RATIONALITY OF THE MODULI SPACE OF STABLE PAIRS OVER A COMPLEX CURVE

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Abstract. Let X be a smooth complex projective curve of genus $g \geq 2$. A pair on X is formed by a vector bundle $E \to X$ and a global non-zero section $\phi \in H^0(E)$. There is a concept of stability for pairs depending on a real parameter τ, giving rise to moduli spaces $\mathcal{M}_\tau(r, \Lambda)$ of τ-stable pairs of rank r and fixed determinant Λ. In this paper we prove that the moduli spaces $\mathcal{M}_\tau(r, \Lambda)$ are in many cases rational.

Dedicated to the 60th Anniversary of Themistocles M. Rassias

1. Introduction

Let X be a compact connected Riemann surface of genus $g \geq 2$, which we may interpret as a smooth projective complex curve. Fix a Kähler form ω on X. Consider a C^∞ hermitian vector bundle $E \to X$ of rank r and degree d (cf. [9]). A unitary connection A on E endows it with a holomorphic structure $\overline{\partial}_A$, given by the $(0,1)$-part of $A = \partial_A + \overline{\partial}_A$. The connection is said to be *Hermitian-Einstein*, or *Hermitian-Yang-Mills*, if

$$F_A = c \cdot \mathrm{Id} \cdot \omega \,.$$

The constant c is constrained by the topology to be $c = \frac{d}{r}\mathrm{Vol}_\omega(X)$. This provides a link between gauge theory and the theory of holomorphic bundles. The fundamental theorem given by the Hitchin-Kobayashi correspondence establishes that a holomorphic structure $\overline{\partial}$ on E arises from a (unique up to unitary gauge automorphism of the bundle) connection A if and only if the holomorphic vector bundle $(E, \overline{\partial})$ is *polystable*; the definition of polystability is recalled below.

For a holomorphic bundle E, we define the slope $\mu(E) := d/r$, where d is its degree and r its rank. We say that E is stable if $\mu(E') < \mu(E)$ for all *holomorphic* proper subbundles $E' \subset E$. A vector bundle E is polystable if it is a direct sum of stable vector bundles of the same slope.

Of much interest is the extension to the case of pairs (E, ϕ) formed by a hermitian vector bundle E together with a global smooth section $\phi \in \Gamma(E)$. In this case, we look

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for unitary connections A satisfying a *vortex equation*

$$(1) \quad \begin{cases} \frac{2}{\sqrt{-1}} F_A = (\phi \otimes \phi^* - \tau \cdot \mathrm{Id})\, \omega \\ \overline{\partial}_A \phi = 0 \end{cases}$$

where ϕ^* is the adjoint of ϕ with respect to the unitary metric, so that $\phi \otimes \phi^* \in \Gamma(\mathrm{End}\, E)$, and τ is a real parameter. In this situation τ is not constrained. The pair (A, ϕ) induces a holomorphic structure $\overline{\partial}_A$ on E, and ϕ is a *holomorphic section* of the holomorphic vector bundle $(E, \overline{\partial}_A)$.

A holomorphic pair (also called a Bradlow pair) over X is a pair (E, ϕ), where $E \longrightarrow X$ is a holomorphic vector bundle, and $\phi \in H^0(E)$, i.e. a holomorphic section. There is a notion of stability of pairs depending on a parameter $\tau \in \mathbb{R}$. A holomorphic pair is τ-stable whenever the following conditions are satisfied:

- for any subbundle $E' \subset E$, we have $\mu(E') < \tau$,
- for any subbundle $E' \subset E$ such that $\phi \in H^0(E')$, we have $\mu(E/E') > \tau$.

A holomorphic pair is said to be τ-semistable if in the above definition the weak inequalities hold instead of the strict ones. A τ-semistable pair is τ-polystable if it is the direct sum of a τ-stable pair and a polystable vector bundle.

The link between this algebro–geometric concept and gauge theory comes from a Hitchin-Kobayashi correspondence which establishes that the solutions to (1) correspond to polystable holomorphic pairs.

There has been much interest in holomorphic pairs in the last fifteen years. The moduli spaces of τ-stable pairs is the space which parametrizes these objects. There is an algebraic construction of it, [2], which gives it the structure of a quasi-projective complex variety. Let $\mathcal{M}_\tau(r, d)$ be the moduli space of τ-stable pairs of rank r and degree d. For any line bundle Λ over X of degree d, we denote by $\mathcal{M}_\tau(r, \Lambda)$ the moduli space of τ-stable pairs of rank r and fixed determinant $\bigwedge^r E = \Lambda$.

