Equivariant motive of the $\mathrm{SL}(3,\mathbb{C})$-character variety of torus knots

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ABSTRACT

Let Γ be the fundamental group of the complement of the torus knot of type (m,n). This has a presentation $\Gamma = \langle x, y \,|\, x^m = y^n \rangle$. Using the geometric description of the character variety $X(\Gamma, G)$ of characters of representations of Γ into $G = \mathrm{SL}(3,\mathbb{C})$, we determine explicitly its associated μ_3-equivariant motive.

Dedicated to José María Montesinos Amilibia, with our deepest admiration.

1. Introduction

Let Γ be a finitely presented group, and let $G = \mathrm{SL}(r,\mathbb{C})$. A *representation* of Γ in G is a homomorphism $\rho : \Gamma \to G$. Consider a presentation $\Gamma = \langle x_1, \ldots, x_k | r_1, \ldots, r_s \rangle$. Then ρ is completely determined by the k-tuple $(A_1, \ldots, A_k) = (\rho(x_1), \ldots, \rho(x_k))$ subject to the relations $r_j(A_1, \ldots, A_k) = \mathrm{Id}$, $1 \le j \le s$. The space of representations is

$$\begin{aligned} R(\Gamma, G) &= \mathrm{Hom}(\Gamma, G) \\ &= \{(A_1, \ldots, A_k) \in G^k \,|\, r_j(A_1, \ldots, A_k) = \mathrm{Id},\, 1 \le j \le s\} \subset G^k . \end{aligned}$$

Therefore $R(\Gamma, G)$ is an affine algebraic set.

We say that two representations ρ and ρ' are equivalent if there exists $P \in G$ such that $\rho'(g) = P^{-1}\rho(g)P$, for every $g \in G$. The moduli space of representations is defined as the GIT quotient

$$M(\Gamma, G) = R(\Gamma, G)//G .$$

Recall that by definition of GIT quotient for an affine variety, if we write $R(\Gamma, G) = \mathrm{Spec}\, A$, then $M(\Gamma, G) = \mathrm{Spec}\, A^G$. For a representation $\rho : \Gamma \to G$, we define its *character* as the map $\chi_\rho : \Gamma \to \mathbb{C}$, $\chi_\rho(g) = \mathrm{tr}\, \rho(g)$. Note that two equivalent representations ρ and

ρ' have the same character. There is a character map $\chi : R(\Gamma, G) \to \mathbb{C}^\Gamma$, $\rho \mapsto \chi_\rho$, whose image

$$X(\Gamma, G) = \chi(R(\Gamma, G))$$

is called the *character variety of* Γ. The traces χ_ρ span a subring $B \subset A^G$, and $X(\Gamma, G) = \operatorname{Spec} B$. Actually, for $G = \mathrm{SL}(r, \mathbb{C})$, the ring of invariant polynomials is generated by characters (see Chapter 1 in [12]), so the natural algebraic map

$$M(\Gamma, G) \to X(\Gamma, G)$$

is an isomorphism.

The character varieties for $\mathrm{SL}(2, \mathbb{C})$ have been extensively studied in the last three decades [3, 4, 12]. Given a manifold M, the moduli of representations of $\pi_1(M)$ into $\mathrm{SL}(2, \mathbb{C})$ contain information of the topology of M. This is specially relevant for 3-dimensional manifolds [3], where the fundamental group and the geometrical properties of the manifold are strongly related. This has been used to study knots $K \subset S^3$, by analysing the $\mathrm{SL}(2, \mathbb{C})$-character variety of the fundamental group of the knot complement $S^3 - K$ (these are called *knot groups*). The case of $\mathrm{SL}(2, \mathbb{C})$-representations of the fundamental group of a surface has also been extensively analysed [5, 7, 11, 15], in this situation focusing more on geometrical properties of the moduli space in itself (cf. non-abelian Hodge theory).

However, much less is known of the character varieties for other groups, notably for $\mathrm{SL}(r, \mathbb{C})$ with $r \geq 3$. The character varieties for $\mathrm{SL}(3, \mathbb{C})$ for free groups have been described in [9, 10]. In the case of 3-manifolds, little has been done. For knot groups, the first case to analyse is clearly that of torus knots. These are defined as follows. Let $T^2 = S^1 \times S^1$ be the 2-torus and consider the standard embedding $T^2 \subset S^3$. Let m, n be a pair of coprime positive integers. Identifying T^2 with the quotient $\mathbb{R}^2/\mathbb{Z}^2$, the image of the straight line $y = \frac{m}{n}x$ in T^2 defines the *torus knot* of type (m, n), which we shall denote as $K_{m,n} \subset S^3$ (see [18, Chapter 3]). The $\mathrm{SL}(3, \mathbb{C})$-character variety of the torus knot $K_{2,3}$ has been described in [6], and for the general torus knot $K_{m.n}$ it is given in [17].

The fundamental group of the knot complement $S^3 - K_{m,n}$ is the group

$$\Gamma_{m,n} = \langle x, y \,|\, x^n = y^m \rangle \,.$$

Therefore the character variety is described explicitly as

$$\mathcal{X}_r = X(\Gamma_{m,n}, \mathrm{SL}(r, \mathbb{C})) = \{(A, B) \in \mathrm{SL}(r, \mathbb{C})^2 \,|\, A^n = B^m\} /\!/ \,\mathrm{SL}(r, \mathbb{C}) \qquad (1.1)$$

Various geometrical properties of character varieties can be studied. Basic properties include connectedness, number of irreducible components, and the dimension. More elaborated properties are the fundamental group or the Poincaré polynomials; such topological properties have been studied for the character varieties for surfaces via non-abelian Hodge theory, which produces a *homeomorphism* of the moduli of representations with the moduli space of Higgs bundles [7]. If one focuses on the algebro-geometric aspects of character varieties, one can try to compute the motives, the Hodge numbers or the

E-polynomials. For instance, for $\mathrm{SL}(3,\mathbb{C})$-character varieties of torus knots, the motive is given in [17].

