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Symplectic Foliations in Four-Manifolds

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Abstract

A detailed study of symplectic foliations in a symplectic manifold of dimension 4 is carried out, proving the analogue of the Baum-Bott formulas and giving restrictions to the existence of symplectic foliations with no singular points.

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1 Introduction

Let (M, ω) be a symplectic manifold of dimension $2n$. Let L be a complex line bundle over M. A (codimension one) symplectic foliation [3] with normal bundle L is given by some $\alpha \in \mathcal{C}^\infty(T^*M \otimes L)$ satisfying the integrability condition $\alpha \wedge d\alpha = 0$. The associated distribution $K_x = \ker \alpha(x) \subset T_xM$ must satisfy that K_x is a symplectic subspace of the tangent space. Moreover the singular set

$$S = \{x \in M | \alpha(x) = 0\}$$

must be a union of symplectic submanifolds of M, all of them of (real) codimension 4 or more, and intersecting transversely along symplectic submanifolds.

Alternatively, a symplectic foliation can be defined as a (possibly singular) foliation (i.e., a locally integrable distribution K_x of real codimension 2, for x in an open dense subset $M - S$ of M) whose leaves are symplectic and whose singular locus S is a union of symplectic submanifolds of real codimension 4 or more, intersecting transversely along symplectic submanifolds. A point in $M - S$ is called a regular point. Let U be a neighborhood of a regular point.

Then the leaves of the foliation in U can be described as the level sets of a differentiable function $f : U \to \mathbb{C}$, so that $K_x = \ker df(x)$ in U. We may take a cover $\{U_i\}$ of $M - S$. Then in the overlap region $U_i \cap U_j$ the functions f_i and f_j compare as $f_i = \phi_{ij} \circ f_j$, where ϕ_{ij} is a smooth map from $\mathbb{C}$ to $\mathbb{C}$. So $df_i = d\phi_{ij}(f_j(x))df_j$, where $d\phi_{ij} \in \mathrm{GL}(2, \mathbb{R})$ and it is constant along leaves. Such $(d\phi_{ij})$ define a cocycle and hence a cohomology class in $H^1(M - S, \mathrm{GL}(2, \mathbb{R}))$. Since S has codimension 4 or more, this group is isomorphic to $H^1(M, \mathrm{GL}(2, \mathbb{R}))$, so there is a line bundle L over M determined by the foliation, named the normal bundle associated to it. The 1-forms df_i glue together to a well-defined L-valued 1-form in $M - S$. After multiplying by a suitable smooth function (vanishing to infinite order in S if necessary), we get a 1-form $\alpha \in \mathcal{C}^\infty(T^*M \otimes L)$ which defines the foliation, i.e., $\ker \alpha(x) \subset T_xM$ is the subspace K_x.

A chart $\phi : U \subset M \to \mathbb{C}^n$ will be called adapted at the point $x \in U$ if $(\phi_*)_x\omega = \omega_0$, where ω_0 is the standard symplectic form in $\mathbb{C}^n$. This means that ϕ satisfies the Darboux condition just at the point x. Most of the foliations constructed following the method in [3] (e.g., Kupka foliations) have singularities of the following kinds:

1. isolated points where there are adapted charts $(z_1, \dots, z_n)$ such that $\alpha = z_1 dz_1 + \cdots + z_n dz_n$, i.e., the leaves are the level sets of the function $f = z_1^2 + \cdots + z_n^2$.

2. a union of smooth symplectic submanifolds such that each point of them has an adapted chart $(z_1, \dots, z_n)$ where α is written as $\eta(z_1, \dots, z_n)$ for a holomorphic 1-form η.

This happens because the symplectic foliations, in [3] are constructed by restricting a holomorphic foliation in $\mathbb{CP}^N$ to $M \subset \mathbb{CP}^N$ under some asymptotically holomorphic embedding (see [10] for definitions). So it is reasonable to treat with:

Definition 1.1 *Let (M, ω) be a symplectic manifold of dimension $2n$. A holomorphically modeled symplectic foliation $\mathcal{F}$ is a foliation such that given any point $x \in M$, there exists an adapted chart $\phi : U \to \mathbb{C}^n$ such that $\phi_*(\mathcal{F})$ is a holomorphic foliation.*

This implies, in particular, that the leaves of the foliation are symplectic and the singular set is symplectic whenever is smooth or stratified.

We are going to study only holomorphically modeled symplectic foliations from now on, since the singularities of generic symplectic foliations can have very complicated topology.

