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Formality and Symplectic Geometry

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1. Introduction

Problem: Find topological properties of a manifold with a particular geometric structure (e.g. complex structure, Riemannian structure with prescribed holonomy group, ...)

This would help on the classification problem: Given a smooth manifold X, we may have topological obstructions for X to admit a particular geometric structure. This can be useful to know when a manifold admits some geometric structure.

Geometric structures we shall focus on:

We consider smooth (oriented) compact manifold M. We are interested in whether M admits either of the following structures:

- Kähler (and complex projective).
- Symplectic.

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2. Kähler vs. symplectic

2.1. Complex projective manifolds

Let $\mathbb{CP}^n = \{(z_0, \ldots, z_n); z_i \in \mathbb{C}\}/\mathbb{C}^*$ be the complex projective space.

A (smooth) complex projective manifold is a smooth submanifold $X \subset \mathbb{CP}^n$ which is described as the zeroes of some complex polinomials $F_1, \ldots, F_n$ in the variables $(z_0, \ldots, z_n)$.

$\mathbb{CP}^n$ has a natural metric, the Fubini-Study metric. In coordinates $z' = (z_1, \ldots, z_n)$ for the open subset $U = \{z_0 \neq 0\} \subset \mathbb{CP}^n$,

$$g = \mathrm{Re}\left(\frac{\sum (1 + \|z'\|^2) dz_i \cdot d\bar{z}_i - \sum \bar{z}_i z_j dz_i \cdot d\bar{z}_j}{(1 + \|z'\|^2)^2}\right).$$

Consider the 2-form $\omega \in \Omega^2(\mathbb{CP}^n)$ given as

$$\omega = \frac{\sqrt{-1}}{2}\left(\frac{\sum (1 + \|z'\|^2) dz_i \wedge d\bar{z}_i - \sum \bar{z}_i z_j dz_i \wedge d\bar{z}_j}{(1 + \|z'\|^2)^2}\right).$$

These satisfy:

- ω determines (and is determined) by g, through $g(u,v)=\omega(u,Jv)$, where J is the complex structure.
- $d\omega=0$, so $[\omega]\in H^2(X,\mathbb{R})$.
- $\omega^n=\mathrm{vol}>0$ in $\Omega^{2n}(\mathbb{CP}^n)$.

The complex projective manifold $X\subset\mathbb{CP}^n$ inherits (J,g,ω) from $\mathbb{CP}^n$. This structure determines X completely.

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2.2. Kähler manifolds

A Kähler manifold (X,J,g,ω) is a manifold X endowed with:
- J a complex structure (i.e. a complex atlas),
- g a J-invariant Riemannian metric, i.e. $g(\cdot\,,\cdot)=g(J(\cdot),J(\cdot))$,
- $\omega\in\Omega^2(X)$, $g(\cdot\,,\cdot)=\omega(\cdot\,,J(\cdot))$,
- and satisfying $d\omega=0$ (automatically $\omega^n=\mathrm{vol}>0$).

Theorem (Kodaira, 1954):

$$X \text{ complex projective } \iff X \text{ Kähler with } [\omega]\in H^2(X,\mathbb{Z}).$$

2.3. Symplectic manifolds

A symplectic manifold (M, ω) is a manifold M with a 2-form $\omega \in \Omega^2(M)$ satisfying:

- $d\omega = 0$,
- $\omega^n = \text{vol} > 0$.

So we do not require the existence of a complex structure.

We may always put an *almost-complex* structure I. This makes the tangent bundle TM into a complex bundle (weaker than having a complex atlas). It allows to have a Riemannian metric g associated to ω as before.

3. Topological properties

Problem: Construct symplectic manifolds which do not admit Kähler structures.

3.1. Topological properties of Kähler manifolds

Kähler (compact) manifolds satisfy many topological restrictions:

(i) The Betti numbers $b_1, b_3, b_5, \ldots$ are even.

(ii) Hard-Lefschetz theorem (Lefschetz): Let $\dim M = 2n$. For any $k = 0, 1, \ldots, n-1$, the map

$$[\omega]^k \cup \ : H^k(M) \longrightarrow H^{2n-k}(M)$$

is an isomorphism. (This implies (i))

(iii) The fundamental group $\pi_1(M)$ is of a particular type (what is known as a *Kähler group*).

(iv) Kähler manifolds are *formal* (Deligne-Griffiths-Morgan-Sullivan, 1975).

3.2. Topological properties of symplectic manifolds

Do symplectic manifolds satisfy the same topological properties?