A large number of topological and geometrical properties of $\mathcal{M}_\tau(r, \Lambda)$ have been studied in the literature. In [10], a construction of the moduli space is given using gauge theoretic techniques (more precisely, by reducing the vortex equation to the Hermitian-Einstein equation on a complex surface). This gives the general properties about the possible values of τ for non-emptiness, smoothness, etc, of the moduli space. Thaddeus [18] studied thoroughly the case of rank $r = 2$, computing the Poincaré polynomial of the moduli space and describing their topology quite explicitly. Later this was extended in [17] to compute the Hodge polynomials, and in [14] to rank $r = 3$. The general properties of the mixed Hodge structures of $\mathcal{M}_\tau(r, \Lambda)$ is found in [16]. In [15], a Torelli type theorem for the moduli spaces $\mathcal{M}_\tau(r, \Lambda)$ is proved; this amounts to the following: the algebraic structure of the moduli space allows to recover the complex structure of X. We also mention that in [4], the authors compute the Brauer group of these moduli spaces.

The focus of the present paper is another geometrical property of $\mathcal{M}_\tau(r, \Lambda)$, namely the *rationality*. A variety Z is rational if there is a birational rational map $Z \dashrightarrow \mathbb{P}^N$, where $\mathbb{P}^N$ is the complex projective space of dimension N. A birational rational map is an isomorphism between two Zariski open subsets of both spaces. We denote $Z \sim \mathbb{P}^N$.

Let us state now the main results of this paper.

A variety Z is called stably rational if $Z \times \mathbb{P}^n$ is rational for some n, so $Z \times \mathbb{P}^n \sim \mathbb{P}^N$. This notion is weaker than rationality. We have the following result, which we prove in Section 3.

Theorem 1.1. *Suppose $(r, g, d) \neq (3, 2, even)$. Then the variety $\mathcal{M}_\tau(r, \Lambda)$ is stably rational, for any τ.*

Regarding the rationality of the moduli space of pairs, we have the following, which is proved in Sections 4 and 5.

Theorem 1.2. *Let X be a smooth complex irreducible and projective curve of genus $g \geq 2$. Then, for any $\tau \in \mathbb{R}$, rank r and line bundle Λ of degree $d > 0$ over X, the moduli space $\mathcal{M}_\tau(r, \Lambda)$ is rational, in the following cases:*

- $d > rg$,
- $\gcd(r-1, d) = 1$,
- $\gcd(r, d) = 1$, $d > r(g-1)$.

This result is related to the work of Hoffman [11], where by very different techniques, there are some general results which prove the rationality of most moduli spaces $\mathcal{M}_\tau(r, \Lambda)$. The novelty of the proof given here lies in the fact that it uses only elementary techniques.

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2. Moduli spaces of pairs

We collect here some known results about the moduli spaces of pairs; the details can be found in [6], [7], [17], [15] and [18].

Let X be an irreducible smooth projective curve, defined over the field of complex numbers, of genus $g \geq 2$. A *holomorphic pair* (E, ϕ) over X consists of a holomorphic bundle on X and a nonzero holomorphic section $\phi \in H^0(E)$. Let $\mu(E) := \deg(E)/\operatorname{rk}(E)$ be the slope of E. Take any $\tau \in \mathbb{R}$. A holomorphic pair (E, ϕ) is called *τ-stable* (respectively, *τ-semistable*) whenever the following conditions are satisfied:

- for any nonzero proper subbundle $E' \subset E$, we have $\mu(E') < \tau$ (respectively, $\mu(E') \leq \tau$),
- for any proper subbundle $E' \subset E$ such that $\phi \in H^0(E')$, we have $\mu(E/E') > \tau$ (respectively, $\mu(E/E') \geq \tau$).

A *critical value* of the parameter $\tau = \tau_c$ is one for which there are strictly τ-semistable pairs. There are only finitely many critical values.

Fix an integer $r \geq 2$, and also fix a holomorphic line bundle Λ over X. Let d be the degree of Λ. We denote by $\mathcal{M}_\tau(r,\Lambda)$ (respectively, $\overline{\mathcal{M}}_\tau(r,\Lambda)$) the moduli space of τ-stable (respectively, τ-polystable) pairs (E,ϕ) of rank $\mathrm{rk}\,(E) = r$ and determinant $\det(E) = \Lambda$. The moduli space $\overline{\mathcal{M}}_\tau(r,\Lambda)$ is a normal projective variety, and $\mathcal{M}_\tau(r,\Lambda)$ is a smooth quasi-projective variety contained in the smooth locus of $\overline{\mathcal{M}}_\tau(r,\Lambda)$.

For non-critical values of the parameter, there are no strictly τ-semistable pairs, so $\mathcal{M}_\tau(r,\Lambda) = \overline{\mathcal{M}}_\tau(r,\Lambda)$, and it is a smooth projective variety. For a critical value τ_c, the variety $\overline{\mathcal{M}}_{\tau_c}(r,\Lambda)$ is in general singular.