Suppose that M is a symplectic manifold of dimension 4. Then the singular set S is a collection of finitely many points. At any of the singular points there is, at least, one preferred trivialization in which the foliation is equal to a holomorphic foliation. This will be called holomorphic trivialization of $\mathcal{F}$ at the singular point (but it is not holomorphic in any sense, since M is not assumed to have an integrable complex structure). This means that $\phi_*(\mathcal{F})$ is given by a holomorphic vector field

$$v(z,w) = F(z,w)\frac{\partial}{\partial z} + G(z,w)\frac{\partial}{\partial w},$$

i.e., $\{v, Jv\}$ generate $\phi_*(\mathcal{F})_x$ for all x off the singular set. The 1-form

$$G(z,w)dz - F(z,w)dw$$

generates the foliation. This does not mean that α is necessarily holomorphic, since changing α by an element of $\mathrm{GL}(2,\mathbb{R})$ does not change the foliation. For instance, consider the symplectic Lefschetz pencils of [4]: these have two kinds of singular points, over which the holomorphic trivializations give either $\alpha = z_1 dz_1 + z_2 dz_2$ or $\alpha = z_2 dz_1 - z_1 dz_2$.

We want to start a detailed study of the (holomorphically modeled) symplectic foliations for manifolds of dimension 4. In Section 2 we prove the existence of almost complex structures making the foliation a complex distribution. In Section 3 we define the tangent bundle $T_{\mathcal{F}}$ and the normal bundle $\mathcal{N}_{\mathcal{F}}$ to the symplectic foliation $\mathcal{F}$, so that we can prove the Baum-Bott formulas in Section 5. Section 4 gives formulas relating the tangency number of a pseudo-holomorphic curve with a symplectic foliation in terms of $T_{\mathcal{F}}$ and $\mathcal{N}_{\mathcal{F}}$. Finally, Section 6 initiates the study of the question on the existence of regular symplectic foliations (those with empty singular set) in four-manifolds. This is aimed to a (future) possible classification of all regular symplectic foliations in a four-manifold, extending the classification of regular holomorphic foliations in complex manifolds given by Brunella [2].

Some of the results that follow may be extended to higher dimensions, but the classification results that we get in Section 6 are specific to dimension four. In what follows we will assume that (M, ω) is a symplectic four-manifold and $\mathcal{F}$ is a (holomorphically modeled) symplectic foliation with singular set S.

2 Adapted almost complex structures

The process of construction of symplectic foliations in [3] goes through choosing an almost complex structure J compatible with the symplectic structure

ω. Consider the foliation α as an application $\alpha : TM \to L$. Using the almost complex structure on TM, we decompose α in complex linear and complex anti-linear parts,

$$\alpha = \alpha_{1,0} + \alpha_{0,1}.$$

When $\alpha_{0,1}(x) = 0$ the subspace $\ker \alpha(x) \subset T_x M$ is complex. Still when $|\alpha_{0,1}(x)| < |\alpha_{1,0}(x)|$ the subspace $\ker \alpha(x)$ is symplectic.

It is very convenient to have almost complex structures adapted to a symplectic foliation.

Definition 2.1 *Given a symplectic manifold (M, ω) and a symplectic foliation α on M, we say that an almost complex structure J over M is compatible with the symplectic foliation if J is compatible with the symplectic structure and if $\alpha_{0,1} = 0$.*

Lemma 2.2 *Let $\mathcal{F}$ be a (holomorphically modeled) symplectic foliation with singular set $S \subset M$ on a symplectic four-manifold M. Then we can perturb $\mathcal{F}$ by a C^1-small isotopy in a neighborhood of S to a foliation that satisfies the following: for $p \in S$ there are holomorphic trivializations $\phi : U \subset M \to \mathbb{C}^n$ which are also Darboux charts, i.e., $\phi^* \omega_0 = \omega$ in U.*

Proof. Take an adapted chart at p, $\phi : U \subset M \to \mathbb{C}^n$ such that $\phi_*(\mathcal{F})$ is a holomorphic foliation. Then pull-back the symplectic structure of $\mathbb{C}^2$ to $U \subset M$. For a small enough U, we have that $\phi^* \omega_0$ and ω are arbitrarily close. Therefore Moser's trick produces an isotopy taking one of the symplectic structures to the other one. This gives a diffeomorphism ψ that can be extended to the whole of M and which is very close to the identity for small U. If now we consider the foliation $\psi_*(\mathcal{F})$ in M, then we have a small C^1-perturbation of $\mathcal{F}$ such that in a neighborhood of p it admits a Darboux chart which makes the foliation of holomorphic type. $\square$

Henceforth we shall suppose that the foliations we consider satisfy the conclusion of Lemma 2.2. Any result on a foliation which is invariant by symplectic isotopies continue to hold for the foliation before the perturbation.

Now we can prove the following basic existence result of adapted almost complex structures.