- Kodaira-Thurston manifold (Thurston, 1976). This is a symplectic 4-manifold with $b_1 = 3$.
  McDuff (1984) constructed a simply-connected symplectic manifold with $b_3 = 3$.
- Gompf (1995) constructed symplectic manifolds with $\pi_1(M)$ isomorphic to any given (presentable) group.
- There exist symplectic manifolds not satisfying hard-Lefschetz (e.g. Kodaira-Thurston). There are examples with b_i even and with prescribed fundamental group.
- Also there are non-formal symplectic manifolds:
  - Kodaira-Thurston is non-formal.
  - Babenko-Taimanov (1998) found the first simply-connected example: McDuff's manifold is non-formal. These manifolds have dimension at least 10.
  - Cavalcanti (2004) gave the first example satisfying hard-Lefschetz.
  - Fernández-Muñoz (2005) constructed the first simply-connected example of dimension 8 (the lowest possible dimension).

One should not be misled. There are striking parallelisms between symplectic and Kähler manifolds:

- Theory of pseudo-holomorphic curves, Gromov-Witten invariants, Quantum cohomology (Gromov, 1985, and others).
- Asymptotically holomorphic techniques, Lefschetz pencils (Donaldson, 1996, 1999, and others).

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Kähler $\leftrightarrow$ Symplectic:

Similarities at analytical level

Differences at topological level

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4. The first example

The Kodaira-Thurston manifold was the first example of a symplectic manifold not admitting a Kähler structure.

Let H be the Heisenberg group,

$$H = \left\{ \begin{pmatrix} 1 & b & c \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} ; \; a, b, c \in \mathbb{R} \right\},$$

$$\Gamma = \left\{ \begin{pmatrix} 1 & b & c \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} ; \; a, b, c \in \mathbb{Z} \right\},$$

and $E = \Gamma \backslash H$.

Note that E is the total space of a S^1-bundle over the 2-torus with Chern class 1.

$$\begin{array}{rcl} E & \rightarrow & T^2 \\ [a, b, c] & \mapsto & [a, b]\,. \end{array}$$

A basis for the left invariant 1-forms on E is given by:

$$\alpha = da, \; \beta = db, \; \gamma = dc - b\, da.$$

Note that $\alpha \wedge \beta = d\gamma$.

The Kodaira-Thurston manifold is

$$KT = E \times S^1 \, .$$

Let $\eta = d\theta$ the standard 1-form comming from the S^1-factor. Then the cohomology of KT is

$$\begin{aligned} H^0(KT) &= \langle 1 \rangle, \\ H^1(KT) &= \langle [\alpha], [\beta], [\eta] \rangle, \\ H^2(KT) &= \langle [\alpha \wedge \gamma], [\beta \wedge \gamma], [\alpha \wedge \eta], [\beta \wedge \eta] \rangle, \\ H^3(KT) &= \langle [\alpha \wedge \gamma \wedge \eta], [\beta \wedge \gamma \wedge \eta], [\alpha \wedge \beta \wedge \gamma] \rangle, \\ H^4(KT) &= \langle [\alpha \wedge \beta \wedge \gamma \wedge \eta] \rangle. \end{aligned}$$

KT is sympletic, with symplectic form

$$\omega = \beta \wedge \gamma + \alpha \wedge \eta \, .$$

Clearly, $d\omega = 0$ and $\omega^2 = 2\,\alpha \wedge \beta \wedge \gamma \wedge \eta > 0$.

$b_1(KT) = 3 \implies KT$ does not admit a Kähler structure.

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5. Minimal models (Sullivan, 1977)

Rational homotopy deals with spaces up rational homotopy equivalence, in particular, with

- Rational homotopy groups: $\pi_n(X) \otimes \mathbb{Q}$.
- Rational (co)homology: $H_n(X, \mathbb{Q})$, $H^n(X, \mathbb{Q})$.

(Actually, $\mathbb{Q}$ may be replaced by the field of real numbers $\mathbb{R}$ with no harm.)

If X is a smooth manifold, we consider the differential forms

$$(\Omega X, d) .$$

This is a graded-commutative differential algebra (GCDA for short). We extract an "invariant" from it as follows.

Consider the equivalence relation $\sim$ between GCDAs generated by quasi-isomorphisms, $\psi : (A_1, d_1) \longrightarrow (A_2, d_2)$, i.e. morphisms inducing isomorphisms

$$\psi : H(A_1, d_1) \xrightarrow{\cong} H(A_2, d_2).$$

Then associate to $(\Omega X, d)$ its class in $(\text{GCDAs}/\sim)$.