Denote $\tau_m := \frac{d}{r}$ and $\tau_M := \frac{d}{r-1}$. The moduli space $\mathcal{M}_\tau(r,\Lambda)$ is empty for $\tau \notin (\tau_m,\tau_M)$. In particular, this forces $d > 0$ for τ-stable pairs. Denote by $\tau_1 < \tau_2 < \ldots < \tau_L$ the collection of all critical values in (τ_m,τ_M). Then the moduli spaces $\mathcal{M}_\tau(r,\Lambda)$ are isomorphic for all values τ in any interval (τ_i,τ_{i+1}), $i = 0,\ldots,L$; here $\tau_0 = \tau_m$ and $\tau_{L+1} = \tau_M$.

However, the moduli space changes when we cross a critical value. Let τ_c be a critical value. Denote $\tau_c^+ := \tau_c + \epsilon$ and $\tau_c^- := \tau_c - \epsilon$ for $\epsilon > 0$ small enough such that (τ_c^-,τ_c^+) does not contain any critical value other than τ_c. We define the *flip loci* $\mathcal{S}_{\tau_c^\pm}$ as the subschemes:

- $\mathcal{S}_{\tau_c^+} = \{(E,\phi) \in \mathcal{M}_{\tau_c^+}(r,\Lambda) \mid (E,\phi) \text{ is } \tau_c^-\text{-unstable}\}$,
- $\mathcal{S}_{\tau_c^-} = \{(E,\phi) \in \mathcal{M}_{\tau_c^-}(r,\Lambda) \mid (E,\phi) \text{ is } \tau_c^+\text{-unstable}\}$.

When crossing τ_c, the variety $\mathcal{M}_\tau(r,\Lambda)$ undergoes a birational transformation:

$$\mathcal{M}_{\tau_c^-}(r,\Lambda) - \mathcal{S}_{\tau_c^-} = \mathcal{M}_{\tau_c}(r,\Lambda) = \mathcal{M}_{\tau_c^+}(r,\Lambda) - \mathcal{S}_{\tau_c^+}\,.$$

Proposition 2.1 ([14, Proposition 5.1])**.** *Suppose $r \geq 2$, and let τ_c be a critical value with $\tau_m < \tau_c < \tau_M$. Then*

- $\mathrm{codim}\,\mathcal{S}_{\tau_c^+} \geq 3$ *except in the case $r = 2$, $g = 2$, d odd and $\tau_c = \tau_m + \frac{1}{2}$ (in which case* $\mathrm{codim}\,\mathcal{S}_{\tau_c^+} = 2$*),*
- $\mathrm{codim}\,\mathcal{S}_{\tau_c^-} \geq 2$ *except in the case $r = 2$ and $\tau_c = \tau_M - 1$ (in which case* $\mathrm{codim}\,\mathcal{S}_{\tau_c^-} = 1$*). Moreover, we have that* $\mathrm{codim}\,\mathcal{S}_{\tau_c^-} = 2$ *only for $\tau_c = \tau_M - 2$.*

The codimension of the flip loci is then always positive, hence we have the following corollary:

Corollary 2.2. *The moduli spaces $\mathcal{M}_\tau(r,\Lambda)$, $\tau \in (\tau_m,\tau_M)$, are birational.*

For a complex vector space V, by $\mathbb{P}(V)$ we will denote the projective space parametrizing lines in V.

The moduli spaces for the extreme values τ_m^+ and τ_M^- of the parameter are known explicitly. Let $M(r,\Lambda)$ be the moduli space of *stable* vector bundles or rank r and fixed determinant Λ. Define

$$\mathcal{U}_m(r,\Lambda) = \{(E,\phi) \in \mathcal{M}_{\tau_m^+}(r,\Lambda) \mid E \text{ is a stable vector bundle}\}\,, \tag{2}$$

and denote

$$\mathcal{S}_{\tau_m^+} := \mathcal{M}_{\tau_m^+}(r,\Lambda) - \mathcal{U}_m(r,\Lambda)\,.$$

Then there is a map

$$\pi_1 : \mathcal{U}_m(r,\Lambda) \longrightarrow M(r,\Lambda), \qquad (E,\phi) \longmapsto E\,, \tag{3}$$

whose fiber over any E is the projective space $\mathbb{P}(H^0(E))$. When $d \geq r(2g-2)$, and E is stable, we have $H^1(E) = 0$, and hence (3) is a projective bundle (cf. [17, Proposition 4.10]).