Proposition 2.3 *Given a symplectic manifold (M, ω) and a symplectic foliation α on it, there is a C^1-close symplectically isotopic foliation α' such that the set $\mathcal{J}_{\alpha'}$ of adapted almost complex structures is non-empty and contractible.*

Proof. Applying an C^1-small isotopy we may suppose that α satisfies the conclusion of Lemma 2.2. The condition $\alpha_{0,1} = 0$ is equivalent to impose that

$\ker \alpha(x)$ be complex for any regular point $x \in \hat{M} = M - S$ of the foliation. We know that, given a point $x \in M$, the set $\mathcal{J}(x)_\omega$ of complex structures in T_xM compatible with the symplectic form $\omega(x)$ is non-empty and contractible. This defines a bundle $\mathcal{J}_\omega$ over M.

For any point x in $\hat{M}$ we define $\mathcal{J}(x)_\alpha$ as the set of compatible complex structures in T_xM which preserve the complex structure in $K_x = \ker \alpha(x)$. If we denote by U_x the symplectic complement of K_x, it is easy to show that $\mathcal{J}(x)_\alpha = \mathcal{J}_{\omega,K_x} \times \mathcal{J}_{\omega,U_x}$ where $\mathcal{J}_{\omega,K_x}$ and $\mathcal{J}_{\omega,U_x}$ denote the set of compatible almost complex structures on K_x and U_x, respectively, with respect to the symplectic form ω. Therefore $\mathcal{J}(x)_\alpha$ is contractible. We denote by $\mathcal{J}_\alpha$ the bundle over $\hat{M}$ constructed with fibers $\mathcal{J}(x)_\alpha$.

Using Lemma 2.2 we can suppose that around any critical point $x \in S$ there are holomorphic trivializations such that the charts are of Darboux type. Therefore there is a neighborhood U_i of each point $x_i \in S$, with a preferred section $J_i : U_i \to (\mathcal{J}_\alpha)_{|U_i}$. Now we have a section J of the bundle $\mathcal{J}_\alpha$ defined over the union of the U_i. Recalling that for any point $x \in \hat{M}$ the fiber $\mathcal{J}_\alpha(x)$ is contractible we extend J to a global section that we still denote by the same symbol. This almost complex structure satisfies the needed properties.

We have shown the non-emptiness of the space. To show the contractibility we only need to fix an almost complex structure and to define a deformation from any other one to the fixed one. One starts to deform at the critical points and then extends the deformation all over the manifold. □

3 Bundles associated to a foliation

In the open submanifold $\hat{M} = M - S$, we can always define a bundle associated to the foliation as

$$T_{\mathcal{F}} = \{v \in T_xM : v \in \ker \alpha(x), x \in \hat{M}\},$$

which we call the tangent bundle to the foliation. An adapted almost complex structure, constructed using Proposition 2.3, proves that $T_{\mathcal{F}}$ is a complex line bundle over $\hat{M}$. Now recall that $T_{\mathcal{F}}$ is topologically fixed by $c_1(T_{\mathcal{F}}) \in H^2(\hat{M}; \mathbb{Z})$. Finally S has codimension four (it is a set of points) so we have a canonical isomorphism

$$H^2(M; \mathbb{Z}) \stackrel{\cong}{\to} H^2(\hat{M}, \mathbb{Z}).$$

So $c_1(T_{\mathcal{F}})$ can be understood as a class of $H^2(M; \mathbb{Z})$ canonically associated to a bundle, still denoted by $T_{\mathcal{F}}$, which is a topological extension of $T_{\mathcal{F}}$ to all of M. There is a natural map

$$T_{\mathcal{F}} \to TM$$

which is an injection over the open submanifold $\hat{M}$. Actually, this map is defined over $\hat{M}$ but may be changed by multiplication by non-zero complex valued smooth functions. Multiplying by a smooth function vanishing to infinite order at the critical points, it is not difficult to see that we may extend the map to all of M, in such a way that it is differentiable.

Fixing a compatible almost complex structure, the dual normal bundle is defined over $\hat{M}$ as

$$\mathcal{N}^*_{\mathcal{F}} = \{\gamma \in (T^*_x M)_{1,0} : T_{\mathcal{F},x} \subset \ker \gamma, x \in \hat{M}\}.$$

This is again a complex line bundle and so admits an extension to M satisfying that there is a natural map

$$(T^*M)_{1,0} \to \mathcal{N}^*_{\mathcal{F}},$$

which is a surjection over $\hat{M}$.

Taking the complex duals we can define the cotangent bundle and the normal bundle to the foliation $T^*_{\mathcal{F}}$, $\mathcal{N}_{\mathcal{F}}$. Finally,

Lemma 3.1 *We have that*

$$K_M = T^*_{\mathcal{F}} \otimes \mathcal{N}^*_{\mathcal{F}},$$

where K_M is the canonical bundle associated to the symplectic manifold.