Here, we shall give, using the result of [17], the μ_3-equivariant motive of the $\mathrm{SL}(3,\mathbb{C})$-character varieties of torus knots. Note that the center of $\mathrm{SL}(r,\mathbb{C})$, consisting of the matrices $\varpi\,\mathrm{Id}$, where $\varpi \in \mu_r = \{e^{2\pi i k/r}, k = 0,\ldots,r-1\}$, act on (1.1). Therefore the motive of $X(\Gamma_{m,n},\mathrm{SL}(r,\mathbb{C}))$ has a μ_r-action. This produces a μ_r-equivariant motive as explained in Section 2. Our main result is:

Theorem 1.1 *The μ_3-equivariant motive of the $\mathrm{SL}(3,\mathbb{C})$-character variety of the (m,n)-torus knot is:*

- *If* $n, m \equiv 1,5 \pmod 6$, *then*
$$\begin{aligned} h_{\mu_3}(\mathcal{X}_3) = \Big[&P_0 + \frac{1}{36}(m-1)(m-2)(n-1)(n-2)P_1 + \frac{1}{6}(n-1)(m-1)(n+m-4)P_3 + \\ &+ \frac{1}{4}(n-1)(m-1)P_5\Big]T + \\ &+ \Big[\frac{1}{3}(m-1)(n-1)(n+m-4)P_3 + \frac{1}{18}(m-2)(m-1)(n-2)(n-1)P_1\Big]R \end{aligned}$$

- *If* $n \equiv 2,4 \pmod 6$, $m \equiv 1,5 \pmod 6$, *then*
$$\begin{aligned} h_{\mu_3}(\mathcal{X}_3) = \Big[&P_0 + \frac{1}{36}(m-1)(m-2)(n-1)(n-2)P_1 + \frac{1}{6}(n-1)(m-1)(n+m-4)P_3 + \\ &+ \frac{1}{4}(n-2)(m-1)P_5 + \frac{1}{2}(m-1)P_6\Big]T + \\ &+ \Big[\frac{1}{3}(m-1)(n-1)(n+m-4)P_3 + \frac{1}{18}(m-2)(m-1)(n-2)(n-1)P_1\Big]R \end{aligned}$$

- *If* $n \equiv 3 \pmod 6$, $m \equiv 1,5 \pmod 6$, *then*
$$\begin{aligned} h_{\mu_3}(\mathcal{X}_3) = \Big[&P_0 + \frac{1}{36}(m-1)(m-2)n(n-3)P_1 + \frac{1}{6}(m-1)(m-2)P_2 + \\ &+ \frac{1}{6}(m-1)(mn+n^2-5n-m-2)P_3 + (m-1)P_4 + \frac{1}{4}(n-1)(m-1)P_5\Big]T + \\ &+ \Big[\frac{1}{18}(m-2)(m-1)(n^2-3n+3)P_1 - \frac{1}{6}(m-2)(m-1)P_2 + \\ &+ \frac{1}{3}(m-1)(n^2+mn-5n-m+7)P_3 - (m-1)P_4\Big]R \end{aligned}$$

- *If* $n \equiv 0 \pmod 6$, $m \equiv 1,5 \pmod 6$, *then*
$$\begin{aligned} h_{\mu_3}(\mathcal{X}_3) = \Big[&P_0 + \frac{1}{36}(m-1)(m-2)n(n-3)P_1 + \frac{1}{6}(m-1)(m-2)P_2 + \\ &+ \frac{1}{6}(m-1)(mn+n^2-5n-m-2)P_3 + (m-1)P_4 + \\ &+ \frac{1}{4}(n-2)(m-1)P_5 + \frac{1}{2}(m-1)P_6\Big]T + \\ &+ \Big[\frac{1}{18}(m-2)(m-1)(n^2-3n+3)P_1 - \frac{1}{6}(m-2)(m-1)P_2 + \\ &+ \frac{1}{3}(m-1)(n^2+mn-5n-m+7)P_3 - (m-1)P_4\Big]R \end{aligned}$$

- *If* $n \equiv 2, 4 \pmod 6$, $m \equiv 3 \pmod 6$, *then*

$$\begin{aligned} h_{\mu_3}(\mathcal{X}_3) = \Big[& P_0 + \frac{1}{36} m(m-3)(n-1)(n-2)P_1 + \frac{1}{6}(n-1)(n-2)P_2 + \\ & + \frac{1}{6}(n-1)(mn+m^2-n-5m-2)P_3 + (n-1)P_4 + \\ & + \frac{1}{4}(n-2)(m-1)P_5 + \frac{1}{2}(m-1)P_6 \Big] T + \\ & + \Big[\frac{1}{18}(n-2)(n-1)(m^2-3m+3)P_1 - \frac{1}{6}(n-2)(n-1)P_2 + \\ & + \frac{1}{3}(n-1)(m^2+mn-5m-n+7)P_3 - (n-1)P_4 \Big] R \end{aligned}$$

where T *is the trivial representation and* R *is the non-trivial two-dimensional rational representation. Here,* $P_0 = \mathbb{L}^2$, $P_1 = \mathbb{L}^4 + 4\mathbb{L}^3 - 3\mathbb{L}^2 - 15\mathbb{L} + 12$, $P_2 = \mathbb{L}^4 + 2\mathbb{L}^3 - 3\mathbb{L}^2 - \mathbb{L} + 4$, $P_3 = \mathbb{L}^2 - 3\mathbb{L} + 3$, $P_4 = \mathbb{L}^2 - \mathbb{L} + 1$, $P_5 = \mathbb{L}^2 - 3\mathbb{L} + 2$, $P_6 = \mathbb{L}^2 - 2\mathbb{L} + 1$.