The theory of minimal models tells us that this codifies the rational homotopy type of X in most cases.
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The good news is that there is a canonical representative, called the minimal model, for any (A, d). The minimal model $(\mathcal{M}, d)$ satisfies:

- $\mathcal{M} = \bigwedge(x_1, x_2, \dots)$ is free.
  $\bigwedge$ means the "graded-commutative algebra freely generated by"
- $dx_i \in \bigwedge(x_1, \dots, x_{i-1})$.
- dx_i contains no linear term.
- $(\mathcal{M}, d) \longrightarrow (A, d)$ is a quasi-isomorphism.

A minimal model $(\mathcal{M}_X, d)$ for X is a minimal model for $(\Omega X, d)$.

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Example

The minimal model of the Kodaira-Thurston manifold is the following:

$$\begin{array}{rcl} \psi : (\mathcal{M}_{KT}, d) = (\bigwedge(x,y,z,u), d) & \stackrel{\sim}{\longrightarrow} & (\Omega(KT)_L, d) \subset (\Omega(KT), d) \\ x & \mapsto & \alpha \\ y & \mapsto & \beta \\ z & \mapsto & \gamma \\ u & \mapsto & \eta \end{array}$$

where $dz = x\,y$.

Clearly, $(\mathcal{M}_{KT}, d)$ is a minimal algebra, ψ is a CDGA map, and it is a quasi-isomorphism, since the cohomology of $(\mathcal{M}_{KT}, d)$ is

$$\begin{aligned} H^0(\mathcal{M}_{KT}, d) &= \langle 1 \rangle, \\ H^1(\mathcal{M}_{KT}, d) &= \langle [x], [y], [u] \rangle, \\ H^2(\mathcal{M}_{KT}, d) &= \langle [x\,z], [y\,z], [x\,u], [y\,u] \rangle, \\ H^3(\mathcal{M}_{KT}, d) &= \langle [x\,z\,u], [y\,z\,u], [x\,y\,z] \rangle, \\ H^4(\mathcal{M}_{KT}, d) &= \langle [x\,y\,z\,u] \rangle. \end{aligned}$$

So $(\mathcal{M}_{KT}, d)$ is *the* minimal model.

Theorem (S): If either:

- X is simply-connected,
- X is nilpotent:
  1. $\Gamma = \pi_1(X)$ is nilpotent: $\Gamma_1 = \Gamma$, $\Gamma_n = [\Gamma_{n-1}, \Gamma]$, $n \geq 1$, then $\exists n_0$ such that $\Gamma_{n_0} = 0$,
  2. Γ acts nilpotently on each $\pi_k(X)$: $G_{k,1} = \pi_k(X)$, $G_{k,n} = [\Gamma, G_{k,n-1}] \subset \pi_k(X)$, $n \geq 1$, then $\exists n_0$ such that $G_{k,n_0} = 0$,

then the minimal model

$$(\mathcal{M}_X, d) \longrightarrow (\Omega X, d)$$

codifies the rational homotopy of X. More specifically, $\mathcal{M}_X = \bigwedge(V)$,

$$V = \bigoplus_{n \geq 1} V^n$$

(V^n is the vector space corresponding to the degree n generators), then

$$V^n \cong (\pi_n(X) \otimes \mathbb{R})^* .$$

6. Formality

6.1. Definition

A CDGA (A,d) is formal if $(A,d) \sim (H,0)$.

Obviously $H = H(A,d)$. Explicitly,

$$\begin{array}{ccc} & (\mathcal{M},d) & \\ \swarrow & & \searrow \\ (A,d) & & (H,0) \end{array}$$

So the minimal model can be deduced from $H = H(A,d)$.

All the information is in the cohomology algebra.

A space X is formal if $(\Omega X, d)$ is formal.

Theorem (DGMS): Kähler manifolds are formal.

Example

The Kodaira-Thurston manifold $KT = E \times S^1$ is non-formal.

It is enough to see that E is non-formal. Let's try to construct a quasi-isomorphism $\psi : (\mathcal{M}_E, d) \longrightarrow (H^*(E), 0)$.

- ψ must un algebra map.
- ψ must commute with the differentials.
- If $a \in \mathcal{M}_E$ is closed, $\psi(a) = [a]$.