Proof. Let us start with a regular point $x \in \hat{M}$. Given any $\phi \in (K_M)_x = (T^*_x M)_{1,0} \wedge (T^*_x M)_{1,0}$, contraction with ϕ defines a morphism $\phi : T_{\mathcal{F}} \to \mathcal{N}^*_{\mathcal{F}}$. So the contraction map yields an isomorphism from K_M to $\mathrm{Hom}(T_{\mathcal{F}}, \mathcal{N}^*_{\mathcal{F}}) \cong T^*_{\mathcal{F}} \otimes \mathcal{N}^*_{\mathcal{F}}$. This isomorphism is defined in $\hat{M}$, but again it canonically extends to M because S has codimension 4. □

Remark 3.2 *The definition of all the precedent bundles does not depend on the choice of almost complex structure, thanks to Proposition 2.3. Note moreover that $c_1(T_{\mathcal{F}})$ and $c_1(N_{\mathcal{F}})$ do not change with the small isotopies of Lemma 2.2, so they are well defined for any (holomorphically modeled) symplectic foliation $\mathcal{F}$.*

4 Tangencies between pseudoholomorphic curves and foliations

We recall the following result of McDuff

Proposition 4.1 (see [8]) *Given two pseudoholomorphic curves C_1, C_2 in a symplectic four-manifold intersecting at a point x. If the intersection is transverse then the intersection number is $+1$. If the intersection is not transverse, but the two curves do not have a common component, then the intersection number is a finite positive number.*

Recall that once we have fixed an adapted almost complex structure J, the leaves $\mathcal{F}$ of the foliation are pseudo-holomorphic curves. Given a J-holomorphic curve C such that $C \subset \hat{M}$ we define for each point $x \in C$ the number $I(x, C, \mathcal{F})$ as the intersection number of the curve and the leaf passing through x minus 1. This number is different from zero at most at a finite number of points of C. We define

$$\mathrm{tang}(C, \mathcal{F}) = \sum_{p \in C} I(p, C, \mathcal{F}),$$

which we will call the tangency number. From Proposition 4.1 this is a finite non-negative number, whenever C has no component contained in a leaf of the foliation.

If we fix a (possibly singular) J-holomorphic curve C we can compute its (virtual) Euler characteristic $\chi(C)$ by the adjunction formula

$$c_1(M, \omega) \cdot C = C \cdot C + \chi(C).$$

This coincides with the usual Euler characteristic for a smooth C.

We have the following formulas to compute the tangency number of a J-holomorphic curve.

Lemma 4.2 *Suppose that C is a J-holomorphic curve without any component inside a leaf of the foliation and not touching the singular set S. Then*

$$c_1(\mathcal{N}_{\mathcal{F}}) \cdot C = \chi(C) + \mathrm{tang}(C, \mathcal{F}),$$

$$c_1(T_{\mathcal{F}}) \cdot C = C \cdot C - \mathrm{tang}(C, \mathcal{F}).$$

Proof. The first and second formulas are equivalent by using Lemma 3.1 and the adjunction formula. To prove the first one, we note that the formula is purely homological and so a perturbation of C does not change the result. So we perturb C in order to make it smooth. To do this we simply observe that in any small neighborhood U we have that $C \bigcap U = \{f(x) = 0\}$, where f is a function $f : U \to \mathbb{C}$. And then $C_\epsilon = \{f(x) = \epsilon\}$ provides a way to perturb. We do not need to keep the perturbation holomorphic so it is very easy to globalize it. Moreover the perturbation can also assure that $I(x, C, \mathcal{F})$

be always $+1$, 0 or -1 (the sign can be minus since the perturbed curve may not be J-holomorphic).

We give the following interpretation to the number $\mathrm{tang}(C, \mathcal{F})$. Consider the bundle $\mathrm{Hom}_{\mathbb{C}}(TC, (\mathcal{N}_{\mathcal{F}})_{|C})$. We define a section ϕ of this bundle as follows. For any point $x \in C$ we have that T_xC, $\mathcal{N}_{\mathcal{F},x}$ and $T_{\mathcal{F},x}$ are complex subspaces of T_xM. For $v \in T_xC$ we define $\phi(v)$ as the intersection of the affine subspace $v + T_{\mathcal{F},x}$ with $\mathcal{N}_{\mathcal{F},x}$. If x is a tangency point, then T_xC and $T_{\mathcal{F},x}$ coincide and we define $\phi(v)$ as zero. This is equivalent to say that ϕ is the restriction to T_xC of the projection $T_xM \to \mathcal{N}_{\mathcal{F},x}$ along $T_{\mathcal{F},x}$.