(Note that we can swap n, m *if necessary to be in one of the cases above.)*

2. Equivariant motives

Let $\mathcal{V}ar_{\mathbb{C}}$ be the category of quasi-projective complex varieties. We denote by $K(\mathcal{V}ar_{\mathbb{C}})$ the Grothendieck ring of $\mathcal{V}ar_{\mathbb{C}}$. This is the abelian group generated by elements $[Z]$, for $Z \in \mathcal{V}ar_{\mathbb{C}}$, subject to the relation $[Z] = [Z_1] + [Z_2]$ whenever Z can be decomposed as a disjoint union $Z = Z_1 \sqcup Z_2$ of a closed and a Zariski open subset. There is a naturally defined product in $K(\mathcal{V}ar_{\mathbb{C}})$ given by $[Y] \cdot [Z] = [Y \times Z]$. We write $\mathbb{L} := [\mathbb{A}^1]$, where $\mathbb{A}^1$ is the affine line, the *Lefschetz object* in $K(\mathcal{V}ar_{\mathbb{C}})$. Clearly $\mathbb{L}^k = [\mathbb{A}^k]$. Finally, let $\mathcal{S}m\mathcal{V}ar_{\mathbb{C}}$ denote the category of *smooth projective* varieties over $\mathbb{C}$. We consider the ring $K^{bl}(\mathcal{S}m\mathcal{V}ar_{\mathbb{C}})$ generated by the smooth projective varieties subject to the relations $[X] - [Y] = [\mathrm{Bl}_Y(X)] - [E]$, where $Y \subset X$ is a smooth subvariety, $\mathrm{Bl}_Y(X)$ is the blow-up of X along Y, and E is the exceptional divisor. By [2, Theorem 3.1], there is an isomorphism

$$K^{bl}(\mathcal{S}m\mathcal{V}ar_{\mathbb{C}}) \cong K(\mathcal{V}ar_{\mathbb{C}}).$$

Now we move to the definition of Chow motives. Given a smooth projective variety X, let $CH^d(X)$ denote the abelian group of $\mathbb{Q}$-cycles on X, of codimension d, modulo rational equivalence. If $X, Y \in \mathcal{S}m\mathcal{V}ar_{\mathbb{C}}$, suppose that X is connected and $\dim(X) = d$. The group of correspondences (of degree 0) from X to Y is $\mathrm{Corr}(X, Y) = CH^d(X \times Y)$. For varieties $X, Y, Z \in \mathcal{S}m\mathcal{V}ar_{\mathbb{C}}$, the composition of correspondences

$$\mathrm{Corr}(X, Y) \otimes \mathrm{Corr}(Y, Z) \to \mathrm{Corr}(X, Z)$$

is defined as

$$g \circ f = p_{XZ*}(p_{XY}^*(f) \cdot p_{YZ}^*(g)),$$

where $p_{XZ} : X \times Y \times Z \to X \times Z$ is the projection, and similarly for p_{XY} and p_{YZ}.

Definition 2.1 *The category of* (effective Chow) motives *is the category* $\mathcal{M}ot$ *such that:*

- *its objects are pairs* (X, p) *where* $X \in \mathcal{S}m\mathcal{V}ar_{\mathbb{C}}$, *and* $p \in \mathrm{Corr}(X, X)$ *is an idempotent* ($p = p \circ p$);
- *if* $(X, p), (Y, q)$ *are effective motives, then the morphisms are* $\mathrm{Hom}((X, p), (Y, q)) = q \circ \mathrm{Corr}(X, Y) \circ p$.

There is a natural functor

$$h : \mathcal{S}m\mathcal{V}ar_{\mathbb{C}}^{\mathrm{opp}} \to \mathcal{M}ot \tag{2.1}$$

such that, for a smooth projective variety X, $h(X) = (X, \Delta_X)$, where $\Delta_X \in \mathrm{Corr}(X, X)$ is the graph of the identity $\mathrm{Id}_X : X \to X$. We say that $h(X)$ is the *motive of* X.

The category $\mathcal{M}ot$ is pseudo-abelian, where direct sums and tensor products are defined by $(X, p) \oplus (Y, q) = (X \sqcup Y, p+q)$ and $(X, p) \otimes (Y, q) = (X \times Y, p^*_{X\times X}(p) \cdot p^*_{Y \times Y}(q))$. In particular

$$\begin{aligned} h(X \sqcup Y) &= h(X) \oplus h(Y), \\ h(X \times Y) &= h(X) \otimes h(Y). \end{aligned}$$

This allows us to define $K(\mathcal{M}ot)$ as the abelian group generated by elements $[M]$, for $M \in \mathcal{M}ot$, subject to the relations $[M] = [M_1] + [M_2]$, when $M = M_1 \oplus M_2$. This is a ring with the product $[M_1] \cdot [M_2] = [M_1 \otimes M_2]$.

