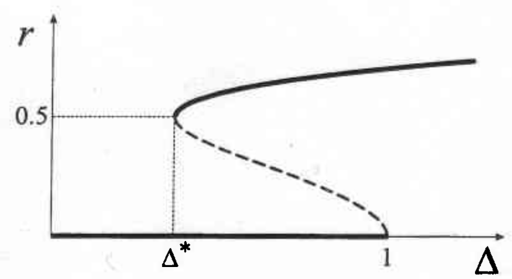
Since $\Delta_j \in [0, 2\delta]$, for $a < 8$ and $\delta > 1/2$, depending on initial phase distribution the values of Δ_j along the chain can be found in all parameter domains shown in Fig. 2(a). Hence for random initial conditions along the rather long chain we get units which are in different regimes (motionless, bistable or oscillatory). Further we shall demand that $\delta > 1/2$. For $a > 8$ we have only bistable and oscillatory regimes.



(a)



(b)



(c)

Fig. 2. (a) Parameter space of the system (7) exhibiting three possible dynamical regimes; (b) Phase planes illustrating these regimes; (c) Bifurcation diagram for fixed parameter value $a^* < 8$ (r is amplitude of oscillations). The solid lines correspond to stable solutions (fixed point, limit cycle) and the dotted line corresponds to an unstable limit cycle.

2.2. *Dynamics of the system (6) when $\delta = 0$*

Now let us put $\delta = 0$ in the system (6) while $d \neq 0$. Besides, by changing variables $\phi_j = \omega t + \varphi_j$ we can exclude from (6) a rotation with constant angular frequency ω. Then the system is gradient [Defontaines *et al.*, 1990]. The stable regimes correspond to local minima of the potential function in the form of either (i) in-phase oscillations [Nekorkin & Makarov, 1995] or (ii) phase waves [Nekorkin *et al.*, 1996]:

(i) *In-phase oscillations.* In this case a trajectory of the system (6) tends to a state with $\phi_{j+1} = \phi_j$ and, consequently, $\varphi_{j+1} = \varphi_j$. For the parameter region $a > 8$ and d small enough, there is *spatial amplitude disorder* [Nekorkin *et al.*, 1997]. In this case there are 2^n different possible stable amplitude distributions determined by suitable initial conditions. Each amplitude pattern can be coded by a sequence of "0" and "1". Symbols "0" and "1" correspond to small and high amplitudes of oscillations, respectively. For $n \to \infty$ such sequences can be periodic, with arbitrary period, or chaotic. For $a < 8$ the motionless state, $r_j = 0$, is the only global attractor.

(ii) *Phase waves.* They are characterized by a constant phase shift between neighboring oscillators, and hence

$$\varphi_{j+1} - \varphi_j = \frac{2\pi k}{n} \tag{8}$$

with $k = \pm 1, \pm 2, \ldots, \pm n/2$. The value $|n/k|$ can be treated as a wavelength and, accordingly, $K = 2\pi k/n$ is a wave vector on the 1D axis. Two types of phase waves can propagate in the system (6). The first one is a uniform wave. For this wave the oscillation amplitudes along the chain are one and the same constant. The second type is a nonuniform or patterned wave. For such waves the oscillation amplitude is a two-periodic function of the spatial coordinate j [Nekorkin *et al.*, 1996].

Thus depending on initial conditions and parameter values in the system (6) either a state of spatial disorder or phase waves appear. In the first case the *spatial* dynamics of the amplitude can be very complex and transitions from some ordered state to a disordered pattern can be observed. To characterize complexity in the spatial behavior of the system, we introduce a quantity which reflects the "level of disorder". Further this quantity can help us follow the time evolution of spatial patterns. There are several quantities characterizing chaotic time series, but usually the calculation of such quantities as entropy, Lyapunov exponents, Fourier power spectra and so on demands long time-consuming evolutions. Here we introduce a simpler quantity, called the *disorder parameter*, that we think best suits to characterize rather short patterns. To do this let us consider some spatial pattern which can be described by variables u_j with $j = 1, 2, \ldots, n$, where n is not high. We assume that the disorder parameter has to be minimal (in fact zero) for patterns having purely periodic structure along the spatial coordinate and maximal for patterns without periodicity. For convenience, we continue considering periodic boundary conditions, $u_{j+n} = u_j$. Then our disorder parameter is:

$$\sigma(t) = \frac{2}{(n+1)U} \min\left\{\sum_{j=1}^{n} |u_j - u_{j-k}|,\ k = 1, 2, \ldots, n-1\right\} \tag{9}$$

where $U = u_{\max} - u_{\min}$ is a normalization constant and $u_{\max,\min}$ are the corresponding maximum and minimum possible values of $u_j(t)$. The quantity σ accounts for the shortest distance between the original pattern and patterns obtained by rotation of the original one for k nodes. Hence for periodic patterns $\sigma = 0$, because after rotating such pattern one period it will be identical to itself, and the distance will be equal to zero. With increasing pattern complexity σ grows and approaches its saturation value. It is equal to 1 and corresponds to a long, completely random pattern with u_j erratically jumping between $u_{\max}$ and $u_{\min}$ because for such pattern probability to find different (by U) values in the nodes of original and shifted patterns is maximal and equal to 1/2.

For a particular illustration of the concept just introduced we have integrated the equations for a ring chain (6) with 40 units and parameter values taken from the region of spatial amplitude disorder. We have chosen a periodic pattern as initial amplitude distribution along the ring [Fig. 3(a)]. The disorder parameter, σ, for such distribution vanishes though in practice due to the discreteness of a pattern it can slightly deviate from zero. The initial phase distribution have been chosen random [phase differences along the ring are shown in Fig. 3(c)]. Hence, its disorder parameter must be high. As expected, we finally obtain some rather complex spatial amplitude pattern which does not change in time [Fig. 3(b)]. Figure 3(d) shows the final distribution of phase differences. All of them vanish. Hence, finally, we get oscillations with homogeneous phase distribution and complex amplitude distribution. Figure 3(e) shows the evolution of the disorder parameter, σ, calculated for the amplitude distribution (solid line) and the distribution of phase differences (dashed line). The level of amplitude disorder starts from zero for the initial pattern and grows fast up to some fixed value. Hence the amplitude distributions soon gets rather complex and after about $t \approx 50$ we have a stationary amplitude pattern with a higher level of complexity than the initial one. At the same time the disorder level of phase differences starts from a high value, which corresponds to a random initial distribution and slowly goes down to zero. Hence the phase distribution is not stationary and keeps changing slowly while the amplitude distribution is practically unaltered. In the experiment, we observed formation and evolution of temporal phase clusters as earlier described [Nekorkin & Makarov, 1995]. Phase differences between clusters decrease in time and, finally, beyond $t \approx 2 \cdot 10^5$ we have the zero level of disorder in phase-difference distribution [Fig. 3(d)] corresponding to the in-phase oscillations of all units.