Such ϕ is a smooth map whose zeroes correspond to the tangency points. After perturbing C we can assure that the zeros are simple (the perturbation makes that T_xC is not a complex subspace anymore, but all the objects will be homotopic to the initial ones). Now we have

$$\langle c_1(\mathrm{Hom}_{\mathbb{C}}(TC, (\mathcal{N}_{\mathcal{F}})_{|C}), [C]\rangle = Z(\phi),$$

where $Z(\phi)$ is the number of zeroes of the section ϕ, counted taking into account orientations. So $Z(\phi)$ coincides with $\mathrm{tang}(C, \mathcal{F})$. Now we get

$$-\chi(C) + c_1(\mathcal{N}_{\mathcal{F}}) \cdot C = \mathrm{tang}(C, \mathcal{F}).$$

□

Suppose now that we have a J-holomorphic curve C included in the foliation. The singular points of C are a subset of the singular points of the foliation. For each singular point p of C we define an index as follows. Let (U, ϕ) a holomorphic trivialization of $\mathcal{F}$ at p. Let $\{F = 0\}$ be the (holomorphic) equation locally defining C and let v be a vector field locally defining $\mathcal{F}$. The gradient of F gives a submersion $\{F = \varepsilon\} \to \{F = 0\}$ (which is a diffeomorphism off p). We lift the field v to $\{F = \varepsilon\}$ to get a field with zeroes over the preimage of p, which we perturb to get a generic vector field v'. We define $Z(p, C, \mathcal{F})$ to be the sum of the Poincaré-Hopf indices of the critical points of v' appearing near p in $\{F = \varepsilon\}$.

Again we define

$$Z(C, \mathcal{F}) = \sum_{p \in C \bigcap S} Z(p, C, \mathcal{F}).$$

We can show, analogously to the proof of Lemma 4.2 the following

Lemma 4.3 *Let C be a J-holomorphic curve included inside the foliation. Then*

$$c_1(\mathcal{N}_{\mathcal{F}}) \cdot C = C \cdot C + Z(C, \mathcal{F}),$$
$$c_1(T_{\mathcal{F}}) \cdot C = \chi(C) - Z(C, \mathcal{F}).$$

□

5 Baum-Bott formulas

Let $p \in M$ be a singular point of the foliation α. We may write $\mathcal{F}$, for a certain trivialization, in local coordinates as

$$v(z,w) = F(z,w)\frac{\partial}{\partial z} + G(z,w)\frac{\partial}{\partial w}. \tag{1}$$

Let $J(z,w)$ be the jacobian matrix of (F,G). We can consider the following indices

$$\mathrm{Det}(p,\mathcal{F}) = \mathrm{Res}_0 \left\{ \frac{\det J(z,w)}{FG} dz \wedge dw \right\},$$

$$\mathrm{Tr}(p,\mathcal{F}) = \mathrm{Res}_0 \left\{ \frac{(\mathrm{tr}\, J(z,w))^2}{FG} dz \wedge dw \right\}.$$

The number $\mathrm{Det}(p,\mathcal{F})$ is the multiplicity of the intersection of $\{F=0\}$ and $\{G=0\}$ and so it is clearly independent of the choice of holomorphic trivialization.

Let us interpret the second expression. In local coordinates we have

$$\alpha = B(z,w)dz - A(z,w)dw,$$

with A and B defined in a neighborhood U of $p=(0,0)$ and vanishing simultaneously at p. Set

$$\beta = f(z,w)\frac{A_z + B_w}{|A|^2+|B|^2}(\bar{A}dz + \bar{B}dw),$$

where f is a cut-off function being 0 in a small neighborhood of the origin and 1 outside a slightly bigger neighborhood $V \subset U$. Remark that

$$d\alpha = \beta \wedge \alpha, \quad \text{on} \ \ U - V. \tag{2}$$

Putting together the local constructions we find an open covering $\{U_j\}_{j\in I}$ of M, $(1,0)$-forms α_j and $(1,0)$-forms β_j satisfying (2) in $U_j - V_j$. We set $V_j = \emptyset$ in the case that U_j does not contain a singular point. Moreover we can assure that $U_j \bigcap V_i = \emptyset$.

We have that in the intersections $U_j \bigcap U_i$

$$\alpha_i = g_{ij}\alpha_j,$$

where $\{g_{ij}\}$ is the cocycle defining $\mathcal{N}_{\mathcal{F}}$. We have also

$$\beta_i \wedge \alpha_i = \left(\frac{dg_{ij}}{g_{ij}} + \beta_j\right) \wedge \alpha_i, \text{ i.e.,}$$

$$\left(\frac{dg_{ij}}{g_{ij}} + \beta_j - \beta_i\right) \wedge \alpha_i = 0.$$

Now $\{\frac{dg_{ij}}{g_{ij}} + \beta_j - \beta_i\}$ is a cocycle of smooth sections of $\mathcal{N}^*_{\mathcal{F}}$ (which is a fine sheaf). Hence it is possible to find γ_i such that

$$\gamma_j \wedge \alpha_j = 0, \quad \text{on } U_j$$

$$\frac{dg_{ij}}{g_{ij}} = \beta_i - \beta_j + \gamma_i - \gamma_j, \quad \text{on } U_j \cap U_i.$$

We have $d\alpha_j = (\beta_j + \gamma_j) \wedge \alpha_j$ on $U_j - V_j$. Moreover we can define the global closed 2-form

$$\Omega = \frac{1}{2\pi i} d(\beta_j + \gamma_j),$$

on U_j. It represents the first Chern class of $\mathcal{N}_{\mathcal{F}}$ in De Rham sense.