In $\mathcal{M}ot$, we have that $\mathbf{1} = h(pt)$ is the identity of the tensor product, so it is called the *unit motive*. It is easily seen that there is an isomorphism $\mathbf{1} \cong (\mathbb{P}^1, \mathbb{P}^1 \times pt)$. Set $\mathbb{L} = (\mathbb{P}^1, pt \times \mathbb{P}^1)$, which is called the *Lefschetz motive*. Therefore $h(\mathbb{P}^1) = \mathbf{1} \oplus \mathbb{L}$, and more generally,

$$h(\mathbb{P}^n) = \mathbf{1} \oplus \mathbb{L} \oplus \cdots \oplus \mathbb{L}^n.$$

Denote also by $\mathbb{L} \in K(\mathcal{M}ot)$ the class of the Lefschetz motive $\mathbb{L} \in \mathcal{M}ot$.

In [13] it is shown that the motive of the blow-up of a smooth projective variety X along a codimension r smooth subvariety Y is $h(\mathrm{Bl}_Y(X)) = h(X) \oplus \left(\bigoplus_{i=1}^{r-1} (h(Y) \otimes \mathbb{L}^i)\right)$, being thus compatible with the relation defining $K^{bl}(\mathcal{S}m\mathcal{V}ar_{\mathbb{C}})$. So the map h in (2.1) descends to $K^{bl}(\mathcal{S}m\mathcal{V}ar_{\mathbb{C}}) \to K(\mathcal{M}ot)$, hence defining a ring homomorphism

$$\chi : K(\mathcal{V}ar_{\mathbb{C}}) \to K(\mathcal{M}ot). \tag{2.2}$$

When X is smooth and projective, we have

$$\chi([X]) = [h(X)],$$

so we can think of the map χ as the natural extension of the notion of motives to all quasi-projective varieties. Notice that $\chi(\mathbb{L}) = \mathbb{L}$, which justifies the use of the same notation for the Lefschetz object and the Lefschetz motive.

Let G be a finite group. We have the category $\mathcal{V}ar_{\mathbb{C}}^G$ of quasi-projective complex varieties with a G-action, and the category $\mathcal{S}m\mathcal{V}ar_{\mathbb{C}}^G$ of smooth projective complex varieties endowed with a G-action. As before, we have well-defined Grothendieck rings $K(\mathcal{V}ar_{\mathbb{C}}^G)$ and $K^{bl}(\mathcal{S}m\mathcal{V}ar_{\mathbb{C}}^G)$, which are isomorphic [2].

Let X be a smooth projective variety with an action of a finite group G. Let $CH^d_{\mathbb{C}}(X\times X)=CH^d(X\times X)\otimes\mathbb{C}$ denote the Chow ring with complex coefficients. The action of G on X defines a morphism

$$\varphi:\mathbb{C}[G]\longrightarrow CH^d_{\mathbb{C}}(X\times X),$$

given by $g\mapsto\Gamma_g$. By a theorem of Maschke [8, XVIII, Thm, 1.2], the group ring $\mathbb{C}[G]$ is semisimple. Every semisimple ring R admit a decomposition in simple rings $R=\prod_{i=1}^s R_i$, where $R_i=R\cdot e_i$. Such $e_i\in R$ are the idempotents of R_i and $e_i\cdot e_j=0$ for $i\neq j$. Furthermore, the sum of these elements is

$$1=e_1+e_2+\cdots+e_s. \tag{2.3}$$

In our case,

$$\mathbb{C}[G]=\prod_{i=1}^s\mathbb{C}[G]\cdot e_i \tag{2.4}$$

where $e_i^2=e_i$ and $e_i\cdot e_j=0$ whenever $i\neq j$. If we let $p_i=\varphi(e_i)\in CH^d_{\mathbb{C}}(X\times X)$, then $p_i^2=p_i$ and $p_i\cdot p_j=0$ for $i\neq j$. The equality (2.3) gives the decomposition of the motive $h(X)$ of the variety

$$h(X)=\bigoplus_{i=1}^s(X,p_i).$$

Definition 2.2 *We define the* equivariant motive *of X as*

$$h_G(X):=\sum(X,p_i)e_i\in K(\mathcal{M}ot)\otimes\mathbb{C}[G].$$

This means that $h_G(X)$ is the image of $\sum e_i\otimes e_i\in\mathbb{C}[G]\otimes\mathbb{C}[G]$ under the natural map $\varphi\otimes\mathrm{Id}:\mathbb{C}[G]\otimes\mathbb{C}[G]\to CH^d_{\mathbb{C}}(X\times X)\otimes\mathbb{C}[G]$.

The proof of [13] can be carried out for a smooth projective variety X endowed with a G-action and a smooth subvariety $Y\subset X$ which is G-invariant. This gives that $h_G(\mathrm{Bl}_Y(X))=h_G(X)\oplus\left(\bigoplus_{i=1}^{r-1}(h_G(Y)\otimes\mathbb{L}^i)\right)$. Thus the map h_G in Definition 2.2 descends to a map

$$K(\mathcal{V}ar^G_{\mathbb{C}})=K^{bl}(\mathcal{S}m\mathcal{V}ar^G_{\mathbb{C}})\to K(\mathcal{M}ot)\otimes\mathbb{C}[G].$$

The proof of this fact follows the same arguments presented in [13], taking into account the equivariance of the Chern classes x_k of the projective bundle $E\to Y$, where E denotes the exceptional divisor of $\mathrm{Bl}_Y X$, and hence the classes p_i commute with x_k.