Regarding the right-most moduli space $\mathcal{M}_{\tau_M^-}(r,\Lambda)$: any τ_M^--stable pair (E,ϕ) sits in an exact sequence

$$0 \longrightarrow \mathcal{O} \stackrel{\phi}{\longrightarrow} E \longrightarrow F \longrightarrow 0\,,$$

where F is a semistable bundle of rank $r-1$ and $\det(F) = \Lambda$. Let

$$\mathcal{U}_M(r,\Lambda) \;=\; \{(E,\phi) \in \mathcal{M}_{\tau_M^-}(r,\Lambda) \,|\, F \text{ is a stable vector bundle}\}\,,$$

and denote

$$\mathcal{S}_{\tau_M^-} := \mathcal{M}_{\tau_M^-}(r,\Lambda) - \mathcal{U}_M(r,\Lambda)\,.$$

Then there is a map

$$\pi_2 : \mathcal{U}_M(r,\Lambda) \longrightarrow M(r-1,\Lambda), \qquad (E,\phi) \longmapsto E/\phi(\mathcal{O})\,, \tag{4}$$

whose fiber over any $F \in M(r-1,\Lambda)$ is the projective spaces $\mathbb{P}(H^1(F^*))$ (cf. [8, Theorem 7.7]). Note that $H^0(F^*) = 0$, because $d > 0$. So the map in (4) is always a projective bundle.

In the particular case of rank $r = 2$, the right-most moduli space is

$$\mathcal{M}_{\tau_M^-}(2,\Lambda) = \mathbb{P}(H^1(\Lambda^{-1}))\,, \tag{5}$$

since $M(1,\Lambda) = \{\Lambda\}$. In particular, Corollary 2.2 shows that all $\mathcal{M}_\tau(2,\Lambda)$ are rational quasi-projective varieties.

We have the following:

Lemma 2.3 ([15, Lemma 5.3]). *Let S be a bounded family of isomorphism classes of strictly semistable bundles of rank r and determinant Λ. Then $\dim M(r,\Lambda) - \dim S \geq (r-1)(g-1)$.*

Proposition 2.4. *The following two hold:*

- $\operatorname{codim} \mathcal{S}_{\tau_m^+} \geq 2$ *except in the case $r = 2$, $g = 2$, d even (in which case* $\operatorname{codim} \mathcal{S}_{\tau_m^+} = 1$*).*
- *Suppose $r \geq 3$. Then* $\operatorname{codim} \mathcal{S}_{\tau_M^-} \geq 2$ *except in the case $r = 3$, $g = 2$, d even (in which case* $\operatorname{codim} \mathcal{S}_{\tau_M^-} = 1$*).*

Proof. For any $(E,\phi) \in \mathcal{M}_{\tau_m^+}(r,\Lambda)$, the vector bundle E is semistable. Therefore, Lemma 2.3 implies that $\operatorname{codim} \mathcal{S}_{\tau_m^+} \geq (r-1)(g-1)$. Now the first statement follows.

As the dimension of $H^1(F^*)$ is constant, the codimension of $\mathcal{S}_{\tau_M^-}$ in $\mathcal{M}_{\tau_M^-}(r,\Lambda)$ is at least the codimension of a locus of semistable bundles. Applying Lemma 2.3 to $M(r-1,\Lambda)$ we conclude that $\operatorname{codim} \mathcal{S}_{\tau_M^-} \geq (r-2)(g-1)$. Now the second statement follows. □

3. Stable rationality

A variety Z is said to be stably rational if $Z \times \mathbb{P}^n$ is rational for some n. We prove here Theorem 1.1.

Let $\mathrm{Br}(\mathcal{M}_\tau(r,\Lambda))$ denote the Brauer group of $\mathcal{M}_\tau(r,\Lambda)$. In [4] the authors computed this group.

Theorem 3.1 ([4, Theorem 1.1])**.** *Assume that $(r,g,d) \neq (3,2,2)$. Then*

$$\mathrm{Br}(\mathcal{M}_\tau(r,\Lambda)) \,=\, 0\,.$$

Theorem 3.2. *Let Λ be a line bundle over X. Suppose $(r,g,d) \neq (3,2,\mathit{even})$. Then the moduli space $\mathcal{M}_\tau(r,\Lambda)$ of τ-stable pairs over X of rank $r \geq 2$ and fixed determinant Λ is stably rational.*

Proof. We already know that $\mathcal{M}_\tau(2,\Lambda)$ are rational varieties. So for $r = 2$, the result holds. Also, the birational class of the moduli spaces $\mathcal{M}_\tau(r,\Lambda)$ are independent of τ (for fixed r and Λ).

Now let $r \geq 2$, and fix the line bundle Λ. Let μ be a line bundle on X of degree at least $2g-2$. Consider the variety

$$M := \{(E,\phi,\psi)\,|\, E \in M(r,\Lambda),\ \phi \in \mathbb{P}(H^0(E\otimes\mu)),\ \psi \in \mathbb{P}(H^1(E^*))\}\,.$$

Since $\deg(\Lambda\otimes\mu^r) > r(2g-2)$, it follows that

$$M \longrightarrow \mathcal{U}_m(r,\Lambda\otimes\mu^r), \qquad (E,\phi,\psi) \mapsto (E\otimes\mu,\phi)$$

(see (2)) is a projective bundle.