Now it is easy to check [5] that

$$\mathrm{Tr}(p, \mathcal{F}) = \frac{1}{(2\pi i)^2} \int_\Gamma \beta \wedge d\beta,$$

for any form β satisfying that $d\alpha = \beta \wedge \alpha$, where Γ is any small 3-sphere around the origin. This formula shows that $\mathrm{Tr}(p, \mathcal{F})$ is well defined independently of the particular holomorphic trivialization.

Define the global indices

$$\begin{aligned} \mathrm{Det}(\mathcal{F}) &= \sum_{p \in S} \mathrm{Det}(p, \mathcal{F}), \\ \mathrm{Tr}(\mathcal{F}) &= \sum_{p \in S} \mathrm{Tr}(p, \mathcal{F}). \end{aligned}$$

The Baum-Bott formulas are the natural extension of [1] to the case of symplectic foliations.

Theorem 5.1 *Let α be a (holomorphically modeled) symplectic foliation in a compact symplectic four-manifold, then*

$$\begin{aligned} \mathrm{Det}(\mathcal{F}) &= c_2(M) - c_1(T_{\mathcal{F}}) \cdot c_1(M) + c_1^2(T_{\mathcal{F}}), \\ \mathrm{Tr}(\mathcal{F}) &= c_1^2(M) - 2c_1(T_{\mathcal{F}}) \cdot c_1(M) + c_1^2(T_{\mathcal{F}}). \end{aligned}$$

Proof. The first formula does not depend on the integrability of the topological distribution associated to α. It is equivalent to

$$\mathrm{Det}(\mathcal{F}) = c_2(M) - c_1(T_{\mathcal{F}}) \cdot c_1(\mathcal{N}_{\mathcal{F}}) = c_2(TM) - c_2(T_{\mathcal{F}} \oplus \mathcal{N}_{\mathcal{F}}), \qquad (3)$$

since $c_1(T_{\mathcal{F}} \oplus \mathcal{N}_{\mathcal{F}}) = c_1(TM)$, because $T_{\mathcal{F}} \oplus \mathcal{N}_{\mathcal{F}}$ is isomorphic to TM off the singular set. To prove (3) we take a vector field of the form (1) in a neighborhood U_j of each critical point x_j. Then we extend this to a global (differentiable) vector field v of TM with transverse zeroes. In $M - \cup U_j$, we have that v is also a section of $T_{\mathcal{F}} \oplus \mathcal{N}_{\mathcal{F}}$. Since in $U_j - \{x_j\}$, v is a section of $T_{\mathcal{F}}$ and the line bundle $T_{\mathcal{F}}$ extends over the whole of M, we can modify v over each U_j to get a section of $T_{\mathcal{F}}$ with no zeroes in U_j. This gives a global section $\tilde{v}$ of $T_{\mathcal{F}} \oplus \mathcal{N}_{\mathcal{F}}$ which coincides with v off the union of U_j and has no zeroes inside the U_j's. Therefore $c_2(TM) - c_2(T_{\mathcal{F}} \oplus \mathcal{N}_{\mathcal{F}})$ is a sum of the order of vanishing $\mathrm{Det}(x_j, \mathcal{F})$ of v over each critical point x_j.

The second equality depends on the integrability and expresses a localization property of the first Chern number of the normal bundle $\mathcal{N}_{\mathcal{F}}$. To see this fact, note that this formula is equivalent to

$$\mathrm{Tr}(\mathcal{F}) = c_1^2(\mathcal{N}_{\mathcal{F}}).$$

To prove this, recall that the closed 2-form Ω represents $c_1(\mathcal{N}_{\mathcal{F}})$. Therefore

$$c_1(\mathcal{N}_{\mathcal{F}})^2 = \int_M \Omega \wedge \Omega.$$

By (2) we have

$$\Omega \wedge \Omega = 0,$$

outside V_j. Now, we use Stokes theorem to compute

$$\int_{V_j} \Omega \wedge \Omega = \int_{\partial V_j} (\beta_j + \gamma_j) \wedge (d\beta_j + d\gamma_j) = \mathrm{Tr}(p, \mathcal{F}).$$

This concludes the proof. $\square$

6 Topological restrictions for the existence of regular symplectic foliations

A regular symplectic foliation is one that satisfies that the singular set is $S = \emptyset$. We will show in this Section some obstructions for the existence of regular symplectic foliations in certain classes of symplectic manifolds.