Recall $(\mathcal{M}_E, d) = (\bigwedge(x, y, z), d)$ with $dz = x\, y$.

$$\begin{array}{rcll} (\bigwedge(x,y,z),d) & \longrightarrow & (H^*(E),0) & \\ x & \mapsto & [x] & \\ y & \mapsto & [y] & \\ x\,y & \mapsto & \psi(x)\cup\psi(y) = [x]\cup[y] = 0, & \quad x\cdot y = dz \\ z & \mapsto & \psi(z) = a[x] + b[y], & \quad \text{for some } a, b \in \mathbb{R} \\ x\,z & \mapsto & \psi(x)\cup\psi(z) = [x]\cup(a[x]+b[y]) = 0 & \\ & & \psi(x\,z) = [x\,z] \neq 0 \in H^2(E) & \end{array}$$

Contradiction! So (KT, ω) is symplectic and non-formal.

6.2. Massey products

There is a quick way to check non-formality (it often works, but not always).

Let $a_1, a_2, a_3 \in H^*(X)$ be cohomology classes such that $a_1 \cup a_2 = 0$ and $a_2 \cup a_3 = 0$. Take forms α_i in X with $a_i = [\alpha_i]$ and write

$$\alpha_1 \wedge \alpha_2 = d\xi, \ \alpha_2 \wedge \alpha_3 = d\zeta \,.$$

Then

$$d(\alpha_1 \wedge \zeta - (-1)^{\deg(a_1)} \xi \wedge \alpha_3) = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 - \alpha_1 \wedge \alpha_2 \wedge \alpha_3 = 0 \,.$$

The Massey product of the classes a_i is defined as

$$\langle a_1, a_2, a_3 \rangle = [\alpha_1 \wedge \zeta - (-1)^{\deg(a_1)} \xi \wedge \alpha_3] \in H^*(X)/\text{choices} \,.$$

If $\langle a_1, a_2, a_3 \rangle \neq 0$ then X is non-formal. (Basically, the Massey products can be transferred through quasi-isomorphisms, and in $(H, 0)$ they are automatically vanishing.)

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Example

For the Kodaira-Thurston manifold, take $a_1 = [\alpha]$, $a_2 = [\alpha]$, $a_3 = [\beta]$.

Let $\alpha_1 = \alpha, \alpha_2 = \alpha, \alpha_3 = \beta$, so

$$\alpha_1 \wedge \alpha_1 = 0 \implies \xi = 0,$$

$$\alpha_1 \wedge \alpha_2 = d\gamma \implies \zeta = \gamma.$$

The Massey product is

$$\langle [\alpha], [\alpha], [\beta] \rangle = [\alpha \wedge \gamma] \neq 0\,.$$

This confirmes again that KT is non-formal (without having to compute the minimal model).

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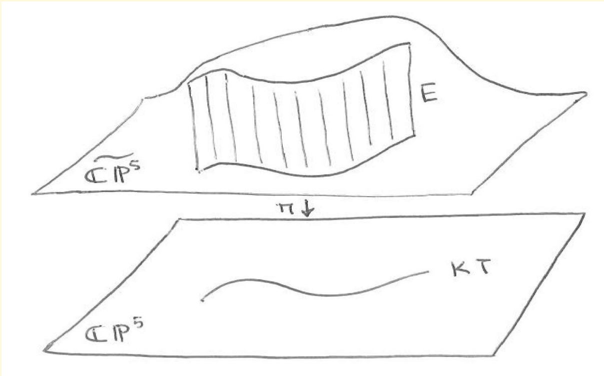
7. Non-formal simply-connected symplectic manifolds

7.1. First example (Babenko-Taimanov, 1998)

Embed $KT \subset \mathbb{CP}^5$ symplectically and consider the symplectic blow-up (Gromov, McDuff, 1984)

$$\pi : M = \widetilde{\mathbb{CP}^5} \to \mathbb{CP}^5$$



By analogy with the Kähler situation, $E = \pi^{-1}(KT)$ is called the exceptional divisor.

$$E = \mathbb{P}_{\mathbb{C}}(\nu_{KT})$$

where ν_{KT} is the normal bundle (with a complex structure, coming from a suitable almost-complex structure on the ambient manifold).

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Let $u \in \Omega^2(M)$ be a Thom form for E, i.e. $[u] = \text{P.D.}[E]$ and u is supported in a neighborhood of E.

Take $\alpha, \beta, \gamma \in \Omega^1(KT)$ as before.

There are 3-forms $\alpha \wedge u, \beta \wedge u, \gamma \wedge u \in \Omega^3(M)$.

The following Massey product:

$$\langle [\alpha \wedge u], [\alpha \wedge u], [\beta \wedge u] \rangle = [(\alpha \wedge u) \wedge (\gamma \wedge u^2)] \in H^8(M)/\text{choices}$$

is non-zero.