By Theorem 3.1, $\mathrm{Br}(\mathcal{M}_\tau(r,\Lambda\otimes\mu^r)) \,=\, 0$. If $(r,g,d) \neq (2,2,\mathrm{even})$, then Proposition 2.4 says that

$$\mathrm{codim}\,(\mathcal{M}_{\tau_m^+}(r,\Lambda\otimes\mu^r) - \mathcal{U}_m(r,\Lambda\otimes\mu^r)) \geq 2\,.$$

By the Purity Theorem [13, VI.5 (Purity)], this implies that $\mathrm{Br}(\mathcal{U}_m(r,\Lambda\otimes\mu^r)) = 0$.

So

$$M \ \text{ is birational to } \ \mathbb{P}^s \times \mathcal{U}_m(r,\Lambda\otimes\mu^r) \tag{6}$$

for some natural number s. On the other hand, the map

$$M \longrightarrow \mathcal{U}_M(r+1,\Lambda)$$

that sends any (E,ϕ,ψ) to the pair $(\tilde{E},\tilde{\psi})$ defined by the extension

$$0 \longrightarrow \mathcal{O} \longrightarrow \tilde{E} \longrightarrow E \longrightarrow 0,$$

given by $\psi \in H^1(E^*)$, is again a projective fibration.

For $(r,g,d) \neq (2,2,\mathrm{even})$, we know that $\mathrm{Br}(\mathcal{M}_\tau(r+1,\Lambda)) \,=\, 0$ (see Theorem 3.1), and

$$\mathrm{codim}\,(\mathcal{M}_{\tau_M^-}(r+1,\Lambda) - \mathcal{U}_M(r+1,\Lambda)) \geq 2$$

by Proposition 2.4. Then the Purity Theorem yields that $\mathrm{Br}(\mathcal{U}_M(r+1,\Lambda)) = 0$. So

$$M \ \text{ is birational to } \ \mathbb{P}^t \times \mathcal{U}_M(r+1,\Lambda) \tag{7}$$

for some natural number t.

From (6) and (7) it follows that

$$\begin{gathered}\mathbb{P}^s \times \mathcal{M}_\tau(r, \Lambda \otimes \mu^r) \text{ and } \mathbb{P}^t \times \mathcal{M}_\tau(r+1, \Lambda) \\ \text{are birational, for } (r, g, d) \neq (2, 2, \text{even}).\end{gathered} \tag{8}$$

Hence, if $(g, d) \neq (2, \text{even})$, we see by an easy induction that $\mathcal{M}_\tau(r+1, \Lambda)$ is stably rational, for any $r+1 \geq 3$.

Finally, if $(g, d) = (2, \text{even})$, we proceed as follows. For $r+1 = 4$, we use a line bundle μ of *odd* degree. Then we already know that $\mathcal{M}_\tau(r, \Lambda \otimes \mu^r)$ is stably rational (because $\deg(\Lambda \otimes \mu^r)$ is odd). From (8), the variety $\mathcal{M}_\tau(r+1, \Lambda)$ is also stably rational. For $r+1 \geq 5$, we use induction and (8). □

4. Rationality for d large

In this section we shall suppose that $d/r \geq 2g-1$.

Let $M(r, \Lambda)$ be the moduli space of stable vector bundles of rank r and fixed determinant Λ. Fix a point $x \in X$. Let $M(r, \Lambda(-x))$ be the moduli space of stable vector bundles with rank r and fixed determinant $\Lambda(-x) := \Lambda \otimes \mathcal{O}_X(-x)$.

On $M(r, \Lambda)$, there are three projective bundles associated to the three vector spaces of the following short exact sequence

$$0 \longrightarrow H^0(E \otimes \mathcal{O}_X(-x)) \longrightarrow H^0(E) \longrightarrow E_x \longrightarrow 0 \tag{9}$$

for any $E \in M(r, \Lambda)$. Note that this is exact because $H^1(E \otimes \mathcal{O}_X(-x)) = H^0(E^* \otimes \mathcal{O}_X(x) \otimes K_X)^* = 0$, as $-\frac{d}{r} + 1 + 2g - 2 \leq 0$.

First, there is a universal projective bundle $\mathcal{P}$ over $X \times M(r, \Lambda)$. Restricting the universal bundle to $\{x\} \times M(r, \Lambda)$ we get a projective bundle

$$f : \mathcal{P}_x \longrightarrow M(r, \Lambda) . \tag{10}$$

The fiber of $\mathcal{P}_x$ over any $E \in M(r, \Lambda)$ is the projective space $\mathbb{P}(E_x)$ of lines in E_x.