A symplectic Lefschetz fibration of genus g is a map $f : M \to S$ to a Riemann surface S such that there are only a finite number of critical points x_i around each of which there are adapted charts (z, w) where $f = zw$, the fibers are smooth symplectic surfaces off the critical points and the generic fiber is a connected genus g surface.

Lemma 6.1 *Let M be a symplectic manifold admitting a rational or elliptic Lefschetz fibration over a Riemann surface. Then $c_1^2(M) \leq c_2(M)$. Moreover if the fibration does not have singular fibers then $c_1^2(M) = c_2(M)$.*

Proof. First we do the rational case, i.e., the case of a symplectic Lefschetz fibration of genus $g = 0$. Suppose that F is a singular fiber (a fiber containing some of the critical points). By the local model, F is a complex curve with nodes. Therefore $F = C_1 \cup \cdots \cup C_r$, for some smooth complex curves C_i meeting transversely. Since there is a retraction from the general fiber S^2 to F, we have that F is simply connected. Therefore the graph determined by the C_i is a tree. Thus we may suppose that $C_1 \cdot C_2 = 1$ and $C_1 \cdot C_i = 0$, $i > 2$. Then

$$0 = C_1 \cdot (C_1 + C_2 + \cdots + C_r) = C_1^2 + 1.$$

So there is a (-1)-symplectic sphere contained in a fiber that we may blow down. We continue the process until we get a ruled surface, for which $c_1^2(M) = c_2(M)$. To conclude one needs only to observe that a blow up makes c_1^2 decrease and c_2 increase (strictly).

Now consider the elliptic case, i.e., a symplectic Lefschetz fibration of genus $g = 1$. Let $F = C_1 \cup \cdots \cup C_r$ be a singular fiber. If there is a component, say C_1 which is a torus, then $\pi_1(\mathbb{T}^2) \cong \pi_1(C_1)$ and the other components form a tree from which we find a (-1)-sphere to blow down as above. If there is a component C_1 which is a nodal rational curve, then $\pi_1(\mathbb{T}^2) \twoheadrightarrow \pi_1(C_1)$ and the other components cannot form a cycle, and we find a (-1)-sphere. Finally if all the curves C_i are rational and smooth and there is a cycle, say $C_1, \ldots, C_s$, then $s = r$ or else we find a (-1)-sphere. We get $C_i^2 = -2$ and hence the curve has $\chi(F) = s$. This implies that the number $\chi(M) = c_2(M)$ equals the number of critical points.

On the other hand, for a fiber as above, $c_1(M) \cdot C_i = C_i^2 + \chi(C_i) = -2 + 2 = 0$. Also $c_1(M) \cdot F' = 0$ for a regular fiber. The remainder of the homology is obtained by a section of the fibration, rim tori (tori which map to a circle) and (-2)-spheres which are obtained by gluing two vanishing (-1)-discs. Tori of self-intersection zero and (-2)-spheres have zero intersection with $c_1(M)$. Therefore $c_1(M)$ is a multiple of the fiber class and thus $c_1(M)^2 = 0$. $\square$

Now we give a topological restriction for the existence of a regular symplectic foliation. This can be combined with Lemma 6.1 to obtain restrictions on the possible four-manifolds admitting regular foliations. We need to use the following result.

Theorem 6.2 ([11], [6] Thm 10.1.15) *Suppose that (M, ω) is a symplectic manifold with $b_2^+(M) > 1$ and $SW_M(K) \neq 0$ for a given $K \in \mathcal{C}_M$. Assume* *furthermore that the class $c = \frac{1}{2}(K - c_1(M,\omega))$ is nonzero in $H^2(X,\mathbb{Z})$. Then for a generic compatible almost-complex structure J on M, the class $PD(c) \in H^2(M,\mathbb{Z})$ can be represented by a pseudo-holomorphic manifold.* □

Theorem 6.3 *Let $\mathcal{F}$ be a regular foliation over a symplectic manifold such that $b_2^+(M) > 1$, $c_1(M,\omega) + 2c_1(T^*_{\mathcal{F}})$ is a Seiberg-Witten basic class and $c_1(T^*_{\mathcal{F}}) \neq 0$. Then $c_1^2(M) \geq 2c_2(M)$.*

Proof. The class $c_1(T^*_{\mathcal{F}})$ satisfies the hypothesis of Theorem 6.2. The only problem arises with the genericity of the adapted almost-complex structure. Let us suppose for a moment that J is generic. Then Theorem 6.2 says that there is a pseudo-holomorphic curve S that represents the dual class of $c_1(T^*_{\mathcal{F}})$. As $\mathcal{F}$ is a regular foliation, obviously S does not pass through any critical point. The curve is a sum of irreducible components $S = \sum m_j S_j$ (c.f. [8]). By Theorem 5.1 the inequality that we are proving is equivalent to $c_1^2(T^*_{\mathcal{F}}) \geq 0$. We just need to prove that $c_1(T^*_{\mathcal{F}}) \cdot S_j \geq 0$ for all j. If we prove that $S_j \cdot S_j \geq 0$ we are done because, by Proposition 4.1, we have that $S_i \cdot S_j \geq 0$ if $i \neq j$.