The idempotents e_i are associated in a one-to-one way to the irreducible representations of G. For an irreducible representation R_i, let χ_i be its character. Then

$$e_i=\frac{1}{|G|}\sum_{g\in G}\chi_i(g)g,$$

so $h_i(X)=(X,p_i)$, where

$$p_i=\frac{1}{|G|}\sum_{g\in G}\chi_i(g)\Gamma_g.$$

For the trivial representation R_1, we recover the quotient motive of X/G by the result in [1],

$$h(X/G) = h_1(X) = \left(X, \frac{1}{|G|} \sum_{g \in G} \Gamma_g \right). \tag{2.5}$$

This holds for smooth projective varieties by [1], hence it holds for all quasi-projective varieties since $K(\mathcal{V}ar_{\mathbb{C}}^G) = K^{bl}(\mathcal{S}m\mathcal{V}ar_{\mathbb{C}}^G)$.

Finally, from $h(X) = \sum h_i(X)$ the equivariant motive recovers the usual motive of a quasi-projective variety.

Now we analyse the case of a cyclic group $G = C_r$ of order r.

Lemma 2.1 *Let ξ be an r-th primitive root of unity and let g be a generator for the group C_r. Then, the decomposition* (2.4) *is*

$$\mathbb{C}[C_r] = \bigoplus_{a=0}^{r-1} \mathbb{C}\, e_a, \quad \textit{where the projectors are} \quad e_a = \frac{1}{r} \sum_{k=0}^{r-1} (\xi^a g)^k .$$

Proof. First, we compute the product $e_a \cdot e_b$. By definition

$$\begin{aligned} e_a \cdot e_b &= \frac{1}{r^2} \left(\sum_{i=0}^{r-1} (\xi^a g)^i \right) \left(\sum_{j=0}^{r-1} (\xi^b g)^j \right) \\ &= \frac{1}{r^2} \sum_{c=0}^{r-1} \sum_{\substack{i+j \equiv c \\ (\mathrm{mod}\ r)}} \xi^{ai+bj} g^{i+j} = \frac{1}{r^2} \sum_{c=0}^{r-1} g^c \sum_{\substack{i+j \equiv c \\ (\mathrm{mod}\ r)}} \xi^{ai+bj}. \end{aligned}$$

We focus on the sum $\sum_{i+j=c} \xi^{ai+bj}$. If $a \neq b$, this sum is zero, since the sequence $\{ai+bj \pmod r\}_{i+j=c}$ is nothing but $\{0, 1, \dots, r-1\}$. Thus, $e_a \cdot e_b = 0$ if $a \neq b$. The case $a = b$, the sum is non-zero and it is $r \cdot (\xi^a)^c$, and we conclude that $e_a \cdot e_a = e_a$. □

In Lemma 2.1, the element corresponding to the trivial representation is $e_0 = \frac{1}{r} \sum g^k$. So $h_0(X) = h(X/C_r)$. Suppose that we are in the situation

$$h_a(X) = h_b(X), \text{ when } \gcd(r,a) = \gcd(r,b). \tag{2.6}$$

Then we can recover the equivariant motive from the quotients $X/\langle g^d \rangle$, for $d|r$. We start with the case that r is prime. Then $h_1(X) = \ldots = h_{r-1}(X)$. Hence

$$h_{C_r}(X) = h_0(X) e_0 + h_1(X)(e_1 + \ldots + e_{r-1}), \tag{2.7}$$

where

$$\begin{aligned} h_0(X) &= h(X/C_r), \\ h_1(X) &= \frac{1}{r-1} (h(X) - h(X/C_r)). \end{aligned}$$

If r is not prime, then

$$h_{C_r}(X) = h_0(X)e_0 + \sum_{d|r} h_d(X) \left(\sum_{\substack{1\leq l\leq r/d-1 \\ \gcd(l,r/d)=1}} e_{ld} \right).$$

To determine $h_{C_r}(X)$ we need as many equations as divisors of r. These are provided by the following result.

Lemma 2.2 *For any $d|r$, we have* $\sum_{k=0}^{r/d-1} h_{kd}(X) = h(X/\langle g^{r/d}\rangle)$.

Proof. Let ξ be an r-th primitive root of the unity and let $g \in G$ be a generator of the group C_r. Then,

$$\sum_{k=0}^{r/d-1} e_{kd} = \frac{1}{r}\sum_{k=0}^{r/d-1}\sum_{i=0}^{r-1}\left(\xi^{kd}g\right)^i = \frac{1}{r}\sum_{k=0}^{r/d-1}\sum_{i=0}^{r-1}\xi^{kdi}g^i = \frac{1}{r}\sum_{i=0}^{r-1} g^i \sum_{k=0}^{r/d-1}\xi^{kdi}$$

The sum $\sum_{k=0}^{r/d-1}\xi^{kdi}$ is zero if and only if di is multiple of r, that is, $i = \frac{r}{d}b$ for some integer number b. Then, the sum becomes

$$\sum_{k=0}^{r/d-1} e_{kd} = \frac{1}{d}\sum_{b=0}^{d-1} g^{\frac{r}{d}b}.$$

Now take the image under $\varphi : \mathbb{C}[C_r] \to CH^d_{\mathbb{C}}(X \times X)$. This produces the motive

$$\left(X, \frac{1}{d}\sum_{b=0}^{d-1}\Gamma_{g^{\frac{r}{d}b}}\right) = h(X/\langle g^{r/d}\rangle).$$

The result follows. □

3. Character varieties of torus knots

Let

$$\Gamma_{m,n} = \langle x, y \,|\, x^n = y^m \rangle$$

be the torus knot group, and consider the character varieties for $\mathrm{SL}(r,\mathbb{C})$ and $\mathrm{PGL}(r,\mathbb{C}) = \mathrm{SL}(r,\mathbb{C})/\mu_r$,

$$\mathcal{X}_r = X(\Gamma_{m,n}, \mathrm{SL}(r,\mathbb{C})),$$
$$\bar{\mathcal{X}}_r = X(\Gamma_{m,n}, \mathrm{PGL}(r,\mathbb{C})),$$