Secondly, as $\frac{d}{r} \geq 2g-2$, we have the projective bundle

$$\mathcal{P}_0 \longrightarrow M(r, \Lambda) ,$$

whose fiber over any $E \in M(r, \Lambda)$ is the projective space $\mathbb{P}(H^0(E))$ of lines in $H^0(E)$. Note that we have $H^1(E) = 0$ because $d \geq r(2g-2)$.

Finally, consider as a third projective bundle

$$\mathcal{P}_1 \longrightarrow M(r, \Lambda)$$

whose fiber over any $E \in M(r, \Lambda)$ is the projective space $\mathbb{P}(H^0(E \otimes \mathcal{O}_X(-x)))$, as $\frac{d}{r} - 1 \geq 2g-2$.

From (9), there is a natural embedding

$$\mathcal{P}_1 \hookrightarrow \mathcal{P}_0 ,$$

and a projection

$$\pi : \mathcal{P}_0 - \mathcal{P}_1 \longrightarrow \mathcal{P}_x . \tag{11}$$

Recall that the projective bundle $\mathcal{P}_0$ coincides [17, Proposition 4.10] with an open subset of the moduli space of pairs for the extreme value of the parameter $\tau_m^+ = \tau_m + \epsilon$, $\epsilon > 0$,

$$\mathcal{U}_m(r, \Lambda) = \{(E, \phi) \in \mathcal{M}_{\tau_m^+}(r, \Lambda) \mid E \text{ is a stable vector bundle}\}.$$

There is a map

$$\mathcal{P}_0 = \mathcal{U}_m(r, \Lambda) \longrightarrow M(r, \Lambda), \qquad (E, \phi) \mapsto E$$

whose fiber is the projective space $\mathbb{P}(H^0(E))$.

If $\gcd(r, d) = 1$, then the rationality of $\mathcal{M}_\tau(r, \Lambda)$ is easy to deduce, as shown by the following proposition.

Proposition 4.1. *Let* $\gcd(r, d) = 1$*, and* $d > r(2g-2)$*. Then* $\mathcal{M}_\tau(r, \Lambda)$ *is rational for any* τ*.*

Proof. It is know that when $\gcd(r, d) = 1$, the moduli space $M(r, \Lambda)$ is rational [12, Theorem 1.2]. Since $M(r, \Lambda)$ is a smooth projective rational variety, the Brauer group $\mathrm{Br}(M(r, \Lambda)) = 0$. Hence the projective bundle

$$\mathcal{P}_0 \longrightarrow M(r, \Lambda)$$

is the projectivization of a vector bundle over $M(r, \Lambda)$. Since any vector bundle is Zariski locally trivial, it follows that $\mathcal{P}_0$ is birational to $\mathbb{P}^N \times M(r, \Lambda)$ for some N. Therefore, $\mathcal{P}_0$ is rational. Hence $\mathcal{M}_{\tau_m^+}(r, \Lambda)$ is rational (recall that $\mathcal{P}_0$ is a Zariski open subset of $\mathcal{M}_{\tau_m^+}(r, \Lambda)$). So $\mathcal{M}_\tau(r, \Lambda)$, being birational to $\mathcal{M}_{\tau_m^+}(r, \Lambda)$ (see Corollary 2.2), is rational. □

Proposition 4.2. *For any* r *and* Λ*, the Brauer group* $\mathrm{Br}(\mathcal{P}_x)$ *of the variety* $\mathcal{P}_x$ *vanishes. Furthermore, the variety* $\mathcal{P}_x$ *is rational.*

Proof. The Brauer group $\mathrm{Br}(M(r, \Lambda))$ is generated by the Brauer class $cl(\mathcal{P}_x)$ of the projective bundle $\mathcal{P}_x$ (cf. [1]). On the other hand, we have an exact sequence

$$\mathbb{Z} \cdot cl(\mathcal{P}_x) \longrightarrow \mathrm{Br}(M(r, \Lambda)) \longrightarrow \mathrm{Br}(\mathcal{P}_x) \longrightarrow 0$$

(see [1]). Hence $\mathrm{Br}(\mathcal{P}_x) = 0$.

Note that $\mathcal{P}_x$ is an open subset of the moduli space of parabolic bundles with one marked point x and small parabolic weight α with quasi-parabolic structure of the type $E_x \supset l$, where l is a line in E_x. Hence the rationality is given by a theorem of Boden and Yokogawa [5, Theorem 6.2]. □

Theorem 4.3. *If* $d > r(2g-1)$*, then the moduli space* $\mathcal{M}_\tau(r, \Lambda)$ *is rational, for any* τ*.*

Proof. By the argument in Proposition 4.1, it is enough to see that $\mathcal{P}_0$ is a rational space.