Suppose that $S_j \cdot S_j < 0$, then S_j cannot be contained in the foliation because its self-intersection would be zero. So we apply Lemma 4.2 to obtain

$$c_1(T^*_{\mathcal{F}}) \cdot S_j = -S_j \cdot S_j + \mathrm{tang}(S_j, \mathcal{F}) > 0.$$

Now if J is not generic, then recall that what we need is smoothness in the associated moduli of pseudo-holomorphic curves. This smoothness can be obtained by a more restrictive perturbation given by Lemma 6.4 below. Certainly we only obtain smoothness for the moduli spaces of pseudo-holomorphic curves before compactifying them, but this is enough to develop the Taubes' formalism and conclude the proof. □

Lemma 6.4 *Given a symplectic four-manifold (M,ω) and a regular symplectic distribution on it. Fix a compatible almost complex structure J on M. Then, for an arbitrary C^∞-close to the identity isotopy of the symplectic foliation and a C^∞-small change of the compatible almost-complex structure, the moduli spaces of pseudo-holomorphic curves associated to the manifold are smooth and of the expected dimension.*

Proof. Following notations and ideas of [9] we want to prove that the space of maps of Riemann surfaces of fixed genus and complex structure with an image in a fixed homology class in M which are J-holomorphic is smooth. In the definition we assume implicitly that a curve in the moduli is simple, i.e., it is not a branched covering over an intermediate curve. If this does not hold then we cannot have smoothness, but in this case the method of [7] can be used to get an orbifold structure). Nonetheless, for the clarity of exposition we will restrict ourselves to the case of simple curves.

We use suitable Sobolev completions so that all the spaces we are working with are endowed with a structure of Banach manifold. We denote by $\mathcal{X}$ the space of all the maps of the fixed Riemann surface Σ to M in the given homology class. We denote by $\mathcal{J}$ the space of almost complex structures on M which are compatible with respect to some distribution $\phi_*(\mathcal{F})$, for $\mathcal{F}$ the given symplectic foliation and ϕ a diffeomorphism close to the identity. Over $\mathcal{X} \times \mathcal{J}$ we define a bundle $\mathcal{E}$ whose fiber at a point (u, J) is

$$\mathcal{E}_{(u,J)} = \Gamma(\Lambda^{0,1}T^*\Sigma \otimes_J u^*TM).$$

Given a point $(u, J) \in \mathcal{X} \times \mathcal{J}$, the operator $\bar{\partial} u$ defines a section of $\mathcal{E}$. So we have a map $\mathcal{F} : \mathcal{X} \times \mathcal{J} \to \mathcal{E}$. For fixed J, the zeros of this section give the moduli of pseudo-holomorphic curves. So to have smoothness we need the surjectivity of the differential of $\mathcal{F}$ at the zeroes

$$D\mathcal{F}(u, J) : \Gamma(u^*TM) \times T\mathcal{J} \to \Gamma(\Lambda^{0,1}T^*\Sigma \otimes_J u^*TM)$$

whenever the curve is simple. In this case, the zeroes of $\mathcal{F}$ are a smooth Banach manifold and by Smale-Sard-Bertini theorem the moduli space is smooth for a generic choice $J \in \mathcal{J}$.

Following the proof of Proposition 3.4.1 in [9], we only need to get a generic perturbation at an injective point of the image of the curve u. This is because the obstruction to be surjective is measured by a function η (see [9]) which is a solution of a laplacian. Therefore the obstruction is globally zero if it is zero in a small neighborhood of a point (Aronszajn's theorem). But now, for a given point $x \in M$ the space $\mathcal{J}$ generates all the possible local perturbations of the almost complex structure as in [9]. Given a point $x \in M$, in [9] the authors use an element $Y \in T_J\mathcal{J}_\omega$ to show that the obstruction to the surjectivity η is zero at x. In our case, we choose an element $Y' \in T_J\mathcal{J}$ such that $Y(u(x)) = Y'(u(x))$ (and keeping the condition that Y and Y' are very concentrated near x so that there is no effect from points far from x). This is enough to get that η is zero. $\square$

Very likely, more results in the lines of Theorem 6.3 can be obtained, which may allow to mimic the classification results of Brunella [2] for regular symplectic foliations on four-manifolds.

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