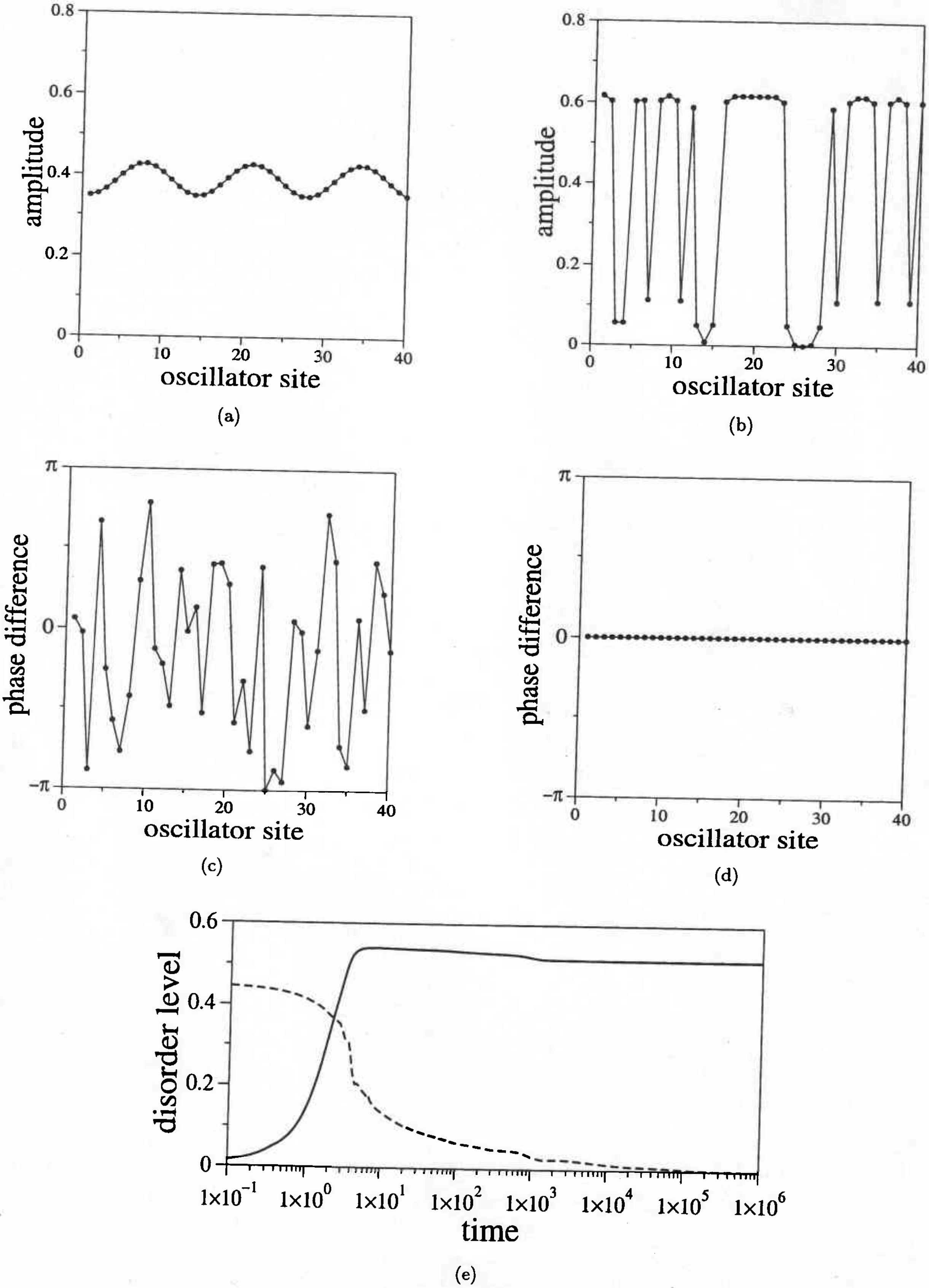


Fig. 3. Formation of spatial amplitude disorder. (a) Initial amplitude distribution; (b) Final steady amplitude distribution; (c) Initial distribution of phase differences; (d) Final steady distribution of phase differences; (e) Time evolution of *disorder* parameter, σ, for evolution of amplitude pattern (solid line) and for phase-difference pattern (dotted line) ($a = 11$, $d = 0.1$, $\delta = \epsilon = 0$).

Thus for $a > 8$, the diffusive coupling allows well-defined phase relations between oscillators in the lattice: (i) for rather weak coupling in the system there exists amplitude disorder with in-phase oscillations; and (ii) choosing appropriately the initial conditions we can excite either uniform or nonuniform phase waves [Nekorkin *et al.*, 1996]. For $a < 8$ all oscillators decay to the state of rest.

2.3. *Dynamics of the system (6) in the general case*

Since for in-phase oscillations $\chi_j = 0$ and $\Phi(0) = \Phi'(0) \equiv 0$, the introduction of a new term $\delta[\Phi(\chi_j) + \Phi(\chi_{j-1})]$ into the system (6) does not change the existence conditions and local stability of such solutions. Thus, this term does not affect the disorder in spatial amplitude. We just get a different transient process to the behavior described in Sec. 2.2.

Let us now consider phase waves following the methodology described by earlier authors [Li & Erneux, 1994; Nekorkin *et al.*, 1996]. Seeking a stationary solution of the system (6) satisfying (8) we obtain the amplitude of the uniform waves

$$\rho^{\pm} = \sqrt{\frac{a \pm [a^2 - 8a\{1 - 2[\delta\Phi(K) - d(1-\cos K)]\}]^{1/2}}{4a}} \qquad (10)$$

Two waves with different amplitudes are possible for a single wave vector K. From (10) we get the conditions for the existence of a wave with amplitude ρ^+,

$$\delta > \frac{8 - a + 16d(1-\cos K)}{16\Phi(K)} \qquad (11)$$

and for wave with amplitude ρ^-

$$\frac{8 - a + 16d(1-\cos K)}{16\Phi(K)} < \delta < \frac{1 + 2d(1-\cos K)}{2\Phi(K)} \qquad (12)$$