We will show that the projection π in (11) is an affine bundle for a vector bundle over $\mathcal{P}_x$.

Consider the projective bundle $f^*\mathcal{P}_0 \longrightarrow \mathcal{P}_x$, where f is the projection in (10). Since $\mathrm{Br}(\mathcal{P}_x) = 0$ (see Proposition 4.2), there is a vector bundle

$$\mathcal{W}_0 \longrightarrow \mathcal{P}_x$$

such that $f^*\mathcal{P}_0$ is the projective bundle $\mathbb{P}(\mathcal{W}_0)$ parametrizing the lines in $\mathcal{W}_0$. Fix one such vector bundle W_0.

Consider the projective subbundle $\mathcal{P}_1$ of $\mathcal{P}_0$ in (11). The pullback $f^*\mathcal{P}_1 \subset f^*\mathcal{P}_0$ is the projectivization of a unique subbundle

$$\mathcal{W}_1 \subset \mathcal{W}_0 \,. \tag{12}$$

Let

$$\mathcal{W} := \mathcal{W}_0/\mathcal{W}_1 \longrightarrow \mathcal{P}_x$$

be the quotient bundle. Note that $\mathbb{P}(\mathcal{W}) = f^*\mathcal{P}_x$; the isomorphism is given by π in (11). Let

$$\mathcal{O}_{\mathbb{P}(\mathcal{W})}(-1) \longrightarrow \mathbb{P}(\mathcal{W}) = f^*\mathcal{P}_x \tag{13}$$

be the tautological line bundle.

We have a tautological section

$$\sigma : \mathcal{P}_x \longrightarrow f^*\mathcal{P}_x$$

of the projective bundle $f^*\mathcal{P}_x \longrightarrow \mathcal{P}_x$; for any point $z \in \mathcal{P}_x$, the image $\sigma(z)$ is the point of $f^*\mathcal{P}_x$ defined by $(z\,,z)$. Let

$$\mathcal{L} := \sigma^*\mathcal{O}_{\mathbb{P}(\mathcal{W})}(-1) \longrightarrow \mathcal{P}_x \tag{14}$$

be the pullback, where $\mathcal{O}_{\mathbb{P}(\mathcal{W})}(-1)$ is the line bundle in (13).

It is straight forward to check that the the projection π in (11) is an affine bundle for the vector bundle

$$\mathcal{W}_1 \otimes \mathcal{L}^* \longrightarrow \mathcal{P}_x \,,$$

where $\mathcal{W}_1$ and $\mathcal{L}^*$ are constructed in (12) and (14) respectively.

The isomorphism classes of affine bundles over a variety Z for a vector bundle $V \longrightarrow Z$ are parametrized by $H^1(Z,\, V)$. If Z is an affine variety, then $H^1(Z,\, V) = 0$. Hence affine bundles over an affine variety are trivial (the trivial affine bundle for V is V itself).

Fix a nonempty affine open subset $U_0 \subset \mathcal{P}_x$ such that the vector bundle $(\mathcal{W}_1 \otimes \mathcal{L}^*)|_{U_0}$ is trivial. Since π in (11) is an affine bundle for the vector bundle $\mathcal{W}_1 \otimes \mathcal{L}^*$, and $\mathcal{W}_1 \otimes \mathcal{L}^*$ is trivial over U_0, we conclude that $\pi^{-1}(U_0)$ is isomorphic to $U_0 \times \mathbb{C}^N$, where N is the relative dimension of the fibration π. From Proposition 4.2 we know that U_0 is rational. Hence we now conclude that $\mathcal{P}_0$ is rational. □

5. Rationality for small d

We want to analyze the cases where $d \leq r(2g-1)$.

We start with the following remark.

Fix a point $x \in X$. By [3, Lemma 2.1], we know that if $d + r \leq r(g-1)$, then a general vector bundle $E \in M_X(r, \Lambda)$ satisfies the condition that

$$H^0(E \otimes \mathcal{O}_X(x)) = 0 \,. \tag{15}$$

Moreover, let $U \subset M_X(r,\Lambda)$ be the subset of the bundles E satisfying (15). Then the proof of [3, Lemma 2.1] shows that

$$\mathrm{codim}\,(M_X(r,\Lambda) - U) \geq r(g-1) - d - r + 1\,.$$

Now let $d' = -d + r(2g-2)$, $\Lambda' = \Lambda^{-1} \otimes K_X^r$, and consider

$$U' = \{E' = E^* \otimes K_X | E \in U\} \subset M_X(r,\Lambda').$$

Then for any $E' \in U'$,

$$H^1(E' \otimes \mathcal{O}_X(-x)) = H^0(E'^* \otimes K_X \otimes \mathcal{O}_X(x))^* = H^0(E \otimes \mathcal{O}_X(x))^* = 0\,.$$

We rewrite the codimension estimate as

$$\mathrm{codim}\,(M_X(r,\Lambda') - U') \geq d' - r(g-1) - r + 1\,. \tag{16}$$

We are now ready to prove the following extension of Theorem 4.3.