By [17, Section 4], we have that μ_r acts on $\mathcal{X}_r$ via $\varpi \cdot (A,B) = (\varpi^m A, \varpi^n B)$, $\varpi \in \mu_r$, and

$$\bar{\mathcal{X}}_r \cong \mathcal{X}_r/\mu_r\,.$$

Let us see that Condition (2.6) is satisfied for this action. Take a, b such that $\gcd(a, r) = \gcd(b, r) = d$. Then there is a Galois automorphism $\sigma : \mathbb{C} \to \mathbb{C}$ such that $\sigma(\xi^b) = \xi^a$, where $\xi = e^{2\pi i/r}$. We let σ act on $\mathcal{X}_r$: this means that σ acts on all entries of the matrices A and B. Then $\sigma : \mathcal{X}_r \to \mathcal{X}_r$ interchanges the action of g to the action of $\sigma(g) = g^p$, where p is defined by $\xi^{bp} = \xi^a$, i.e. $\frac{b}{d}p \equiv \frac{a}{d} \pmod{\frac{r}{d}}$ (the integer p is coprime to r). Therefore $(\mathcal{X}_r, p_a) \cong (\mathcal{X}_r, \sigma(p_a)) = (\mathcal{X}_r, p_b)$, since

$$\sigma(p_a) = \frac{1}{r}\sum_{k=0}^{r-1} (\xi^a \sigma(g))^k = \frac{1}{r}\sum_{k=0}^{r-1} (\xi^a g^p)^k = \frac{1}{r}\sum_{k=0}^{r-1} \left(\xi^{bp} g^p\right)^k = \frac{1}{r}\sum_{k=0}^{r-1} (\xi^b g)^k = p_b.$$

We have the following result for SL(2, $\mathbb{C}$)-character varieties.

Theorem 3.1 *The μ_2-equivariant motive $h_{\mu_2}(\mathcal{X}_2)$ is equal to*

$$\begin{cases} (\mathbb{L} + \frac{1}{4}(n-1)(m-1)(\mathbb{L}-2))T + \frac{1}{4}(n-1)(m-1)(\mathbb{L}-2))N, & n, m \text{ odd.} \\ (\mathbb{L} + \frac{1}{4}(n-2)(m-1)(\mathbb{L}-2) + \frac{1}{2}(m-1)(\mathbb{L}-1))T + \\ \quad +(\frac{1}{4}(n-2)(m-1)(\mathbb{L}-2) - \frac{1}{2}(m-1)(\mathbb{L}-1))N, & n \text{ even, } m \text{ odd.} \end{cases}$$

where T is the trivial representation and N is the non-trivial one.

Proof. The character variety $\mathcal{X}_2$ is described in [14] by finding a set of equations satisfied by the traces of the matrices of the images by the representation. In [16] the same variety $\mathcal{X}_2$ is described by a geometric method based on the study of eigenvectors and eigenvalues of the matrices. The variety $\mathcal{X}_2$ consists of the following irreducible components: one component consisting of reducible representations, isomorphic to $\mathbb{C}$; and $(n-1)(m-1)/2$ components forming the irreducible locus, each of them isomorphic to $\mathbb{C} - \{0, 1\}$. Therefore the motive is $[\mathcal{X}_2] = \mathbb{L} + \frac{1}{2}(n-1)(m-1)(\mathbb{L}-2)$.

As described in [17], the PGL(2, $\mathbb{C}$)-character variety $\bar{\mathcal{X}}_2$ consists of: one component consisting of reducible representations, isomorphic to $\mathbb{C}$; $[\frac{n-1}{2}][\frac{m-1}{2}]$ components of the irreducible locus, each of them isomorphic to $\mathbb{C} - \{0, 1\}$; and if n is even and m is odd, $(m-1)/2$ components of the irreducible locus, each of them isomorphic to $\mathbb{C}^*$ (the case m even and n odd is analogous). Note that we can always assume, by swapping n, m if necessary, that m is odd. Therefore we have that for m, n odd, $[\bar{\mathcal{X}}_2] = \mathbb{L} + \frac{1}{4}(n-1)(m-1)(\mathbb{L}-2)$. For n even and m odd, we have $[\bar{\mathcal{X}}_2] = \mathbb{L} + \frac{1}{4}(n-2)(m-1)(\mathbb{L}-2) + \frac{1}{2}(m-1)(\mathbb{L}-1)$.

By (2.7),

$$h_{\mu_2}(\mathcal{X}_2) = [\bar{\mathcal{X}}_2]T + ([\mathcal{X}_2] - [\bar{\mathcal{X}}_2])N\,,$$

where T is the trivial representation, and N is the non-trivial representation ($T = e_0$ and $N = e_1$ in the notation of Section 2). The result follows. □

Now we move to the description of the SL(3, $\mathbb{C}$)-character variety $\mathcal{X}_3$. The following description appears in [17, Sections 8 and 10].