Let us now investigate the (linear) stability of the wave solutions (8), (10) in the system (6). Introducing perturbations to a wave solution with amplitude ρ^+ or ρ^- and wave vector K, $\zeta_j = r_j - \rho^{\pm}$, $\eta = \varphi_j - (Kj + \varphi^0)$, and using their Fourier representation

$$\zeta_j = \sum_{m=1}^{n} \tilde{\zeta}_m \exp(iMj), \qquad \eta_j = \sum_{m=1}^{n} \tilde{\eta}_m \exp(iMj), \qquad (13)$$

with $M = 2\pi m/n$, we get n second-order linear systems for the Fourier modes

$$\begin{aligned} \dot{\tilde{\zeta}}_m &= C_1(\rho^{\pm}, K, M)\tilde{\zeta}_m + iC_2(\rho^{\pm}, K, M)\tilde{\eta}_m \\ \dot{\tilde{\eta}}_m &= iC_3(\rho^{\pm}, K, M)\tilde{\zeta}_m + C_4(\rho^{\pm}, K, M)\tilde{\eta}_m, \end{aligned} \qquad (14)$$

with

$$\begin{aligned} C_1 &= \rho^{\pm}F'(\rho^{\pm}) + F(\rho^{\pm}) + 2[\delta\Phi(K) - d(1 - \cos K \cos M)] \\ C_2 &= 2\rho^{\pm}(\delta\Phi'(K) - d\sin K)\sin M \\ C_3 &= \frac{2d}{\rho^{\pm}}\sin K \sin M \\ C_4 &= -2d\cos K(1 - \cos M). \end{aligned}$$

The eigenvalues corresponding to (14) are

$$\lambda_{1,2}^{m} = \frac{C_1 + C_4 \pm \sqrt{(C_1 + C_4)^2 - 4(C_1C_4 + C_2C_3)}}{2} \qquad (15)$$

One of the eigenvalues vanishes due to the invariance of the solution relative to a constant phase shift or translation symmetry. Hence for the stability of phase waves we must require that real parts of the remaining eigenvalues be negative. Figure 4 shows the domain of existence and (linear) stability of waves with high amplitude, $\rho = \rho^+$ in a ring of 45 units. The waves with $|k| > 11$ are unstable and the waves with $|k| \leq 11$ are stable in the full domains of their existence. The waves with $\rho = \rho^-$ are unstable. Thus for large enough δ one of the waves with amplitude $\rho = \rho^+$ and $k = \pm 1, \pm 2, \ldots, \pm 11$ can propagate around the ring. To get a prescribed wave we must choose appropriate initial conditions.

3. The Role of Noise

We restrict consideration to the case of weak noise. Here we follow the structure of the previous section and, sequentially, consider the various cases investigated above taking into account the influence of the noise.

3.1. *Nonlinear phase coupling with noise*

We start by neglecting the diffusive coupling between units, $d = 0$. Thus we have an array of units coupled together by the nonlinear phase coupling

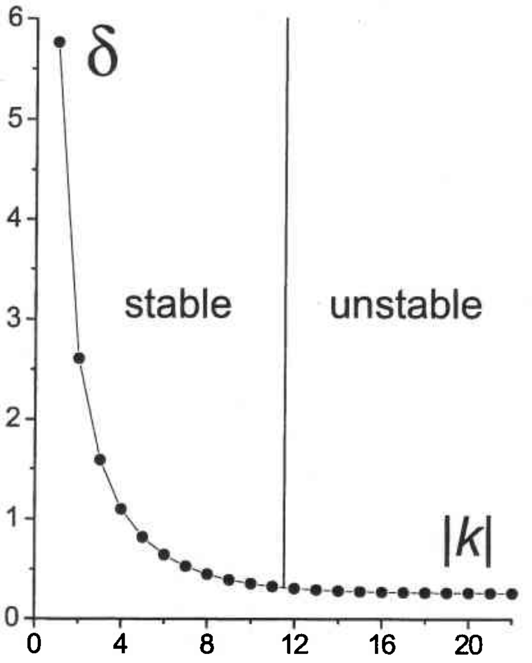


Fig. 4. Stability diagram for phase waves with amplitude ρ^+. The area above dark dots correspond to existence of waves with specified wave number ($n = 45$, $d = 0.02$, $a = 5$, $\alpha = 2$, $\tilde{\chi} = 1$).

functional, Φ, and seek the effect of noise. In polar coordinates, Eqs. (2) become

$$\dot{r}_j = r_j[(F(r_j) + \delta(\Phi(\chi_j) + \Phi(\chi_{j-1})] + \epsilon(\eta_j^1(t)\cos\varphi_j + \eta_j^2(t)\sin\varphi_j) \tag{16}$$

$$\dot{\varphi}_j = \omega + \frac{\epsilon}{r_j}(\eta_j^2(t)\cos\varphi_j - \eta_j^1(t)\sin\varphi_j)\,,$$

where η_j^1 and η_j^2 account for the corresponding real and imaginary parts of the noise $\xi_j(t)$.