Theorem 5.1. *If $d > rg$, then the moduli space $\mathcal{M}_\tau(r,\Lambda)$ is rational, for any τ.*

Proof. By the previous comments, there is an open subset $U \subset M_X(r,\Lambda)$ where

$$H^1(E \otimes \mathcal{O}_X(-x)) = 0$$

for all $E \in U$. Moreover, (16) says that

$$\mathrm{codim}\,(M_X(r,\Lambda) - U) \geq d - r(g-1) - r + 1 \geq 2\,.$$

For all $E \in U$, we have an exact sequence (9).

There is a projective bundle

$$\mathcal{P}_0|_U \longrightarrow U$$

whose fiber over any $E \in U$ is the projective space $\mathbb{P}(H^0(E))$. The universal projective bundle (10) gives a corresponding projective bundle

$$f|_U : \mathcal{P}_x|_U \longrightarrow U\,.$$

By Proposition 4.2, the variety $\mathcal{P}_x|_U$ is rational. Moreover, as $\mathrm{codim}\,(M_X(r,\Lambda) - U) \geq 2$, we have that $\mathrm{codim}\,(\mathcal{P}_x - \mathcal{P}_x|_U) \geq 2$. Therefore Proposition 4.2 and the Purity Theorem [13, VI.5 (Purity)] show that $\mathrm{Br}(\mathcal{P}_x|_U) = 0$. Now the arguments in the proof of Theorem 4.3 can be carried out verbatim. □

A simple extra case, which follows by the argument above, is the following:

Corollary 5.2. *Assume $d > r(g-1)$ and $\gcd(r,d) = 1$. Then the moduli space $\mathcal{M}_\tau(r,\Lambda)$ is rational, for any τ.*

Proof. This is similar to Proposition 4.1, upon noting that, for $d \geq r(g-1)+1$, the open set

$$U = \{E \in M(r,\Lambda) | H^1(E) = 0\}$$

is non-empty and $\mathrm{codim}\,(M(r,\Lambda) - U) \geq 2$, [3, Lemma 2.1] (see the arguments in the proof of Theorem 5.1). □

Another case that can be covered is the following:

Theorem 5.3. *Let* $\gcd(r-1,d)=1$ *and* $d>0$. *Then* $\mathcal{M}_\tau(r,\Lambda)$ *is rational for any* τ.

Proof. For this we shall consider the moduli space of pairs $\mathcal{M}_{\tau_M^-}(r,\Lambda)$ for the extreme value of the parameter $\tau_M^- = \tau_M - \epsilon$, $\epsilon > 0$. By [8, Section 7.2], any τ_M^--stable pair (E,ϕ) sits in an exact sequence

$$0 \longrightarrow \mathcal{O} \stackrel{\phi}{\longrightarrow} E \longrightarrow F \longrightarrow 0\,,$$

where F is a semistable vector bundle of rank $r-1$ with $\det(F)=\Lambda$. Let

$$\mathcal{U}_M(r,\Lambda) \,:=\, \{(E,\phi)\in \mathcal{M}_{\tau_M^-}(r,\Lambda) \,|\, F \text{ is a stable vector bundle}\}\,.$$

Then there is a map

$$\pi_2 : \mathcal{U}_M(r,\Lambda) \longrightarrow M(r-1,\Lambda), \qquad (E,\phi) \longmapsto E/\phi(\mathcal{O})\,, \tag{17}$$

whose fiber over $F\in M(r-1,\Lambda)$ is the projective spaces $\mathbb{P}(H^1(F^*))$ (cf. [8, Theorem 7.7]). Note that $H^0(F^*)=0$ since $d>0$. So the morphism in (17) is always a projective bundle.

When $\gcd(r-1,d)=1$, it must be $\mathcal{U}_M(r,\Lambda) = \mathcal{M}_{\tau_M^-}(r,\Lambda)$. Moreover, the moduli space $M(r-1,\Lambda)$ is rational [12, Theorem 1.2]. Since $M(r-1,\Lambda)$ is a smooth projective rational variety, the Brauer group $\mathrm{Br}(M(r-1,\Lambda))=0$. Hence the projective bundle (17) must be a product, i.e., $\mathcal{M}_{\tau_M^-}(r,\Lambda)$ is isomorphic to $\mathbb{P}^N \times M(r-1,\Lambda)$ for some N. Thus $\mathcal{M}_\tau(r,\Lambda)$ is rational for any τ. □

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