Therefore M is a simply-connected, symplectic and non-formal manifold.

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We need $\operatorname{rank}_{\mathbb{C}}(\nu_{KT}) \geq 3$, so

$$\dim M \geq 6 + \dim KT \geq 10.$$

7.2. Example of dimension 8 (Fernández-Muñoz, 2005)

Start with a non-simply-connected 8-dimensional symplectic manifold:

Take a lattice $\Lambda \subset \mathbb{C}$, so that $T = \mathbb{C}/\Lambda$ is a 2-torus. Note that $H^1(\mathbb{C}/\Lambda) = \langle x_1, x_2 \rangle \cong \mathbb{C}$. Let $E_{\mathbb{C}}$ be a complex version of E (constructed starting with the complex Heisenberg group), so that

$$T \to E_{\mathbb{C}} \to T \times T$$

is a non-trivial fiber bundle,

$$\mathcal{M}_{E_{\mathbb{C}}} = \bigwedge(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2)$$

and setting $\alpha = \alpha_1 + \sqrt{-1}\alpha_2$, $\beta = \beta_1 + \sqrt{-1}\beta_2$, $\gamma = \gamma_1 + \sqrt{-1}\gamma_2$, we have $d\gamma = \alpha \wedge \beta$. Then consider

$$X = E_{\mathbb{C}} \times T\,,$$

with $\mathcal{M}_X = \bigwedge(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \eta_1, \eta_2)$.

The 8-manifold X is symplectic choosing

$$\omega = \sqrt{-1}\,\alpha \wedge \bar{\alpha} + \beta \wedge \gamma + \bar{\beta} \wedge \bar{\gamma} + \sqrt{-1}\,\eta \wedge \bar{\eta}\,.$$

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Choosing $\Lambda \subset \mathbb{C}$ to be the lattice generated by 1 and $\zeta = e^{2\pi i/3}$, there is a group $\mathbb{Z}_3$ by rotations on X as

$$(a, b, c, d) \mapsto (\zeta a, \zeta b, \zeta^2 c, \zeta d).$$

The symplectic orbifold $\widehat{X} = X/\mathbb{Z}_3$ is simply-connected. It is easy to see what happens to the degree 1 cohomology:

$$H^1(X) \cong \langle \alpha_1, \alpha_2, \beta_1, \beta_2, \eta_1, \eta_2 \rangle \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}\,,$$

and $\mathbb{Z}_3$ acts by rotations, so

$$H^1(\widehat{X}) = H^1(X)^{\mathbb{Z}_3} = 0.$$

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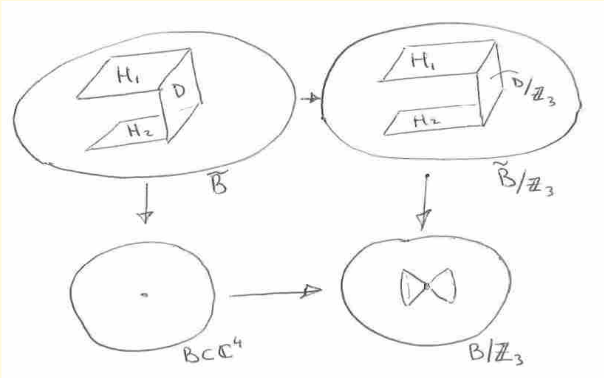
There is a symplectic resolution of singularities (FM):



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Then $\widetilde{X}$ is simply-connected, symplectic, of dimension 8 and ...

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Then $\widetilde{X}$ is simply-connected, symplectic, of dimension 8 and ...

$\widetilde{X}$ is non-formal.

We can check non-formality for $\widehat{X}$. But unfortunately we can't use Massey products.

The following is an useful substitute for Massey products which works in the current situation:

Let $a, x_1, x_2, x_3 \in H^2(M)$ be cohomology classes satisfying $a \cup x_i = 0$, $i = 1, 2, 3$. Choose forms $\alpha, \beta_i \in \Omega^2(M)$ and $\xi_i \in \Omega^3(M)$, with $a = [\alpha]$, $x_i = [\beta_i]$ and $\alpha \wedge \beta_i = d\xi_i$, $i = 1, 2, 3$. If the cohomology class

$$[\xi_1 \wedge \xi_2 \wedge \beta_3 + \xi_2 \wedge \xi_3 \wedge \beta_1 + \xi_3 \wedge \xi_1 \wedge \beta_2] \in H^8(M)/\text{choices}$$

is non-zero, then M is non-formal.

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