Proposition 3.2 *The components of $\mathcal{X}_3$ are the following:*

- *The component of totally reducible representations, isomorphic to $\mathbb{C}^2$.*

- $[\frac{n-1}{2}][\frac{m-1}{2}]$ *components of partially reducible representations, each isomorphic to* $(\mathbb{C}-\{0,1\})\times\mathbb{C}^*$.
- *If n is even, there are $(m-1)/2$ extra components of partially reducible representations, each isomorphic to $\{(u,v)\in\mathbb{C}^2 | v\neq 0, v\neq u^2\}$. (The case m even and n odd is analogous.)*
- $\frac{1}{12}(n-1)(n-2)(m-1)(m-2)$ *componens of irreducible representations, of maximal dimension 4, which are isomorphic to $\mathcal{M}/(T\times_{\mathbb{C}^*}T)$, where $\mathcal{M}\subset \mathrm{GL}(3,\mathbb{C})$ are the stable points for the $(T\times_{\mathbb{C}^*}T)$-action (here T are the diagonal matrices acting by multiplication on $\mathrm{GL}(3,\mathbb{C})$ on the left and on the right).*
- $\frac{1}{2}(n-1)(m-1)(n+m-4)$ *components of irreducible representations, each isomorphic to* $(\mathbb{C}^*)^2-\{x+y=1\}$.

From here, we can read off the motive of the character variety $\mathcal{X}_3$ (cf. [17, Theorem 8.3]):

$$\begin{aligned}
[\mathcal{X}_3] &= \frac{1}{12}(n-1)(n-2)(m-1)(m-2)(\mathbb{L}^4+4\mathbb{L}^3-3\mathbb{L}^2-15\mathbb{L}+12) \\
&\quad + \mathbb{L}^2 + \frac{1}{4}(n-1)(m-1)(\mathbb{L}^2-3\mathbb{L}+2) \\
&\quad + \frac{1}{2}(n-1)(m-1)(n+m-4)(\mathbb{L}^2-3\mathbb{L}+3), \quad m,n \text{ odd}, \\
[\mathcal{X}_3] &= \frac{1}{12}(n-1)(n-2)(m-1)(m-2)(\mathbb{L}^4+4\mathbb{L}^3-3\mathbb{L}^2-15\mathbb{L}+12) \\
&\quad + \mathbb{L}^2 + \frac{1}{4}(n-2)(m-1)(\mathbb{L}^2-3\mathbb{L}+2) + \frac{1}{2}(m-1)(\mathbb{L}^2-2\mathbb{L}+1) \\
&\quad + \frac{1}{2}(n-1)(m-1)(n+m-4)(\mathbb{L}^2-3\mathbb{L}+3), \quad n \text{ even}, m \text{ odd}.
\end{aligned}$$

Now we describe the $\mathrm{PGL}(3,\mathbb{C})$-character varieties $\bar{\mathcal{X}}_3$.

Proposition 3.3 *The components of the $\mathrm{PGL}(3,\mathbb{C})$-character variety $\bar{\mathcal{X}}_3$ are:*

- *The component of totally reducible representations, which is isomorphic to $\mathbb{C}^2/\mu_3\cong\{(x,y,z)\in\mathbb{C}^3 \mid x\,y=z^3\}$.*
- $[\frac{n-1}{2}][\frac{m-1}{2}]$ *components of partially reducible representations, each isomorphic to* $(\mathbb{C}-\{0,1\})\times\mathbb{C}^*$.
- *When n is even, there are $(m-1)/2$ additional components of partially reducible representations, each isomorphic to $\{(u,v)\in\mathbb{C}^2 | v\neq 0, v\neq u^2\}$.*
- *When $m,n\notin 3\mathbb{Z}$, there are the following components of irreducible representations:*
  - $(n-1)(m-1)(n+m-4)/6$ *components isomorphic to* $(\mathbb{C}^*)^2-\{x+y=1\}$
  - *and $(m-1)(m-2)(n-1)(n-2)/36$ components of maximal dimension isomorphic to $\mathcal{M}/(T\times_{\mathbb{C}^*}T)$.*

- *When $n \in 3\mathbb{Z}$, there are the following components of irreducible representations:*
  - $(m-1)(m\,n+n^2-5n-m+2)/6$ *components isomorphic to* $(\mathbb{C}^*)^2-\{x+y=1\}$,
  - $m-1$ *components isomorphic to* $\{(x,y,z)\in\mathbb{C}^3 \mid x\,y=z^3,\ x+y+3z\neq 1\}$,
  - $(m-1)(m-2)n(n-3)/36$ *components of maximal dimension isomorphic to* $\mathcal{M}/(T\times_{\mathbb{C}^*}T)$,
  - *and* $(m-1)(m-2)/6$ *components of maximal dimension isomorphic to* $\mathcal{M}/(T\times_{\mathbb{C}^*}T\rtimes\mu_3)$, *where μ_3 acts by cyclic permutation of columns in $\mathcal{M}$.*

  The case $m\in 3\mathbb{Z}$ is symmetric.

The motive of the character variety $\bar{\mathcal{X}}_3$ is as follows (see [17, Corollary 10.3]):