The noise alters the instantaneous frequencies of oscillations and, consequently, affects their phases. Note that the action of the noise is stronger near the origin, when the amplitude of oscillations, r_j, is close to zero because in the system (16) for $\dot{\varphi}$ the noisy term is proportional to ϵ/r_j. Hence a weak noise can drastically alter the phase differences between nearest neighbors while the amplitude of one of them is small. Phase differences remain almost unaffected while oscillation amplitudes are high. Thus now the values $\Delta_j = \delta[\Phi(\chi_j) + \Phi(\chi_{j-1})]$ are not constants but bounded functions of time, $0 \leq \Delta_j(t) < 2\delta$.

First we fix $a < 8$ and δ large enough ($\delta > 1/2$). Besides, as earlier said, we assume that the intensity of the noise is weak enough so that the transition probability to go over the amplitude potential barrier in the bistable case is negligible. As all Δ_j depend on time, each oscillator in the system (16) can alter its excitation regimes, by moving along a vertical line (a = const, $0 \leq \Delta \leq 2\delta$) on the plane (a, Δ) (Fig. 2). The behavior of an oscillator strongly depends on its amplitude and the phase difference with its neighbors:

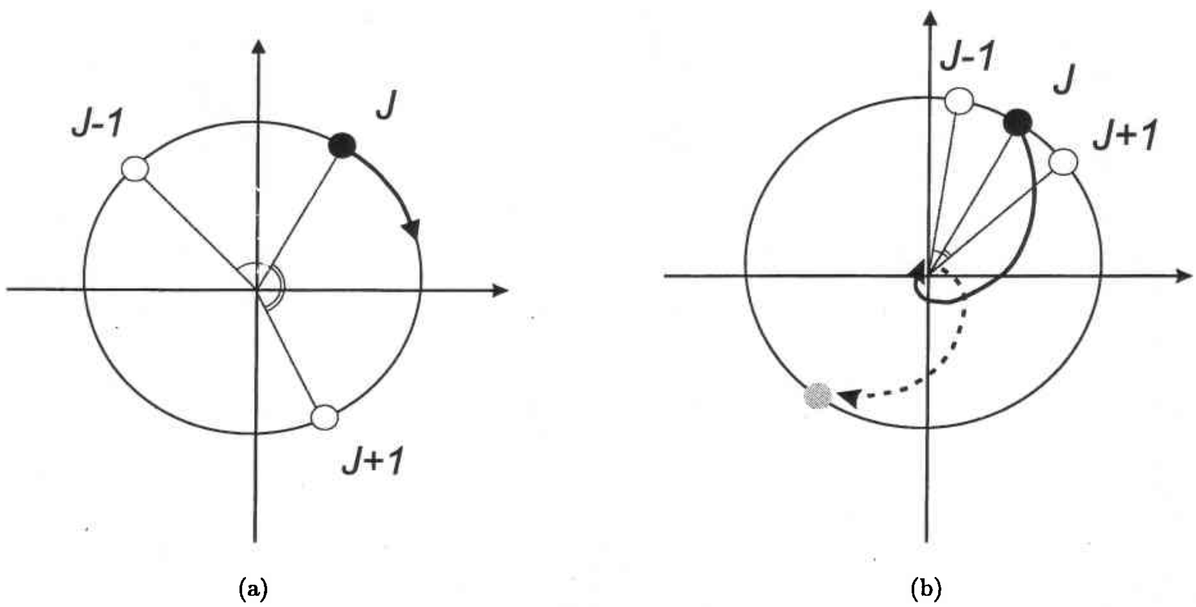


Fig. 5. Qualitative illustration of the evolution of a unit starting from different initial conditions. (a) High value and (b) Low value of phase differences (solid circle corresponds to initial position of the unit, shaded circle corresponds to its position after phase resetting).