- If $n,m\equiv 1,5 \pmod 6$, then $[\bar{\mathcal{X}}_3]=P_0+\frac{1}{36}(m-1)(m-2)(n-1)(n-2)P_1+\frac{1}{6}(n-1)(m-1)(n+m-4)P_3+\frac{1}{4}(n-1)(m-1)P_5$.
- If $n\equiv 2,4 \pmod 6$, $m\equiv 1,5 \pmod 6$, then $[\bar{\mathcal{X}}_3]=P_0+\frac{1}{36}(m-1)(m-2)(n-1)(n-2)P_1+\frac{1}{6}(n-1)(m-1)(n+m-4)P_3+\frac{1}{4}(n-2)(m-1)P_5+\frac{1}{2}(m-1)P_6$.
- If $n\equiv 3 \pmod 6$, $m\equiv 1,5 \pmod 6$, then $[\bar{\mathcal{X}}_3]=P_0+\frac{1}{36}(m-1)(m-2)n(n-3)P_1+\frac{1}{6}(m-1)(m-2)P_2+\frac{1}{6}(m-1)(mn+n^2-5n-m-2)P_3+(m-1)P_4+\frac{1}{4}(n-1)(m-1)P_5$.
- If $n\equiv 0 \pmod 6$, $m\equiv 1,5 \pmod 6$, then $[\bar{\mathcal{X}}_3]=P_0+\frac{1}{36}(m-1)(m-2)n(n-3)P_1+\frac{1}{6}(m-1)(m-2)P_2+\frac{1}{6}(m-1)(mn+n^2-5n-m-2)P_3+(m-1)P_4+\frac{1}{4}(n-2)(m-1)P_5+\frac{1}{2}(m-1)P_6$.
- If $n\equiv 2,4 \pmod 6$, $m\equiv 3 \pmod 6$, then $[\bar{\mathcal{X}}_3]=P_0+\frac{1}{36}m(m-3)(n-1)(n-2)P_1+\frac{1}{6}(n-1)(n-2)P_2+\frac{1}{6}(n-1)(mn+m^2-n-5m-2)P_3+(n-1)P_4+\frac{1}{4}(n-2)(m-1)P_5+\frac{1}{2}(m-1)P_6$.

Here $P_0=\mathbb{L}^2$, $P_1=\mathbb{L}^4+4\mathbb{L}^3-3\mathbb{L}^2-15\mathbb{L}+12$, $P_2=\mathbb{L}^4+2\mathbb{L}^3-3\mathbb{L}^2-\mathbb{L}+4$, $P_3=\mathbb{L}^2-3\mathbb{L}+3$, $P_4=\mathbb{L}^2-\mathbb{L}+1$, $P_5=\mathbb{L}^2-3\mathbb{L}+2$, $P_6=\mathbb{L}^2-2\mathbb{L}+1$.

Now, to compute the μ_3-equivariant motive, we use (2.7):

$$h_{\mu_3}(\mathcal{X}_3)=[\bar{\mathcal{X}}_3]T+\frac{1}{2}([\mathcal{X}_3]-[\bar{\mathcal{X}}_3])(R_1+R_2)$$

where T is the trivial representation of μ_3 and R_1,R_2 are the non-trivial representations (T corresponds to e_0 and R_1,R_2 correspond to e_1,e_2). Note that R_1,R_2 are representations defined over $\mathbb{C}$, but R_1+R_2 is a representation defined over the rationals. Theorem 1.1 follows from this.

References

[1] S. del Baño Rollín and V. Navarro-Aznar, On the motive of a quotient variety. Collectanea Math. **49** (1998), 203–226.

[2] F. Bittner, The universal Euler characteristic for varieties of characteristic zero, Compositio Math. **140** (2004), 1011–1032.

[3] M. Culler and P. Shalen, Varieties of group representations and splitting of 3-manifolds, Annals of Math. **117** (1983), 109–146.

[4] F. González-Acuña and J.-M. Montesinos-Amilibia, On the character variety of group representations in $\mathrm{SL}(2,\mathbb{C})$ and $\mathrm{PSL}(2,\mathbb{C})$, Math. Z. **214** (1993), 627–652.

[5] T. Hausel and M. Thaddeus, *Mirror symmetry, Langlands duality and Hitchin systems*, Invent. Math. **153** (2003), 197–229.

[6] M. Heusener and J. Porti, Representations of knot groups into $\mathrm{SL}(n,\mathbb{C})$ and twisted Alexander polynomials, ArXiv:1406.3705.

[7] N. Hitchin, The self-duality equations on a Riemann surface, Proc. London Math. Soc. (3) **55** (1987), 59–126.

[8] S. Lang, *Algebra (Revised Third Edition)*, Graduate Text in Mathematics **211**, 2005.

[9] S. Lawton, Minimal affine coordinates for $\mathrm{SL}(3,\mathbb{C})$-character varieties of free groups, J. Algebra **320** (2008), 3773–3810.

[10] S. Lawton and V. Muñoz, E-polynomial of the $\mathrm{SL}(3,\mathbb{C})$-character variety of free groups, ArXiv:1405.0816.

[11] M. Logares, V. Muñoz and P. Newstead, Hodge polynomials of $\mathrm{SL}(2,\mathbb{C})$-character varieties for curves of small genus, Rev. Mat. Complut. **26** (2013), 635–703.

[12] A. Lubotzky and A. Magid, *Varieties of representations of finitely generated groups*, Mem. Amer. Math. Soc. **58**, 1985.

[13] Y. Manin, Correspondences, motifs and monoidal transformations, Math. USSR Sb. **6** (1968), 439–470.

[14] J. Martín-Morales and A-M. Oller-Marcén, On the varieties of representations and characters of a family of one-relator subgroups, Topol. Appl. **156** (2009), 2376–2389.

[15] J. Martínez and V. Muñoz, E-polynomials of the $\mathrm{SL}(2,\mathbb{C})$-character varieties of surface groups, Arxiv:1407.6975.

[16] V. Muñoz, The $\mathrm{SL}(2,\mathbb{C})$-character varieties of torus knots, Rev. Mat. Complut. **22** (2009), 489–497.

[17] V. Muñoz and J. Porti, Geometry of the $\mathrm{SL}(3,\mathbb{C})$-character variety of torus knots, Arxiv.org:1409.4784.

[18] D. Rolfsen, *Knots and links*, Publish or Perish, 1990.